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Sparse Grid Spectral Methods and some Results from Approximation Theory

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INTRODUCTION

The sparse grid approach for the solution of partial differential equations (PDEs) was recently extended to spectral and pseudospectral methods. Here, we summarize how results from the approximation theory for Sobolev spaces of functions with bounded mixed derivative can be used to derive asymptotic error estimates for the approximation and interpolation error of functions when either the basis functions are from a *step hyperbolic cross* or when the interpolation is done on a *sparse grid*. For the special case of periodic boundary conditions we discuss how the classical convergence analysis for Galerkin and pseudospectral methods can be used for sparse grid spectral methods. Applying the new methods to simple equations has allowed a comparison with exact solutions. Some results of this comparison are summarized as well as possible applications and extensions to other variants of sparse grid methods.

Zenger [Zen91] has suggested a *sparse grid finite element method* with an approximation order of $O(h^2|\log h|)$ for a finite element subspace of size $\dim \check{X}_N = O(N \log N)$ in comparison to $O(h^2)$ obtained with a subspace of size $\dim X_N = O(N^2)$ by a traditional *full grid finite element method*. His result requires that the mixed derivative $\partial^4 u(x,y)/\partial x^2 \partial y^2$ of the solution u be bounded. The approach was extended to higher order finite element methods and more general domains. Later,

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Eleventh International Conference on Domain Decomposition Methods
Editors Choi-Hong Lai, Petter E. Bjørstad, Mark Cross and Olof B. Widlund ©1999 DDM.org

the sparse grid concept was also applied to finite difference methods (cf. Griebel and Schiekofer [GS98]).

The basic idea behind sparse grid techniques for PDEs has been applied in other branches of mathematics: while classes of functions with bounded mixed derivatives were studied for the definition of good lattice points for number theoretical cubature formulas (see Temlyakov [Tem93] for a review), hyperbolic cross points have played a role in the development of Boolean algebraic methods for interpolation on rectangular domains in two dimensions (cf. the review of Delvos and Schempp [DS89]). Beginning with Babenko [Bab60] and Smolyak [Smo63], approximation theory has seen particular efforts in the description of both approximation and interpolation of (mostly) periodic functions with bounded mixed derivatives. Many results and original references can be found in Temlyakov [Tem89] and [Tem93].

Thus, approximation theory provides a solid basis for the development of *sparse grid spectral methods*. Interpolation of — particularly bivariate — periodic functions and interpolation by Gauß–Chebyshev methods on sparse grids were studied for various Banach type Sobolev spaces with dominating mixed derivatives by Pöplau and Sprengel in [Pöp95], [PS97], [Spr97a], and [Spr97b]. Their results can be used for the numerical analysis of pseudospectral (collocation) methods. In turn, periodic approximation and interpolation on the hyperbolic cross and on sparse grids for some Hilbert type Sobolev spaces were investigated independently by Kupka [Kup97] who used the results for the convergence analysis of spectral and pseudospectral methods for the Helmholtz, linear advection, and heat equation. These equations are also used for numerical tests of the current new method. In what follows, we review some of the results.

APPROXIMATION AND INTERPOLATION ON SOBOLEV SPACES WITH DOMINANT MIXED DERIVATIVE

To motivate the results for Sobolev spaces with dominant mixed derivative, we recall that the standard Sobolev spaces of Lebesgue integrable, 2π–periodic functions $v \in L^q(\Omega)$ on the torus $\boldsymbol{T}^d = \Omega$ are defined by

$$H_p^{r,q}(\Omega) = \{v \in L^q(\Omega) \mid D^\alpha v \in L^q(\Omega) \; \forall \; \alpha : 0 \leq |\alpha| \leq r\} \,, \tag{1}$$

where $D^\alpha v = \partial^{|\alpha|} v / (\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d})$ with $0 \leq |\alpha| \leq m$, m is a non-negative integer, and $|\alpha| := \sum_{i=1}^d \alpha_i$ for $\alpha = (\alpha_1, \ldots, \alpha_d)$, is taken in the sense of periodic distributions (indicated by a subscript p). Moreover, $H_p^{r,q}(\Omega)$ is equipped with the norm

$$\|v\|_{r,q} := \|v\|_{H_p^{r,q}(\Omega)} = \left(\sum_{|\alpha| \leq r} \|D^\alpha v(\mathbf{x})\|_{L^q(\Omega)}^q \right)^{1/q} . \tag{2}$$

For the $q = 2$ case, one can alternatively define a norm for functions $v \in L^2(\Omega)$ by ($\boldsymbol{Z}^d$ denotes the Cartesian product of the field of integers $\boldsymbol{Z}$)

$$|||v|||_r = (2\pi)^{d/2} \left(\sum_{\mathbf{k} \in \boldsymbol{Z}^d} (1 + \mathbf{k}^2)^r |\hat{v}_{\mathbf{k}}|^2 \right)^{1/2} . \tag{3}$$

The space $\dot{H}^r(\Omega)$ consists of all $v \in L^2(\Omega)$ with finite norm $|||\cdot|||_r$,

$$\dot{H}^r(\Omega) = \left\{ v \in L^2(\Omega) \mid |||v|||_r < +\infty \right\} . \tag{4}$$

Both spaces $\dot{H}^r_p(\Omega)$ and $H^r_p(\Omega) \equiv H^{r,2}_p(\Omega)$ and their norms are equivalent.

