25

FETI-H: a scalable domain decomposition method for high frequency exterior Helmholtz problems

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Introduction

The finite element discretization of the exterior Helmholtz problem leads to a system of equations that can be written as

$$\begin{aligned} \widetilde{\mathbf{K}}\mathbf{u} &= \mathbf{f}, \quad \text{where} \\ \widetilde{\mathbf{K}} &= \mathbf{K} - k^2\mathbf{M} + ik\mathbf{M}_S \end{aligned} \tag{1}$$

Matrices $\mathbf{K}$ and $\mathbf{M}$ are the so-called stiffness and mass matrix of the problem, and $\mathbf{f}$ its right-hand side vector. Matrix $\mathbf{M}_S$ is induced by the Sommerfeld radiation condition and is non zero only at the degrees of freedom lying on the outer boundary of the computational domain. In the absence of the Sommerfeld condition — that is, for the interior Helmholtz problem — $\widetilde{\mathbf{K}} = \mathbf{K} - k^2\mathbf{M}$ is usually an indefinite matrix. In this sense, $\widetilde{\mathbf{K}} = \mathbf{K} - k^2\mathbf{M} + ik\mathbf{M}_S$ is also often called an indefinite matrix. The large scale systems of equations resulting from realistic acoustic scattering applications have led to a great interest in the development of Krylov-subspace, multigrid and domain decomposition (DD) based iterative methods [1, 2, 3, 4, 19, 20, 24] for solving problem

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(1).

Here, we present a Lagrange multiplier based two-level DD method for solving iteratively large-scale systems of equations arising from the finite element discretization of high-frequency exterior Helmholtz problems. The proposed method, which is introduced in Section 25 and presented in its simplest form in Section 25 of this paper, is essentially an extension of the regularized version [11] of the FETI (Finite Element Tearing and Interconnecting) method [13, 14, 15, 21] to indefinite problems. Its two key ingredients are the regularization of each subdomain matrix by a complex interface mass matrix, and the preconditioning of the interface problem by an auxiliary coarse problem constructed in Section 25 to enforce at each iteration the orthogonality of the residual to a set of carefully chosen planar waves. In Section 25, we show numerically that the proposed method is scalable with respect to the subdomain size, and the wavenumber.

Domain decomposition with Lagrange multipliers

The DD method presented in this paper is based on the two-level FETI method [13, 9]. Our focus on FETI is motivated by our experience with dual domain decomposition algorithms, and justified by the optimal convergence properties of the FETI method for second-order elasticity and fourth-order plate and shell problems [9, 13, 22, 21]. More specifically, our objective is the extension of the FETI method to exterior Helmholtz problems.

For the sake of clarity, we consider first the case where Ω is partitioned into two non-overlapping subdomains Ω^1 and Ω^2, and the formulation of the problem does not include the Sommerfeld condition (i.e., the interior Helmholtz problem). In Section 25, we generalize the proposed method to the case of arbitrary mesh decompositions and the exterior Helmholtz problem. (In practice, it is the mesh associated with Ω that is decomposed into subdomains, and therefore in this paper we consider only subdomains with matching interfaces).

Let $\mathbf{K}^s$, $\mathbf{M}^s$, $\hat{\mathbf{K}}^s = \mathbf{K}^s - k^2\mathbf{M}^s$, and $\mathbf{f}^s$ denote respectively the stiffness matrix, mass matrix, problem matrix, and right-hand side vector associated with subdomain Ω^s, $s = 1, 2$, and let $\mathbf{u}^s$ denote the restriction to Ω^s of the solution of problem (1). We partition each vector $\mathbf{u}^s$ into two components

$$\mathbf{u}^s = \begin{bmatrix} \mathbf{u}_i^s \\ \mathbf{u}_b^s \end{bmatrix} \tag{2}$$

where the subscripts i and b designate the internal and interface boundary unknowns of a given subdomain, respectively.

