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A domain decomposition method with Lagrange multipliers for linear elasticity

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INTRODUCTION

In the last decade a lot of research has been carried out on nonoverlapping domain decomposition methods with Lagrange multipliers. In these methods the original domain is decomposed into nonoverlapping subdomains. The intersubdomain continuity is then enforced by Lagrange multipliers across the interface defined by the subdomain boundaries. A computationally very efficient member of this class of domain decomposition algorithms is the **F**inite **E**lement **T**earing and **I**nterconnecting (FETI) method introduced by Farhat and Roux [FR91]. In a variant of the FETI method introduced in Farhat, Mandel, and Roux [FMR94] a Neumann and a Dirichlet finite element problem is solved exactly on each subdomain, in each iteration.

In this paper, a new domain decomposition method with Lagrange multipliers is introduced by reformulating the preconditioned system of the FETI algorithm as a saddle point problem with both, primal and dual variables as unknowns. The resulting system is then solved using block–structured preconditioners in combination with a suitable Krylov space method. This approach avoids costly exact subdomain solves

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since it allows inexact subdomain solvers. Good features of the FETI method such as scalability and efficiency are preserved.

The remainder of this paper is organized as follows. In sect. 2, we present the equations of linear elasticity and in sect. 3 a finite element discretization thereof. In sect. 4, we review the FETI method and we introduce our new method in sect. 5. Finally, in sect. 6, we present numerical results for a cantilever problem.

THE EQUATIONS OF LINEAR ELASTICITY

The equations of linear elasticity model the displacement of a linear elastic material under the action of external and internal forces. We denote the elastic body by $\Omega \subset \mathbf{R}^d, d = 2, 3$, and its boundary by $\partial\Omega$ and we assume that one part of the boundary, Γ_0, is clamped, i.e. with homogeneous Dirichlet boundary conditions, and that the rest, $\Gamma_1 := \partial\Omega \setminus \Gamma_0$, is subject to a surface force $\mathbf{g}$, i.e. an inhomogeneous natural boundary condition. We can also introduce an internal volume force $\mathbf{f}$, e.g. gravity. The appropriate space for a variational formulation is the Sobolev space $H^1_{\Gamma_0}(\Omega) := \{\mathbf{v} \in \mathbf{H^1(\Omega)^d} \ : \ \mathbf{v}_{|\Gamma_0} = \mathbf{0}\}$. The linear elasticity problem consists in finding the displacement $\mathbf{u} \in \mathbf{H^1_{\Gamma_0}(\Omega)}$ of the elastic body Ω, such that

$$2\mu \int_\Omega \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) \mathbf{dx} \ + \ \lambda \int_\Omega \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} \, \mathbf{dx} = \langle \mathbf{F}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{H^1_{\Gamma_0}(\Omega)}. \tag{1}$$

Here μ and λ are the Lamé constants, $\varepsilon_{ij}(\mathbf{u}) := \frac{\mathbf{1}}{\mathbf{2}}(\frac{\partial \mathbf{u_i}}{\partial \mathbf{x_j}} + \frac{\partial \mathbf{u_j}}{\partial \mathbf{x_i}})$ is the linearized strain tensor, $\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) = \sum_{\mathbf{i,j=1}}^{\mathbf{d}} \varepsilon_{\mathbf{ij}}(\mathbf{u}) \varepsilon_{\mathbf{ij}}(\mathbf{v})$, and $\langle \mathbf{F}, \mathbf{v} \rangle := \sum_{\mathbf{i=1}}^{\mathbf{d}} \int_\Omega \mathbf{f_i v_i} \, \mathbf{dx} \ + \ \sum_{\mathbf{i=1}}^{\mathbf{d}} \int_{\Gamma_1} \mathbf{g_i v_i} \, \mathbf{d}\sigma$. The associated bilinear form of linear elasticity is

$$a(\mathbf{u}, \mathbf{v}) = \mathbf{2}\mu \int_\Omega \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) \mathbf{dx} \ + \ \lambda \int_\Omega \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} \, \mathbf{dx}.$$

In this article, we only consider the case of compressible elasticity. This means that the Lamé parameter λ remains bounded.