One way to define Sobolev spaces with dominant mixed derivative on the torus $\boldsymbol{T}^d$ is (we note that permutations P_{rd} and general derivatives D^α are useful in proofs):

Definition 1 *Let P_{rd} be the set of all permutations of $D^\alpha u(\mathbf{x}) = \frac{\partial^{rd} u(\mathbf{x})}{\partial x_1^r \cdots \partial x_d^r}$, $r \geq 0$ be a natural number, $1 \leq q < \infty$, and d be the algebraic dimension of $\boldsymbol{T}^d$. Moreover, let $\alpha = \sum_{j=1}^d \alpha_j$ and $\alpha_j = r$. The periodic Sobolev spaces with dominant mixed derivative on $\boldsymbol{T}^d$ can be defined as*

$$S^{r,d}_{q,p}(\boldsymbol{T}^d) := S^{r,d}_{q,p} = \left\{ u \mid D^\alpha u \in L^q(\boldsymbol{T}^d) \right\} \tag{5}$$

equipped with the norm ($\mathbf{x} \in \boldsymbol{T}^d$)

$$\|u(\mathbf{x})\|_{r,d,q} = \left(\|u(\mathbf{x})\|_q^q + \sum_{j=1}^d \| \frac{\partial^r u(\mathbf{x})}{\partial x_j^r} \|_q^q + \sum_{P_{rd}} \| \frac{\partial^{rd} u(\mathbf{x})}{\partial x_1^r \cdots \partial x_d^r} \|_q^q \right)^{1/q} . \tag{6}$$

For $q = 2$, the definition of an inner product is as straightforward as for $H^r_p(\Omega)$. Likewise, the $S^{r,d}_{q,p}$ are Banach (or even Hilbert) spaces. The new spaces can also be established for the case $q = \infty$. It can be shown that $S^{r,d}_{2,p}$ and $H^r_p(\boldsymbol{T}^d)$ satisfy the embedding relations $H^{rd}_p(\boldsymbol{T}^d) \hookrightarrow S^{r,d}_{2,p} \hookrightarrow H^r_p(\boldsymbol{T}^d)$ for all $r \geq 0$ and for all $d \geq 1$. Moreover, $H^{r+1}_p(\boldsymbol{T}^d) \hookrightarrow H^r_p(\boldsymbol{T}^d)$ and $S^{r+1,d}_{2,p} \hookrightarrow S^{r,d}_{2,p}$, whence in particular $S^{r,d}_{2,p} \hookrightarrow L^2(\boldsymbol{T}^d)$. Just like the classical Sobolev spaces $H^m_p(\boldsymbol{T}^d)$, the spaces $S^{m,d}_{2,p}$ may be defined by means of the Fourier transform:

Definition 2 *Let m be a non-negative integer and let d be as in Definition 1. The periodic Hilbert spaces $\dot{S}^{m,d}$ with dominant mixed derivative on the torus $\boldsymbol{T}^d$ are defined as*

$$\dot{S}^{m,d} := \dot{S}^{m,d}(\boldsymbol{T}^d) = \left\{ u \in L^2(\boldsymbol{T}^d) \mid |||u|||_{m,d} < +\infty \right\} . \tag{7}$$

Here, we use the following norm for $\dot{S}^{m,d}$:

$$|||u|||_{m,d} = (2\pi)^{d/2} \left(\sum_{\mathbf{k} \in \boldsymbol{Z}^d} \left(1 + \sum_{j=1}^d k_j^{2m} + \sum_{P_{rd}} \prod_{j=1}^d k_j^{2m} \right) |\hat{u}_{\mathbf{k}}|^2 \right)^{1/2} . \tag{8}$$

The spaces $S^{m,d}_{2,p}$ and $\dot{S}^{m,d}$ can be identified with each other, i.e., they are equivalent for all $m \geq 0$ and $d \geq 1$ and their norms coincide. Proofs can be found in Kupka [Kup97] together with some references for the preceeding relations.