Given an interface matrix $\mathbf{S}_{bb}$ — that is, a matrix defined on the interface between subdomains Ω^1 and Ω^2 — we construct the following modified Lagrangian [11].

$$\begin{aligned} \mathcal{L}(\mathbf{v}^1, \mathbf{v}^2, \lambda) \quad &= \tfrac{1}{2}{\mathbf{v}^1}^T \hat{\mathbf{K}}^1 \mathbf{v}^1 - {\mathbf{f}^1}^T \mathbf{v}^1 + \tfrac{1}{2}{\mathbf{v}^2}^T \hat{\mathbf{K}}^2 \mathbf{v}^2 - {\mathbf{f}^2}^T \mathbf{v}^2 \\ &\quad + \lambda^T(\mathbf{B}^1\mathbf{v}^1 + \mathbf{B}^2\mathbf{v}^2) + \tfrac{1}{2}({\mathbf{v}_b^1}^T \mathbf{S}_{bb}\mathbf{v}_b^1 - {\mathbf{v}_b^2}^T \mathbf{S}_{bb}\mathbf{v}_b^2) \\ &= L(\mathbf{v}^1, \mathbf{v}^2, \lambda) + \tfrac{1}{2}({\mathbf{v}_b^1}^T \mathbf{S}_{bb}\mathbf{v}_b^1 - {\mathbf{v}_b^2}^T \mathbf{S}_{bb}\mathbf{v}_b^2) \end{aligned}$$

where each of $\mathbf{B}^1$ and $\mathbf{B}^2$ is a signed Boolean matrix that extracts from a subdomain vector its interface boundary component, λ is a vector of discrete Lagrange multipliers defined on the interface between Ω^1 and Ω^2, and the superscript T designates the transpose of a quantity.

Solving problem (1) is equivalent to finding the stationary points $\mathbf{u}^1$ and $\mathbf{u}^2$ of the modified Lagrangian $\mathcal{L}$. Indeed, $L(\mathbf{v}^1, \mathbf{v}^2, \lambda)$ is the classical Lagrangian function of a two-subdomain problem, and the quantity $\frac{1}{2}({\mathbf{v}_b^1}^T \mathbf{S}_{bb} \mathbf{v}_b^1 - {\mathbf{v}_b^2}^T \mathbf{S}_{bb} \mathbf{v}_b^2)$ depends only on the traces of $\mathbf{v}^1$ and $\mathbf{v}^2$ on the interface between Ω^1 and Ω^2. Since the vector of Lagrange multipliers λ enforces the continuity equation $\mathbf{B}^1\mathbf{v}^1 + \mathbf{B}^2\mathbf{v}^2 = 0$ on the interface between Ω^1 and Ω^2, it follows that the stationary points $\mathbf{u}_1$ and $\mathbf{u}_2$ of $\mathcal{L}$ are independent of the choice of the interface matrix $\mathbf{S}_{bb}$.

Let $\mathbf{S}_I^s$ denote the subdomain matrix defined as zero inside Ω^s and as $\mathbf{S}_{bb}$ on the interface boundary between Ω^1 and Ω^2. Using the same partitioning as in (2), $\mathbf{S}_I^s$ can be written as

$$\mathbf{S}_I^s = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{S}_{bb} \end{bmatrix} \qquad s = 1,\ 2$$

The Euler equations associated with the modified Lagrangian $\mathcal{L}$ are then given by

$$\begin{aligned} (\hat{\mathbf{K}}^1 + \mathbf{S}_I^1)\mathbf{u}^1 &= (\mathbf{K}^1 - k^2\mathbf{M}^1 + \mathbf{S}_I^1)\mathbf{u}^1 = \mathbf{f}^1 - {\mathbf{B}^1}^T\lambda \\ (\hat{\mathbf{K}}^2 - \mathbf{S}_I^2)\mathbf{u}^2 &= (\mathbf{K}^2 - k^2\mathbf{M}^2 - \mathbf{S}_I^2)\mathbf{u}^2 = \mathbf{f}^2 - {\mathbf{B}^2}^T\lambda \\ \mathbf{B}^1\mathbf{u}^1 + \mathbf{B}^2\mathbf{u}^2 &= 0 \end{aligned} \tag{3}$$

