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Block preconditioners for nonsymmetric saddle point problems

Piotr Krzyżanowski[1]

Introduction

We discuss a class of methods for preconditioning nonsymmetric, indefinite saddle point problems arising from mixed finite element discretization of partial differential equations, in particular, the linearized Navier–Stokes equation. The corresponding matrix has block structure

$$\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix},$$

where the matrix block A is positive definite, yet not necessarily symmetric, and C is nonnegative. We present new mathematical results concerning block diagonal and block triangular preconditioners based on *symmetric, positive definite* blocks. In both cases, the convergence of iterative method is independent of the mesh parameter h. Our analysis is also valid for inexact preconditioning blocks. Detailed proofs of theorems discussed here may be found in [Krz97].

Block preconditioners for saddle point problems were discussed by many authors before, see for example [D'y87], [BP88], [BP90], [BP97], [RW92], [SW94], [Elm96b], [ES96], [ESW97], [Elm96a], [Kla98b], [Kla98a], [KS97]. The approach we propose gives an application programmer a great opportunity to reuse, in an efficient way, existing very powerful methods (or software) like the domain decomposition or the multigrid method for symmetric positive definite problems. We believe that in certain cases this may be a more robust approach than to use custom domain decomposition preconditioners developed especially for the whole

1 Warsaw University, Institute of Applied Mathematics, Banacha 2, 02–097 Warszawa, Poland
Eleventh International Conference on Domain Decomposition Methods
Editors Choi-Hong Lai, Petter E. Bjørstad, Mark Cross and Olof B. Widlund

saddle point problem.

General assumptions

The generic constant "Const", which appears later in this paper is independent of the parameter h.

Let $\bar{V}, \bar{W}$ be real Hilbert spaces with scalar products denoted by $((\cdot,\cdot))$ and $(\cdot,\cdot)$, respectively. The norms in these spaces, induced by the inner products, will be denoted by $||\cdot||$ and $|\cdot|$. We consider a family of finite dimensional subspaces indexed by parameter $h \in (0,1)$, $V_h \subset \bar{V}$, and $W_h \subset \bar{W}$.

Let us introduce three continuous bilinear forms: $a : \bar{V} \times \bar{V} \to R$, $b : \bar{V} \times \bar{W} \to R$, $c : \bar{W} \times \bar{W} \to R$, and assume that $a(\cdot,\cdot)$ is $\bar{V}$–elliptic, i.e.

$$\exists \alpha > 0 \quad \forall v \in \bar{V} \qquad a(v, v) \geq \alpha ||v||^2 \tag{1}$$

and that $c(\cdot,\cdot)$ satisfies

$$\exists \gamma \geq 0 \quad \forall p \in \bar{W} \qquad c(p, p) \geq \gamma |p|^2 \tag{2}$$

(we allow $\gamma = 0$). Notice that $a(\cdot,\cdot)$ does not need to be symmetric. Throughout this paper we also assume that V_h and W_h satisfy the uniform LBB condition, see [GR86],

$$\exists \beta > 0 \quad \forall h \in (0,1) \quad \forall p \in W_h \qquad \beta |p| \leq \sup_{v \in V_h, v \neq 0} \frac{b(v, p)}{||v||}. \tag{3}$$

In what follows we consider preconditioners for a family of finite dimensional problems (we drop the subscript h for simplicity of notation)

Problem 1 *Find $(u,p) \in V \times W$ such that*

$$M \begin{pmatrix} u \\ p \end{pmatrix} \equiv \begin{pmatrix} A & B^* \\ B & -C \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}. \tag{4}$$

The operators in (4) are:

$$\begin{aligned} A : V \to V, \quad & ((Au, v)) = a(u, v) \quad \forall u, v \in V, \\ B : V \to W, \quad & (Bu, p) = b(u, p) \quad \forall u \in V, p \in W, \\ C : W \to W, \quad & (Cp, q) = c(p, q) \quad \forall p, q \in W, \end{aligned}$$

while the right hand side $F \in V, G \in W$ is defined through $((F, v)) \equiv \langle\langle f, v\rangle\rangle$ and $(G, w) \equiv \langle g, w\rangle$, where f, g are given continuous functionals on $\bar{V}$, $\bar{W}$, and $\langle\langle\cdot,\cdot\rangle\rangle$, $\langle\cdot,\cdot\rangle$ denote the duality pairing in $\bar{V}$, $\bar{W}$, respectively. B^* denotes the formal adjoint operator to B, i.e. $(Bu, p) = ((u, B^*p))$ for all $u \in V$, $p \in W$.