The standard projection operator for functions $v \in H^r_p(\boldsymbol{T}^d)$ is given by $P^f_N v = \sum_{\mathbf{k}\in J^f_N} \hat{v}_{\mathbf{k}} e^{i\mathbf{kx}} = \sum_{\mathbf{k}\in J^f_N} \frac{1}{(2\pi)^d}(v, e^{i\mathbf{kx}})e^{i\mathbf{kx}}$, where $J^f_N = \{-N+1 \leq k_j \leq N\}$ for all $\mathbf{k} = (k_1, \ldots, k_d)$ and $N \in \boldsymbol{N}$ ($\boldsymbol{N}$ denotes the set of natural numbers). The $e^{i\mathbf{kx}}$ span a subspace $S^f_N \subset H^r_p(\boldsymbol{T}^d)$. To recover $v \in H^r_p(\boldsymbol{T}^d)$ on a finite set of points $\mathbf{x} \in \Omega$, the *full* grid $G^f_N := \{\mathbf{x}\} = \{(x_{j_1}, \ldots, x_{j_d})\}$ with $x_{j_m} = \frac{\pi}{N} j_m$, $N \in \boldsymbol{N}$, and $j_m \in H^f_N := \{0 \leq j_m \leq 2N-1\}$ for $m = 1, \ldots, d$, is frequently used together with the interpolation operator $I^f_N v = I_{N_1}(v) \otimes \cdots \otimes I_{N_d}(v)$. Here, $N_k = N_1$ for all $2 \leq k \leq d$. I_{N_j} is the interpolation operator for one dimension. Thus, for a continuous v on $\boldsymbol{T}^d$, $I^f_N v(\mathbf{x}) = \sum_{\mathbf{k}\in J^f_N} \tilde{v}_{\mathbf{k}} e^{i\mathbf{kx}}$ with $\tilde{v}_{\mathbf{k}} = \sum_{\mathbf{j}\in H^f_N} (2N)^{-d} v(\mathbf{x_j}) e^{-i\mathbf{kx_j}}$, and $\sum_{\mathbf{j}\in H^f_N}$ ranges over all $\mathbf{x} \equiv \mathbf{x_j} \in G^f_N$. Taking $C > 0$ to be some real valued constant,[3] we have the well-known error estimate for P^f_N:

Theorem 1 *Let $0 \leq l \leq m$ with $l, m \in \boldsymbol{N}$. If $N > 0$, $\|u - P^f_N v\|_{l,2} \leq C N^{l-m} \|v\|_{m,2}$ and thus $\|v - P^f_N v\|_{L^2(\Omega)} \leq C N^{-m} \|v\|_{H^m_p(\Omega)}$ for all $v \in H^m_p(\Omega)$.*

For the interpolation operator I^f_N there is an equivalent estimate:

Theorem 2 *Let $0 \leq l \leq m$ with $l, m \in \boldsymbol{N}$ and $m > d/2$. If $N > 0$, $\|v - I^f_N v\|_{l,2} \leq C N^{l-m} \|v\|_{m,2}$ and thus $\|v - I^f_N v\|_{L^2(\Omega)} \leq C N^{-m} \|v\|_{H^m_p(\Omega)}$ for all $v \in H^m_p(\Omega)$.*

For the spaces $\dot{S}^{m,d}$, hyperbolic cross and sparse grids allow much more efficient approximation and interpolation operators. Let us define the (even) hyperbolic cross of order $N \in \boldsymbol{N}$ as $J^h_N = \left\{(\prod_{j=1}^d \max(1, |k_j|) \leq N) \wedge (k_j > -N)\right\}$ where $\mathbf{k} \in \boldsymbol{Z}^d$. For convenience, throughout this paragraph we take j and p to be integers ranging from 1 to d. We define the (even) step hyperbolic cross of size $n \in \boldsymbol{N}$ as $J^s_n = \{\mathbf{k} \in \boldsymbol{Z}^d \mid ((-2^{n_j-1}+1 \leq k_j \leq 2^{n_j-1}) \ \forall\ n_j > 0) \ \wedge \ (k_j = 0 \text{ if } n_j = 0)\}$, where $\sum_{j=1}^d n_j \leq n$ for $0 \leq n_j \leq n$. Assuming $N = 2^{m-1}$ we construct a set $J^f_{\mathbf{m}} = \{\mathbf{k} \in \boldsymbol{Z}^d \mid ((-2^{m_j-1}+1 \leq k_j \leq 2^{m_j-1}) \ \forall\ m_j > 0) \ \wedge \ (k_j = 0 \text{ if } m_j = 0)\}$. The $e^{i\mathbf{kx}}$ with $\mathbf{k} \in J^f_{\mathbf{m}}$ and $\mathbf{m} \in \boldsymbol{N}^d_0$ span a subspace $S^f_{\mathbf{m}} \subset \dot{S}^{m,d}$. We can use the sets $J^f_{\mathbf{m}}$ to define projections on the even step hyperbolic cross, $J^s_n = \bigcup_{\|\mathbf{m}\|_1 \leq n} J^f_{\mathbf{m}}$ and $\|\mathbf{m}\|_1 = \sum_{j=1}^d |m_j|$. Hence, the $e^{i\mathbf{kx}}$ with $\mathbf{k} \in J^s_n$ span the subspace $S^s_n \subset \dot{S}^{m,d}$ and for each $u(\mathbf{x}) \in S^s_n$ we have $u(\mathbf{x}) = \sum_{\mathbf{k}\in J^s_n} \hat{u}_{\mathbf{k}} e^{i\mathbf{kx}}$ and $\hat{u}_{\mathbf{k}} = (2\pi)^{-d}(u, e^{i\mathbf{kx}})$. In the same manner we can define the projection operator P^h_N with coefficients $\mathbf{k}$ from J^h_N. For the construction of the sparse grid interpolation operator one proceeds as follows. First, the sets $H^f_{\mathbf{m}} = \{\mathbf{j} \in \boldsymbol{N}^d_0 \mid 0 \leq j_p \leq 2^{m_p} - 1\}$ are defined which index the grid points of the *anisotropic* full grids $G^f_{\mathbf{m}} = G_{m_1} \times \cdots \times G_{m_d} = \{\mathbf{x}\}$, where $\mathbf{x} = (x_{j_1}, \ldots, x_{j_d})$