The role of the interface matrix $\mathbf{S}_{bb}$ is now clear. For some given discretization of both subdomains Ω^1 and Ω^2, the prescribed wavenumber k may correspond to a resonant frequency of Ω^1 and/or Ω^2. In other words, k^2 may coalesce with an eigenvalue of either or both pencils $(\mathbf{K}^1,\ \mathbf{M}^1)$ and $(\mathbf{K}^2,\ \mathbf{M}^2)$. In such an event, either or both local problems described in (3) become ill-posed when $\mathbf{S}_{bb} = 0$. Hence, the purpose of a carefully constructed $\mathbf{S}_{bb}$ is to prevent the singularity of the subdomain matrix problems. We note that this issue has already been addressed in the literature, albeit with a different perspective (for example, see [1, 4]).

However, we would like to emphasize that as far as the design of a DD based iterative solver is concerned, it is not the potential singularity of a subdomain matrix problem that is problematic as much as the characterization of this singularity. In fact, for ill-posed Helmholtz subdomain problems, the solution of (3) can be written using the generalized inverse of the local problems. For elasticity problems ($k = 0$), the subdomain matrices can be singular, and as shown in [7, 13, 14, 21, 15], the ill-posed nature of the subdomain problems can be exploited to construct an auxiliary coarse problem. This "coarse grid" can then be used to propagate the error globally, accelerate convergence, and ensure scalability with respect to the subdomain size H.

However, for Helmholtz problems, the null space of the local matrices cannot be easily determined. Indeed, given a subdomain discretization, it is practically impossible to determine whether k^2 is a true eigenvalue of the pencil $(\mathbf{K}^s,\ \mathbf{M}^s)$, or whether it is numerically "close" to an eigenvalue of that pencil. It is not easy either to determine the multiplicity of that eigenvalue. And most importantly, whether k^2 coalesces or not with an eigenvalue of the pencil $(\mathbf{K}^s,\ \mathbf{M}^s)$ depends on the size of the mesh h, which complicates the issues further. For all these reasons, for Helmholtz problems, it

is preferable to regularize the subdomain matrices $\mathbf{K}^s$ with an interface matrix $\mathbf{S}_{bb}$ as proposed once in the regularized FETI method [11], rather than attempt to compute a general form of the subdomain solutions using null spaces of subdomain matrices. It remains to address the issue of how to construct a regularizing interface matrix $\mathbf{S}_{bb}$.

The regularized FETI method for complex problems

Previous work on stabilized finite element methods for the discretization of the Helmholtz equation [16, 17] suggests that a good choice for $\mathbf{S}_{bb}$ is the complex interface mass matrix

$$[\mathbf{S}_{bb}]_{lm} \;=\; ik[\mathbf{M}_{bb}]_{lm} \;=\; ik\int_{\Omega^1\cap\Omega^2}\phi_l\phi_m d\xi$$

where $i=\sqrt{-1}$, and ϕ_l and ϕ_m are the finite element shape functions associated with node l and node m on the interface between subdomains Ω^1 and Ω^2. Indeed, $\mathbf{M}_{bb}$ is positive definite, and it can be shown (see Theorem 1 in [12]) that

$$\begin{aligned}&\mathbf{K}^s - k^2\mathbf{M}^s \pm ik\mathbf{M}^s_I\\ &\text{where}\\ &\mathbf{M}^s_I \;=\; ik\begin{bmatrix}0 & 0\\ 0 & \mathbf{M}_{bb}\end{bmatrix}\end{aligned} \tag{4}$$

is non singular for any value of k and independently of the value of the mesh size h.