We introduce two more operators, $A_0 : V \to V$ and $J_0 : W \to W$. We assume that they are self-adjoint, their inverses are easy to apply, and there exist positive constants a_0, a_1, b_0, b_1, which are independent of h, such that

$$a_0((u,u)) \leq ((A_0 u, u)) \leq a_1((u,u)) \qquad \forall u \in V, \tag{5}$$

$$b_0(p,p) \leq (J_0 p, p) \leq b_1(p,p) \qquad \forall p \in W. \tag{6}$$

The product space $V \times W$ is equipped with a natural scalar product $\langle \cdot, \cdot \rangle$,

$$\left\langle \begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} v \\ q \end{pmatrix} \right\rangle = ((u,v)) + (p,q),$$

however, we are going to analyse the preconditioned problem using custom inner product $[\cdot, \cdot]$, which depends on the preconditioner being used:

$$\left[\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} v \\ q \end{pmatrix} \right] = \left\langle \begin{pmatrix} A_0 & 0 \\ 0 & J_0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} v \\ q \end{pmatrix} \right\rangle.$$

Lemma 1 *Under the above assumptions,*

$$\begin{aligned} &||A||_{V\to V}, \quad ||B||_{V\to W}, \quad ||C||_{W\to W}, \\ &||A^*||_{V\to V}, \quad ||B^*||_{W\to V}, \quad ||C^*||_{W\to W}, \\ &||A_0||_{V\to V}, \quad ||J_0||_{W\to W}, \quad ||A_0^{-1}||_{V\to V}, \quad ||J_0^{-1}||_{W\to W}, \\ &||M||_{V\times W\to V\times W}, \quad ||M^*||_{V\times W\to V\times W} \leq Const. \end{aligned}$$

Preconditioners leading to symmetric, positive definite problems

In this section we are going to extend the results obtained previously for symmetric saddle point problems e.g. in [D'y87] and [BP90]. This approach is of normal equations type, which influences the convergence speed of the method; though, it has certain integesting advantages, too. The transformed system allows for using the conjugate gradients (PCG) method, which is less memory consuming than, e.g. GMRES; the transformed system is well conditioned, so the number of iterations required to damp the error by a given factor is independent of h. Moreover, no scaling is needed for A_0, in contrast to results for preconditioner for the GMRES in the next section and the analysis remains valid for A only V^0-elliptic, see Remark 2.

Block diagonal preconditioner

Using a block diagonal preconditioner

$$M_D = \begin{pmatrix} A_0 & 0 \\ 0 & J_0 \end{pmatrix},$$

we transform system (4) into the following one:

Problem 2 *Find* $(u,p) \in V \times W$ *such that*

$$M_D^{-1} M^* M_D^{-1} M \begin{pmatrix} u \\ p \end{pmatrix} = M_D^{-1} M^* M_D^{-1} \begin{pmatrix} F \\ G \end{pmatrix}.$$

Clearly, the operator $P = M_D^{-1} M^* M_D^{-1} M$ is selfadjoint with respect to the inner product $[\cdot, \cdot]$. The following theorem guarantees P is well conditioned with respect to h:

Theorem 1 *There exist constants* $0 < m_0 \leq m_1$*, independent of* h*, such that*

$$m_0 \left[\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix} \right] \leq \left[P \begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix} \right] \leq m_1 \left[\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix} \right]. \tag{7}$$

Proof. Observe that

$$\left[\mathcal{P}\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right] = ((A_0^{-1}(Au + B^*p), Au + B^*p)) + (J_0^{-1}(Bu - Cp), Bu - Cp). \tag{8}$$

Defining $\tilde{f} = Au + B^*p$ and $\tilde{g} = Bu - Cp$, we obviously have that (u, p) satisfies

$$\begin{aligned} Au + B^*p &= \tilde{f} \\ Bu - Cp &= \tilde{g}, \end{aligned}$$

so from (5), (6) and the stability result for Problem 1, see [GR86], we obtain

$$\left[\mathcal{P}\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right] \geq \text{Const}\left(||u||^2 + |p|^2\right).$$

This yields the lower bound. The upper bound follows from (8) and Lemma 1. *QED*

Remark 1 *From practical point of view it is important to observe that when using the PCG method for solving Problem 2, one has to solve* only one *system with M_0 per inner product $[Px, y]$.*