FINITE ELEMENTS AND THE DISCRETE PROBLEM

Since we only consider compressible elastic materials, it follows from Korn's first inequality, cf. Ciarlet [Cia88], that the bilinear form $a(\cdot, \cdot)$ is uniformly elliptic. Therefore we can successfully discretize the system (1) with low–order, conforming finite elements, such as linear or bi-/trilinear elements.

We assume that a triangulation τ^h of Ω is given which is shape regular and has a typical element diameter of h. We denote by $\mathbf{H^h(\Omega)}$ the corresponding conforming space of finite element functions, e.g. piecewise linear or bi-/trilinear continuous functions. Our goal is to solve the discrete problem

$$a(\mathbf{u_h}, \mathbf{v_h}) = \langle \mathbf{F}, \mathbf{v_h} \rangle \quad \forall \mathbf{v_h} \in \mathbf{H^h(\Omega)}. \tag{2}$$

In the following sections, we work exclusively with the discrete problem. To avoid unnecessary notation, we drop the subscript h from now on.

A REVIEW OF THE FETI METHOD

In this section, we give a brief review of the original FETI method proposed by Farhat and Roux, cf. [FR91]. For more detailed descriptions and proofs, we refer to [FMR94, RF98] and the references therein.

Let the domain $\Omega \subset \mathbf{R}^d, d = 2, 3$, be decomposed into N non-overlapping subdomains $\Omega_i, i = 1, \ldots, N$, and the finite element nodes of neighboring subdomains match on the interface $\Gamma := \left(\bigcup_{i=1}^N \partial\Omega_i\right) \setminus \partial\Omega$.

Let the corresponding conforming finite element spaces be $\mathbf{W_i} = \mathbf{W^h}(\mathbf{\Omega_i}), \mathbf{i} = \mathbf{1}, \ldots, \mathbf{N}$, and let $\mathbf{W} := \prod_{\mathbf{i=1}}^{\mathbf{N}} \mathbf{W_i}$ denote the associated product space.

The finite element spaces can always be identified with $\mathbf{R}^m$ for a suitable m. We mark these spaces in the following with an underline, e.g. $\mathbf{W}$ is identified with $\underline{\mathbf{W}}$. Analogously, we denote by $\underline{\mathbf{u}}$ the vector of nodal values associated with the finite element function $\mathbf{u}$.

For each subdomain $\Omega_i, i = 1, \ldots, N$, we assemble local stiffness matrices K_i and local load vectors $\underline{f}_i$. We denote by $\underline{\mathbf{u}}_i$ the local vectors of nodal values.

We can now reformulate (2) as a minimization problem with constraints given by intersubdomain continuity conditions:

Find $\underline{\mathbf{u}} \in \underline{\mathbf{W}}$, such that

$$\left.\begin{array}{c} J(\underline{\mathbf{u}}) := \frac{1}{2}\underline{\mathbf{u}}^t K \underline{\mathbf{u}} - \underline{\mathbf{f}}^t \underline{\mathbf{u}} \rightarrow \min \\ B\underline{\mathbf{u}} = 0 \end{array}\right\} \tag{3}$$

where $\underline{\mathbf{u}} = [\underline{\mathbf{u}}_1, \ldots, \underline{\mathbf{u}}_N]^t, \underline{\mathbf{f}} = [\underline{\mathbf{f}}_1, \ldots, \underline{\mathbf{f}}_N]^t$, and $K = diag_{i=1}^N K_i$ is a block–diagonal matrix.

The matrix $B = [B_1, \ldots, B_N]$ is constructed such that the components of any vector $\underline{\mathbf{u}}$, which are associated with the same node on Γ, coincide when $B\underline{\mathbf{u}} = 0$. The local stiffness matrices K_i are positive semidefinite. We can assume that the problem (3) is uniquely solvable which is equivalent to $\ker K \cap \ker B = \{0\}$, i.e. K is invertible on the nullspace of B. This is a consequence of the original finite element model being elliptic.