We now extend our consideration to multiple subdomains and the exterior problem. For $s=1,\ldots,N_s$, we define $\widetilde{\mathbf{K}}^s = \hat{\mathbf{K}}^s$ if the subdomain Ω_s does not touch the external artificial boundary and $\widetilde{\mathbf{K}}^s = \hat{\mathbf{K}}^s + ik\mathbf{M}^s_S$ otherwise, where $\hat{\mathbf{K}}^s$ is defined as in the previous section. The modified Lagrangian formulation presented here can be related to alternative transmission conditions for the subdomain interfaces (see [12]) both within a FETI framework [3] and other approaches [1, 4]. For the case of N_s subdomains, this formulation becomes

$$\begin{aligned}\mathcal{L}(\mathbf{v}^s,\lambda) \;&=\; \sum_{s=1}^{s=N_s}\left(\frac{1}{2}{\mathbf{v}^s}^T\widetilde{\mathbf{K}}^s\mathbf{v}^s - {\mathbf{f}^s}^T\mathbf{v}^s\right) + \lambda^T\sum_{s=1}^{s=N_s}\mathbf{B}^s\mathbf{v}^s\\ &+\; \sum_{s=1}^{s=N_s}\sum_{\Omega^s\cap\Omega^q\neq\{\emptyset\}}\frac{1}{2}\left({\mathbf{v}^s_b}^T\mathbf{S}^{s,q}_{bb}\mathbf{v}^s_b - {\mathbf{v}^q_b}^T\mathbf{S}^{s,q}_{bb}\mathbf{v}^q_b\right)\end{aligned}$$

where $\mathbf{S}^{s,q}_{bb}$ is an interface matrix with nonzero values only on $\Omega^s\cap\Omega^q$, and constructed as the mass matrix associated with the degrees of freedom lying on $\Omega^s\cap\Omega^q$

$$\begin{aligned}\mathbf{S}^{s,q}_{bb} \;&=\; \epsilon^{s,q}ik\mathbf{M}^{s,q}_{bb}\\ [\mathbf{M}^{s,q}_{bb}]_{lm} \;&=\; \int_{\Omega^s\cap\Omega^q}\phi_l\phi_m d\xi\\ \epsilon^{q,s} \;=\; -\epsilon^{s,q} \;&=\; \pm 1\end{aligned}$$

Using a notation similar to that of Eq. (4), the Euler equations associated with the above modified Lagrangian can be written as

$$(\widetilde{\mathbf{K}}^s + ik\mathbf{M}^s_I)\mathbf{u}^s \;=\; \mathbf{f}^s - {\mathbf{B}^s}^T\lambda \tag{5}$$

$$\sum_{s=1}^{s=N_s} \mathbf{B}^s \mathbf{u}^s = 0 \tag{6}$$

where

$$\mathbf{M}_I^s = \begin{bmatrix} 0 & 0 \\ 0 & \sum_{\Omega^s \cap \Omega^q \neq \{\emptyset\}} \epsilon^{s,q} \mathbf{M}_{bb}^{s,q} \end{bmatrix} \tag{7}$$

From Eqs. (5,7) and Theorem 1 in [12], it follows that, if $\mathbf{M}_I^s$ has a constant sign — that is, if $\forall q \ / \ \Omega^s \cap \Omega^q \neq \emptyset \ \epsilon^{s,q} = 1$, or $\forall q \ / \ \Omega^s \cap \Omega^q \neq \emptyset \ \epsilon^{s,q} = -1$ — the subdomain problem matrix $\widetilde{\mathbf{K}}^s + ik\mathbf{M}_I^s$ is non singular for any value of k and any value of the mesh size h. Also, in [12] an algorithm is proposed to determine the signs of the interfaces for arbitrary partitions that, in conjunction with Theorem 1 in [12], ensures that the resulting subdomain problem matrices $\widetilde{\mathbf{K}}^s + ik\mathbf{M}_I^s$ are always non singular.