Block triangular preconditioner

It is also possible to construct a symmetric preconditioner for the operator M, based on a block lower triangular matrix

$$M_T = \begin{pmatrix} A_0 & 0 \\ B & J_0 \end{pmatrix}.$$

We transform system (4) into equivalent one,

$$P\begin{pmatrix} u \\ p \end{pmatrix} \equiv M^* L_0^* K_0^{-1} L_0 M \begin{pmatrix} u \\ p \end{pmatrix} = M^* L_0^* K_0^{-1} L_0 \begin{pmatrix} F \\ G \end{pmatrix},$$

where

$$L_0^* K_0^{-1} L_0 \equiv \begin{pmatrix} I & A_0^{-1}B^* \\ 0 & -I \end{pmatrix}\begin{pmatrix} A_0^{-1} & 0 \\ 0 & J_0^{-1} \end{pmatrix}\begin{pmatrix} I & 0 \\ BA_0^{-1} & -I \end{pmatrix}.$$

Observe that $K_0^{-1}L_0$ is nothing else but M_T^{-1} and that P is symmetric with respect to the scalar product $\langle\cdot, \cdot\rangle$.

Theorem 2 *There exist constants $0 < c_0 \leq c_1$ independent of parameter h, such that*

$$c_0\left\langle\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right\rangle \leq \left\langle P\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right\rangle \leq c_1\left\langle\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right\rangle$$

for all $(u, p) \in V \times W$.

In contrast to the diagonal preconditioner, the triangular symmetrized preconditioner requires three applications of A_0^{-1} per inner product.

Remark 2 *From theoretical point of view it is worth noticing that Theorems 1 and 2 hold if (1) is replaced by a weaker condition,*

$$a(u, u) \geq \alpha||u||^2 \qquad \forall v \in \ker B.$$

Preconditioner for the GMRES method

It is well known that after symmetrizing the system, its condition number increases. This fact is also indicated by numerical experiments below. Instead, we may use efficiently preconditioned GMRES method for our problem. If the preconditioner works so good that we obtain satisfactory approximation after few iterations, then the GMRES memory consumption is less painful, and we can benefit from its better convergence properties.

Let us consider block triangular preconditioner,

$$M_T = \begin{pmatrix} A_0 & 0 \\ B & -J_0 \end{pmatrix},$$

which was previously discussed e.g. in [ES96], [ESW97] and [Kla98a], [KS97].

Theorem 3 *If A_0 is scaled so that the symmetric part*[2] *A_{symm} of A together with $A^*A_0^{-1}A$ satisfy*

$$\big(((A_{symm} - \frac{1}{2}A^*A_0^{-1}A)u, u)\big) \geq \delta \cdot ((A_0 u, u)) \qquad \forall u \in V, \tag{9}$$

for some positive constant δ independent of h, then the convergence rate of the GMRES for $M_T^{-1}M$ is independent of h.

Proof.
Observe that

$$\left[M_T^{-1}M\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right] \geq (((A_{symm} - \frac{1}{2}A^*A_0^{-1}A)u,\, u)) + \frac{1}{2}|||B^*p|||^2_{A_0^{-1}},$$

with $|||v|||^2_{A_0^{-1}} \equiv ((A_0^{-1}v,\, v))$. Obviously, $|||B^*p|||^2_{A_0^{-1}} \geq \mathrm{Const}(J_0 p,\, p)$, by (5), (6) and the LBB condition (3), see [BP88].

Therefore, if only A_0 satisfies (9) with δ independent of h, then we can conclude that

$$\left[M_T^{-1}M\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right] \geq \mathrm{Const}\left[\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right].$$

On the other hand, from Lemma 1 we have

$$\left[M_T^{-1}M\begin{pmatrix} u \\ p \end{pmatrix}, M_T^{-1}M\begin{pmatrix} u \\ p \end{pmatrix}\right] \leq \mathrm{Const}\left[\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}\right].$$

In view of the last two inequalities, the theorem follows from [EES83]. *QED*

Remark 3 *Let us notice that the scaling requirement is similar to that of Bramble and Pasciak [BP88] for the triangular preconditioner in the symmetric case.*

A good scaling factor for A_0^{-1} may be obtained, by finding certain extreme eigenvalues of two generalized eigenproblems. Also observe that for $A = A^* = A_0$ ("exact" preconditioning block), (9) holds with $\delta = 1/2$ without any scaling.

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2 By definition, $A_{symm} = \frac{A+A^*}{2}$.