By introducing Lagrange multipliers $\underline{\lambda} \in \underline{\mathbf{U}} := \operatorname{range} B$ to enforce the constraint $B\underline{\mathbf{u}} = 0$, we obtain a saddle point problem from (3):

Find $(\underline{\mathbf{u}}, \underline{\lambda}) \in \underline{\mathbf{W}} \times \underline{\mathbf{U}}$, such that

$$\left.\begin{array}{rcrcl} K\underline{\mathbf{u}} & + & B^t\underline{\lambda} & = & \underline{\mathbf{f}} \\ B\underline{\mathbf{u}} & & & = & 0 \end{array}\right\} \tag{4}$$

We will also use a full rank matrix R, such that $\operatorname{range} R = \ker K$.

The solution of the first equation in (4) exists if and only if $\underline{\mathbf{f}} - B^t\underline{\lambda} \in \operatorname{range} K$.

Eliminating the primal variables $\underline{\mathbf{u}}$ leads to

$$\left.\begin{array}{rcl} PF\underline{\lambda} & = & P\underline{\mathbf{d}} \\ G^t\underline{\lambda} & = & \underline{\mathbf{e}} \end{array}\right\} \tag{5}$$

with $G := BR, F := BK^\dagger B^t, \underline{\mathbf{d}} := BK^\dagger\underline{\mathbf{f}}, P := I - G(G^tG)^{-1}G^t$, and $\underline{\mathbf{e}} := R^t\underline{\mathbf{f}}$. Here $K^\dagger$ is the pseudoinverse of K which provides a solution orthogonal to the nullspace

of K. Note that P is an orthogonal projection from $\underline{\mathbf{U}}$ onto $\ker G^t$. The constraint $G^t\underline{\lambda} = \underline{\mathbf{e}}$ ensures that $\underline{\mathbf{f}} - B^t\underline{\lambda}$ is in the range of K.

We define the space of admissible increments by

$$\underline{\mathbf{V}} := \{\underline{\mu} \in \underline{\mathbf{U}} \,:\, \underline{\mu} \perp B\underline{\mathbf{w}} = 0 \; \forall \underline{\mathbf{w}} \in \ker K\} = \ker G^t.$$

The original FETI method is a conjugate gradient method in the space $\underline{\mathbf{V}}$ applied to

$$PF\underline{\lambda} = P\underline{\mathbf{d}}, \qquad \underline{\lambda} \in \underline{\lambda}_0 + \underline{\mathbf{V}} \tag{6}$$

with an initial approximation $\underline{\lambda}_0$ chosen such that $G^t\underline{\lambda}_0 = \underline{\mathbf{e}}$. Let D be a diagonal matrix. The preconditioner M^{-1} introduced in Farhat, Mandel, and Roux [FMR94] is of the form

$$M^{-1} = B \begin{bmatrix} O & O \\ O & D^{-1}SD^{-1} \end{bmatrix} B^t.$$

The matrix S is the Schur complement of K obtained by eliminating the interior degrees of freedom. This can obviously be done for all subdomains at the same time. The resulting matrix S is block–diagonal, and operates only on the degrees of freedom on the subdomain boundaries. In the simplest case, the diagonal matrix D is chosen as the identity; this is the choice in the original FETI method by Farhat and Roux. Another possibility, which leads to faster convergence, is to choose D as a diagonal matrix where the main diagonal contains the number of subdomains to which the interface node belongs. This multiplicity scaling is discussed in Rixen and Farhat [RF98].

In the application of this preconditioner, N independent Dirichlet problems have to be solved at each iteration step; it is known as the Dirichlet preconditioner.

To keep the search directions of the conjugate gradient method in the space $\underline{\mathbf{V}}$, the application of the preconditioner M^{-1} has to be followed by another application of the projection P. Hence, the Dirichlet variant of the FETI method is the conjugate gradient algorithm applied to

$$PM^{-1}PF\,\underline{\lambda} = PM^{-1}P\,\underline{\mathbf{d}}, \qquad \underline{\lambda} \in \underline{\lambda}_0 + \underline{\mathbf{V}}. \tag{7}$$

It has been shown by Mandel and Tezaur [MT96] that the condition number of the original FETI method (D=I) satisfies $\kappa(PM^{-1}PF) \leq C\,(1 + \log(H/h))^3$, where H/h is determined by the number of degrees of freedom of a subdomain and C is a constant which is independent of h, and H.