From (6), we find that the interface problem associated with the regularized subdomain equations (5) is given by

$$\mathbf{F}_I \lambda = \mathbf{d} \tag{8}$$

where

$$\mathbf{F}_I = \sum_{s=1}^{s=N_s} \mathbf{B}^s (\widetilde{\mathbf{K}}^s + ik\mathbf{M}_I^s)^{-1} {\mathbf{B}^s}^T$$

$$\mathbf{d} = \sum_{s=1}^{s=N_s} \mathbf{B}^s (\widetilde{\mathbf{K}}^s + ik\mathbf{M}_I^s)^{-1} \mathbf{f}^s$$

Note that $\mathbf{F}_I$ is symmetric even if not Hermitian. For this reason, we choose the generalized conjugate residuals method (GCR) (cf. [23]) to solve the interface problem (8).

The FETI-H method

The methodology we follow here for preconditioning the regularized FETI method for exterior Helmholtz problems is based on the ideas proposed in [8] and [10]. Essentially, we propose to precondition at each iteration the interface residual generated by the GCR algorithm, by solving an auxiliary second-level problem obtained by projecting the interface problem (8) onto a suitable coarse space.

Let $\mathbf{r}^k$ denote the k-th residual associated with the solution by the GCR algorithm of the interface problem (8)

$$\mathbf{r}^k = \mathbf{d} - \mathbf{F}_I \lambda^k$$

The convergence of the regularized FETI method can be accelerated by modifying the GCR algorithm so that, at every iteration k, the interface residual $\mathbf{r}^k$ is orthogonal to a subspace represented by an interface matrix $\mathbf{Q}$

$$\mathbf{Q}^T \mathbf{r}^k = 0 \tag{9}$$

Indeed, condition (9) is a weighted-residual weak form of $\mathbf{r}^k = 0$, and therefore its effect at each iteration k is to reduce the error until $\mathbf{r}^k \to 0$. For example, if n_I

denotes the size of the interface problem, constructing an interface matrix $\mathbf{Q}$ with n_I linearly independent columns guarantees that after modification to enforce at every iteration $\mathbf{Q}^T\mathbf{r}^k = 0$, the regularized FETI method converges in one iteration. However, the subspace represented by the interface matrix $\mathbf{Q}$ must be chosen "coarse" enough to keep the overhead associated with enforcing $\mathbf{Q}^T\mathbf{r}^k = 0$ affordable. Eq. (9) is called in reference [10] an "optional admissible solution" constraint.

A straightforward approach to enforce this condition is to introduce the additional Lagrange multiplier $\mu = \mathbf{Q}\gamma$, where γ is a vector of additional unknowns, and modify the GCR algorithm to compute

$$\tilde{\lambda}^k \;=\; \lambda^k + \mu^k \;=\; \lambda^k + \mathbf{Q}\gamma^k \tag{10}$$

Substituting Eq. (10) into Eq. (9) gives

$$\mathbf{Q}^T\mathbf{F}_I\mathbf{Q}\gamma^k \;=\; \mathbf{Q}^T(\mathbf{d} - \mathbf{F}_I\lambda^k) \tag{11}$$

which shows that at each iteration k, γ^k can be obtained from the solution of an auxiliary "second-level coarse FETI" problem, which represents the projection of the regularized FETI interface problem (8) onto the "coarse" subspace represented by $\mathbf{Q}$.

From Eqs. (10,11), it follows that $\tilde{\lambda}^k$ can be computed as

$$\tilde{\lambda}^k \;=\; \mathbf{P}\lambda^k + \lambda^0$$

where $\mathbf{P}$ is the projector given by

$$\mathbf{P} \;=\; \mathbf{I} - \mathbf{Q}(\mathbf{Q}^T\mathbf{F}_I\mathbf{Q})^{-1}\mathbf{Q}^T\mathbf{F}_I$$

and λ^0 is given by

$$\lambda^0 \;=\; \mathbf{Q}(\mathbf{Q}^T\mathbf{F}_I\mathbf{Q})^{-1}\mathbf{Q}^T\mathbf{d} \tag{12}$$

which transforms the original regularized FETI interface problem into

$$\mathbf{P}^T\mathbf{F}_I\mathbf{P}\lambda \;=\; \mathbf{F}_I\mathbf{P}\lambda \;=\; \mathbf{P}^T\mathbf{d} \tag{13}$$

as $\mathbf{P}^T\mathbf{F}_I\mathbf{P} \;=\; \mathbf{F}_I\mathbf{P}$. We solve this problem by a projected generalized conjugate residuals method with the initial approximation given by (12). We note that this can also be interpreted as the GCR method for the interface problem (8) with a right preconditioner $\mathbf{P}$.