Numerical experiments

Our test problem was to solve the homogeneous Dirichlet boundary value problem for the Oseen equations in $\Omega = (-1, 1) \times (-1, 1)$

$$\begin{aligned} -\Delta u + (k \cdot \nabla) u + \nabla p &= f, \\ \operatorname{div} u &= 0, \end{aligned}$$

discretized with Q_2–Q_1 Taylor–Hood rectangular finite elements. The mesh was uniform in both directions, with nx inner pressure nodes along the x-axis and also y-axis. The function $k(\cdot, \cdot)$ was defined as a simple vortex,

$$k(x, y) = \begin{pmatrix} 2y(1 - x^2) \\ -2x(1 - y^2) \end{pmatrix}.$$

For f, we took a random vector with elements uniformly distributed between $(-1, 1)$. All algorithms were implemented using Krylov iterative solvers and additive Schwarz preconditioners from the PETSc library [SGM97]. We examined three different solver/preconditioner configurations: GMRES(30) method with block diagonal preconditioner (GMRES/diag); GMRES(30) method with block triangular preconditioner (GMRES/triang); the symmetrized conjugate gradient method with block diagonal preconditioner (CG/symm). The stopping criterion was to reduce the initial residual by a factor of 10^6.

We report on two choices of preconditioning blocks. The first was to test the "best" (yet still symmetric, positive definite) preconditioners, i.e. A_0 = Laplacian, J_0 = mass matrix. The second choice was to use inexact preconditioners for A_0, J_0, namely the additive Schwarz method (with no coarse space), with standard black-box decomposition into 6 rectangular subdomains with small and fixed overlap (2 nodes) provided by PETSc. The results are presented in Figure 1 and Table 1.

The experiments confirm theoretical results, showing that the number of iterations reflects the quality of the preconditioning blocks being used. When using the symmetric part of relevant operators as the preconditioning blocks, the number of iterations is virtually independent of the mesh size.

Table 1 Flops count relative to GMRES/triang for different solver/preconditioner strategies. Iteration count in brackets.

	nx	DOF	GMRES/diag	GMRES/triang	CG/symm
"best"	20	4182	1.51 (39)	1.00 (21)	2.14 (46)
ASM	20	4182	1.18 (128)	1.00 (76)	1.62 (127)

For additive Schwarz preconditioning blocks, according to [SBG96], the number of iterations with fixed small overlap and no coarse grid, should increase as the square root of nx, and this is approximately what we actually see. Theorems 1 and 3 ensure that the convergence rate is fully independent of nx, if two-level additive Schwarz method was used. The superiority of GMRES/triang was not so apparent when additive Schwarz method preconditioners were used: the number of iterations required by symmetrized CG was competitive to that of GMRES/triang.

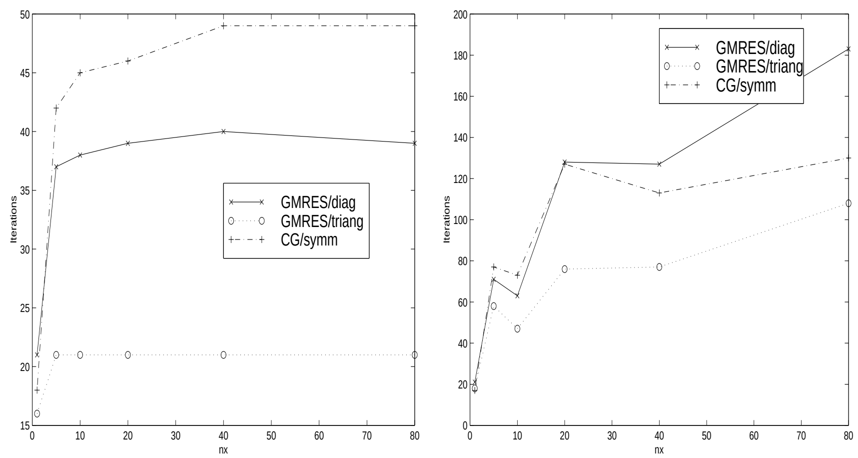

Figure 1 The growth of iterations as a function of problem size for different solvers and preconditioning strategies. **Left**: A_0 = Laplacian, J_0 = Mass matrix. **Right**: additive Schwarz preconditioners (small overlap, no coarse grid).

Acknowledgements

I would like to thank Prof. Maksymilian Dryja and Prof. Krzysztof Moszyński for their valuable remarks and comments. This research was partially sponsored by Polish Scientific Research Committee grant for PhD students, KBN 8T11F01209.

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