In Tezaur [Tez98], a condition number estimate on the order of $(1 + \log(H/h))^2$ is given for an algebraic FETI method developed by Park et al. [PJF97].

For a modification of the FETI preconditioner with almost no extra cost, Klawonn and Widlund [KW99b] show that the condition number is bounded asymptotically by $C\,(1 + \log(H/h)^2$.

THE NEW METHOD USING INEXACT SOLVERS

If $\lambda = \mu + \lambda_0$ with $\mu \in V$ then, we can reformulate (6) as

$$PF\underline{\mu} = P(\underline{\mathbf{d}} - F\underline{\lambda}_0)$$

which is equivalent to

$$PBK^{\dagger}B^t\underline{\mu} = PBK^{\dagger}(\underline{\mathbf{f}} - B^t\underline{\lambda}_0). \tag{8}$$

Keeping in mind that $\underline{\lambda} = \underline{\mu} + \underline{\lambda}_0$ and that $\underline{\mathbf{u}} = K^{\dagger}(\underline{\mathbf{f}} - B^t\underline{\lambda}) + R\underline{\alpha}$, we see immediately that the solution of (8) can also be obtained by solving

$$\begin{bmatrix} K & B^t \\ PB & O \end{bmatrix} \begin{bmatrix} \underline{\mathbf{u}} \\ \underline{\mu} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{f}} - B^t\underline{\lambda}_0 \\ 0 \end{bmatrix}.$$

Since $\underline{\mu} \in \underline{\mathbf{V}}$ and $P\underline{\mu} = \underline{\mu}$ for $\underline{\mu} \in \underline{\mathbf{V}}$, we can make the system matrix symmetric

$$\begin{bmatrix} K & (PB)^t \\ PB & O \end{bmatrix} \begin{bmatrix} \underline{\mathbf{u}} \\ \underline{\mu} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{f}} - B^t\underline{\lambda}_0 \\ 0 \end{bmatrix}. \tag{9}$$

Note that in this formulation we are not enforcing $B\underline{\mathbf{u}} = 0$ but only its projected version $PB\underline{\mathbf{u}} = 0$. The addition of an element of the nullspace of K does not change the solution $\underline{\mathbf{u}}$ of the first equation in (9). We use this fact to post-process $\underline{\mathbf{u}}$, such that $B\underline{\mathbf{u}} = 0$ is finally satisfied. This can be done by setting $\underline{\mathbf{u}}_{cor} := \underline{\mathbf{u}} - R(G^tG)^{-1}G^tB\underline{\mathbf{u}}$. It follows that $B\underline{\mathbf{u}}_{cor} = PB\underline{\mathbf{u}} = 0$.

For the solution of the saddle point problem (9), we propose a preconditioned conjugate residual method with a block–diagonal preconditioner; for a detailed description of this algorithm, see Hackbusch [Hac94] or Klawonn [Kla95, Kla98].

Our preconditioner has the form

$$\mathcal{B} = \begin{bmatrix} \widehat{K} & O \\ O & \widehat{M} \end{bmatrix}$$

Here, $\widehat{K}$ is symmetric and spectrally equivalent to $K + D_H\, Q$ where $Q = diag_{i=1}^{N} Q_i$ is a block–diagonal matrix with Q_i being the mass matrices associated with the discretizations on Ω_i and $D_H = (d_{ii})_{i=1,\ldots,N}$ a diagonal matrix with $d_{ii} = H_i^{-2}, i = 1, \ldots, N$. Here, $H_i, i = 1, \ldots, N$ denotes the diameters of the subdomain $\Omega_i, i = 1, \ldots, N$. We further assume that $\widehat{M}$ is symmetric and spectrally equivalent to M, i.e. we assume the existence of constants $k_0, k_1, m_0, m_1 > 0$, such that