The coarse problem represented by $\mathbf{Q}$ is chosen in the form

$$\mathbf{Q} \;=\; \left[\; \mathbf{B}^1\mathbf{Q}^1 \quad \cdots \quad \mathbf{B}^s\mathbf{Q}^s \quad \cdots \quad \mathbf{B}^{N_s}\mathbf{Q}^{N_s} \;\right]$$

where $\mathbf{Q}^s$ is a matrix of local coarse vectors in subdomain Ω^s. In each subdomain, motivated by the fact that the solution can be approximated by a superposition of planar waves, we define column j of the matrix $\mathbf{Q}^s$ as

$$\mathbf{Q}^s_j(-) = e^{ik\boldsymbol{\Theta}_\mathbf{j}^\mathbf{T}\mathbf{x}(-)},$$

where $-$ indicates a degree of freedom corresponding to a given interface point, $\mathbf{x}(-)$ its nodal coordinates and $\boldsymbol{\Theta}_\mathbf{j}$ is the unitary "coarse direction" vector. In 2D, we choose the coarse directions as

$$\boldsymbol{\Theta}_\mathbf{j} = [\cos\theta_\mathbf{j}, \sin\theta_\mathbf{j}], \quad \text{where} \quad \theta_\mathbf{j} = (\mathbf{j}-\mathbf{1}) \times \frac{\mathbf{2}\pi}{\mathbf{N}_\theta}, \quad \mathbf{j} = \mathbf{1}, \ldots, \mathbf{N}_\theta;$$

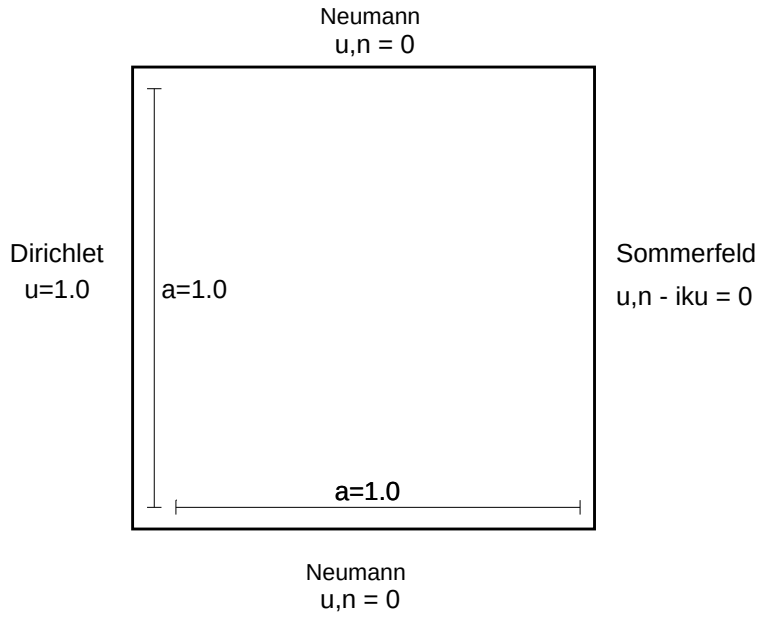

Figure 1 The guided wave problem

We choose N_θ among even integers in order to include opposite directions.

The 3D coarse directions are generated using the following algorithm. At each mesh node of coordinates $[x_n, y_n, z_n]$, we consider a cube centered at this node and uniformly discretize its surface by points of coordinates $[x_j, y_j, z_j], j = 1, \ldots, N_\theta$. The coarse direction vectors $\mathbf{\Theta_j}$ are then obtained by normalizing the vectors $[x_j - x_n, y_j - y_n, z_j - z_n]$. In this case, the choice of N_θ is given by the number of points on the surface of the cube.

We note that $\mathbf{Q}$ has $N_s \times N_\theta$ columns. Depending on the parameters of the problem, some of them may be linearly dependent, in which case they are filtered out during the factorization of the second-level problem matrix $\mathbf{Q}^T \mathbf{F}_I \mathbf{Q}$.

We refer to the regularized FETI method for complex problems equipped with the preconditioner presented herein as the FETI-H method.

Numerical results

To demonstrate the numerical scalability of the FETI-H method, we consider a guided wave problem in two and three dimensions. The 2D version of this problem is depicted in Fig. 25. The 3D version corresponds to a cube where one face is subjected to the Dirichlet boundary condition, the opposite face to the Sommerfeld condition, and the other faces to the Neumann boundary condition. We consider various configurations of this problem and demonstrate numerically the scalability of the FETI-H method with respect to the subdomain size H and the wavenumber k. Results concerning the scalability with respect to the mesh size h can be found in [12].

ka	N_θ	N_S	Size of the coarse problem	Number of iterations
20	4	25	88	33
20	4	49	180	31
20	4	81	304	28
20	8	25	176	19
20	8	49	354	20
20	8	81	600	18
20	16	25	262	18
20	16	49	471	19
20	16	81	713	18
40	4	25	88	70
40	4	49	180	56
40	4	81	304	61
40	8	25	176	25
40	8	49	354	25
40	8	81	600	22
40	16	25	262	19
40	16	49	471	22
40	16	81	713	22
60	4	25	88	148
60	4	49	180	141
60	4	81	304	110
60	8	25	176	50
60	8	49	354	42
60	8	81	600	22
60	16	25	262	18
60	16	49	471	18
60	16	81	713	18

Table 1 The guided wave problem in 2D – $h = 1/315$, total size of the problem = 99225 dofs

In both cases, we perform the same computations but for three different values of the wavenumber corresponding to $ka = 20, 40$, and 60 in 2D, and $ka = 5, 10$, and 15 in 3D. We also consider several $1/H \times 1/H$ mesh partitions with 25, 49 and 81 subdomains. We report the obtained performance results in Table 1 for the 2D case, and Table 2 for the 3D one. Note that in these tables, the size of the coarse problem is computed as the number of linearly independent columns found during the factorization of the second-level problem matrix $\mathbf{Q}^T \mathbf{F}_I \mathbf{Q}$. Hence, this size is less or equal to $N_\theta \times N_s$.

The results summarized in both Table 1 and Table 2 confirm the scalability of the FETI-H method with respect to both the wavenumber and the number of subdomains.

ka	N_θ	N_S	Size of the coarse problem	Number of iterations
5	0	25	0	42
5	0	64	0	99
5	0	125	0	73
5	2	27	178	26
5	2	64	450	26
5	2	125	908	27
5	3	27	428	26
5	3	64	579	27
5	3	125	1211	29
5	4	27	461	28
5	4	64	602	28
5	4	125	1269	34
10	0	27	0	67
10	0	64	0	151
10	0	125	0	116
10	2	27	178	31
10	2	64	450	30
10	2	125	908	33
10	3	27	597	22
10	3	64	1276	31
10	3	125	2537	41
10	4	27	823	29
10	4	64	1441	36
10	4	125	2835	41
15	0	27	0	112
15	0	64	0	331
15	0	125	0	251
15	2	27	178	47
15	2	64	450	38
15	2	125	908	42
15	3	27	598	24
15	3	64	1471	27
15	3	125	2993	26
15	4	27	1067	23
15	4	64	1626	44
15	4	125	3118	47

Table 2 The guided wave problem in 3D – $h = 1/60$, total size of the problem = 226981 dofs

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