$$\begin{array}{ccccccc} k_0 & \underline{\mathbf{u}}^t(K + D_H Q)\underline{\mathbf{u}} & \leq & \underline{\mathbf{u}}^t\widehat{K}\underline{\mathbf{u}} & \leq & k_1 & \underline{\mathbf{u}}^t(K + D_H Q)\underline{\mathbf{u}} \quad \forall \underline{\mathbf{u}} \in \underline{\mathbf{W}}, \\ m_0 & \underline{\lambda}^t M\underline{\lambda} & \leq & \underline{\lambda}^t\widehat{M}\underline{\lambda} & \leq & m_1 & \underline{\lambda}^t M\underline{\lambda} \quad \forall \underline{\lambda} \in \underline{\mathbf{V}}. \end{array} \tag{10}$$

We can assume that the constants k_0, k_1, m_0, and m_1 are independent of the mesh size and the number of subdomains but also note that preconditioners with bounds that are weakly dependent on the mesh size, e.g. ILU methods, are of practical interest.

From these assumptions it is clear that our preconditioner $\mathcal{B}$ is symmetric, positive definite and thus can be used with the preconditioned conjugate residual method. In order to have a computationally efficient preconditioner, we must assume, in addition, that the application of $\widehat{K}^{-1}$ and $\widehat{M}^{-1}$ to a vector can be computed at a low cost.

To guarantee that the iterates belong to range K, we introduce the projection P_R onto range K by

$$P_R := I - R(R^tR)^{-1}R^t.$$

We recall that range $R = \ker K$ and note that P_R is a block matrix with a 3×3 block for each interior subdomain; the expense of applying P_R to a vector is therefore very modest.

The resulting domain decomposition method is the conjugate residual algorithm using the $\mathcal{B}-$inner product applied to the preconditioned system

$$\mathcal{B}^{-1}\mathcal{A}\underline{\mathbf{x}} = \mathcal{B}^{-1}\underline{\mathbf{F}}$$

with

$$\mathcal{A} = \begin{bmatrix} K & (PB)^t \\ PB & O \end{bmatrix}, \mathcal{B}^{-1} = \begin{bmatrix} P_R\widehat{K}^{-1}P_R^t & O \\ O & P\widehat{M}^{-1}P^t \end{bmatrix},$$

$$\underline{\mathbf{x}} = \begin{bmatrix} \underline{\mathbf{u}} \\ \underline{\mu} \end{bmatrix}, \underline{\mathbf{F}} = \begin{bmatrix} \underline{\mathbf{f}} - B^t\underline{\lambda}_0 \\ 0 \end{bmatrix}.$$

We note that it is easy to see that only two matrix-vector products with the projection P and one with the projection P_R are required in each step.

A proof of the following theorem is given in Theorem 2 in [KW99a].

Theorem 1 *For $D = I$, we have*

$$\kappa(\mathcal{B}^{-1}\mathcal{A}) \leq C\,(1 + \log(H/h))^2,$$

with a constant C independent of H, h.

NUMERICAL RESULTS

We have applied our domain decomposition method to a two-dimensional, linear elasticity problem. We consider a plane stress problem with a Poisson ratio $\nu = 0.3$ and an elasticity modulus $E = 2.1 \cdot 10^{11} N/m^2$ which models steel. The domain Ω is a cantilever unit square fixed on the left hand side and free on the other three edges except at the upper right corner, where we impose a point force that has equal components in the positive x and y directions equal to $10^5 N$. We discretize this problem by using bilinear finite elements with a meshsize h.

All computations were performed in MATLAB 5.0. As our Krylov space method, we use the preconditioned conjugate residual method with a zero initial guess. The stopping criterion is the relative reduction of the initial residual by 10^{-6}, i.e. $\|\underline{\mathbf{r}}_n\|_2/\|\underline{\mathbf{r}}_0\|_2 < 10^{-6}$, where $\underline{\mathbf{r}}_n, \underline{\mathbf{r}}_0$ are the $n-$th and initial residual, respectively. In our experiments, we have used a simplified implementation without the projector P_R which is needed for the theoretical analysis. For a more detailed numerical study including this projector, we refer to Klawonn and Widlund [KW99a].

We decompose our domain Ω into $N \times N$ square subdomains with $H := 1/N$. In order to analyze the numerical scalability of the method, we have carried out two different types of experiments for different combinations of preconditioners $\widehat{K}$ and $\widehat{M}$. In our first set of runs, we have kept the subdomain size H/h fixed and increased the number of subdomains and thus the overall problem size. Our second series of experiments is carried out with a fixed number of subdomains and an increasing subdomain size H/h resulting in an increased $1/h$.

Table 1 $H/h = 8$, (I) : $\widehat{K} = K + 1/H^2 M_Q$ and $\widehat{M} = M$, (II) : $\widehat{K} = K + 1/H^2 M_Q$ and $\widehat{M}$ using ILU(0), (III) : $\widehat{K}$ ILU(10^{-4}) of $K + 1/H^2 M_Q$ and $\widehat{M} = M$, (IV) : $\widehat{K}$ using ILU(10^{-4}) of $K + 1/H^2 M_Q$ and $\widehat{M}$ using ILU(0), MS : D= multiplicity scaling, I : D=Identity.

NEW METHOD		Iter (I)		Iter (II)		Iter (III)		Iter (IV)	
$1/h$	$1/H$	MS	I	MS	I	MS	I	MS	I
16	2	11	19	11	19	21	33	21	34
32	4	17	27	17	27	29	60	31	61
64	8	21	33	23	33	37	73	39	76
96	12	21	39	23	39	44	77	45	81
128	16	21	39	25	41	47	80	51	84

Table 2 $H = 1/4$; (I) : $\widehat{K} = K + 1/H^2 M_Q$ and $\widehat{M} = M$, (II) : $\widehat{K} = K + 1/H^2 M_Q$ and $\widehat{M}$ using ILU(0), (III) : $\widehat{K}$ ILU(10^{-4}) of $K + 1/H^2 M_Q$ and $\widehat{M} = M$, (IV) : $\widehat{K}$ using ILU(10^{-4}) of $K + 1/H^2 M_Q$ and $\widehat{M}$ using ILU(0), MS : D= multiplicity scaling, I : D=Identity.

NEW METHOD		Iter (I)		Iter (II)		Iter (III)		Iter (IV)	
$1/h$	H/h	MS	I	MS	I	MS	I	MS	I
16	4	15	25	15	25	27	61	27	59
32	8	17	27	17	27	31	60	32	62
64	16	19	29	21	29	40	77	45	81
128	32	19	33	23	43	98	194	116	194

In order to see how our method behaves in the best possible case, we first report on results for $\widehat{K} = K + \frac{1}{H^2}Q$ and $\widehat{M} = M$, cf. Tables 1,2, Iter (I). To gain insight into the convergence behavior in the case of inexact blocks $\widehat{K}$ and $\widehat{M}$, we have run experiments with preconditioners based on incomplete Cholesky factorizations (ILU). In the following, ILU(0) stands for incomplete Cholesky factorizations with no fill in allowed and ILU(tol) represents a threshold ILU factorization with a threshold of tol as provided in MATLAB 5.0, i.e. all entries in a column of the factor L of the Cholesky factorization are dropped if the magnitude of the entry is smaller than the drop tolerance tol times the norm of that column. Three different combinations are considered, cf. (II)–(IV) in Tables 1,2. In the experiments presented in the Tables 1,2, $\widehat{M}$ using ILU(0) means that the Schur complement is computed inexactly using ILU(0).

For all cases, we present results for $\widehat{M}$ constructed using $D = I$ as well as the multiplicity scaling $D = MS$, cf. section 5. As in the original FETI algorithm, the convergence is considerably faster for the case of multiplicity scaling. With this scaling the asymptotic convergence rate is also reached earlier than for $D = I$.

For a more detailed numerical study, we refer to [KW99a].

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