3 We are not interested here in the numerical value of C, nor is it the same for the various inequalities discussed here. For the case of $d = 2$ and for similar interpolation operators, numerical values can be found in the work of Pöplau and Sprengel cited above.

with $x_{j_p} = \pi j_p / N_p$ and $N_p = 2^{m_p - 1}$ for $\mathbf{j} = (j_1, \ldots, j_d) \in H^f_{\mathbf{m}}$. The construction of an interpolation operator $I^f_{\mathbf{m}} u := I_{m_1}(u) \otimes \cdots \otimes I_{m_d}(u)$ for all sufficiently well-behaved $u \in \dot{S}^{m,d}$ is straightforward. We call $G^s_n := \bigcup_{\|\mathbf{m}\|_1 \leq n} G^f_{\mathbf{m}} = \{\mathbf{x}\} = \{(x_{j_1}, \ldots, x_{j_d})\}$ with $x_{j_p} = \frac{\pi}{2^{m_p - 1}} j_p$ where $\mathbf{j} \in H^s_n := \bigcup_{\|\mathbf{m}\|_1 \leq n} H^f_{\mathbf{m}}$ and $0 \leq j_p \leq 2^{m_p} - 1$ a *Fourier sparse grid* of order n. Naturally, $n \in \boldsymbol{N}_0$. Finally, the interpolation operator I^s_n on G^s_n is defined as $I^s_n = \sum_{\|\mathbf{m}\|_1 \leq n} \Delta_{\mathbf{m}}$ where $\Delta_{\mathbf{m}} = \Delta_{m_1} \otimes \cdots \otimes \Delta_{m_d}$ and $\Delta_{m_j} = \tilde{I}_{m_j} - \tilde{I}_{m_j - 1}$. Here, $\tilde{I}_m = I_{\lfloor 2^{m-1} \rfloor}$ for $m \in \boldsymbol{N}_0$ and $\tilde{I}_{-1} : f \mapsto 0$ for all f defined on G^s_n. The symbol $\lfloor r \rfloor$ means rounding to the nearest integer less than r (relevant for $m = 0$ here). The operator $I_{\lfloor 2^{m-1} \rfloor}$ is again the interpolation operator in one dimension. Sparse grid type interpolation operators have been discussed, e.g., in Smolyak [Smo63], Hallatschek [Hal92], and Temlyakov [Tem93]. For P^h_N, we can derive the following error estimate:

Theorem 3 *Let $u \in \dot{S}^{m,d}$ and let $0 \leq l \leq m$ with $l, m \in \boldsymbol{N}$ and $d \geq 1$. If $N > 0$, $\|P^h_N u - u\|_{l,d,2} \leq C N^{l-m} \|u\|_{m,d,2}$ and in particular $\|P^h_N u - u\|_2 \leq C N^{-m} \|u\|_{m,d,2}$.*

For the interpolation on G^s_n by means of I^s_n the result is only slightly worse:

Theorem 4 *Let $u \in \dot{S}^{m,d}$ and $N > 0$. Moreover, let $m \geq 1$ and $0 \leq l \leq m$ with $l, m \in \boldsymbol{N}$. If $d \geq 2$, then $\|I^s_n u - u\|_{l,d,2} \leq C N^{l-m} (\mathrm{ld}\, N)^{d-1} \|u\|_{m,d,2}$ and in particular $\|I^s_n u - u\|_2 \leq C N^{-m} (\mathrm{ld}\, N)^{d-1} \|u\|_{m,d,2}$.*

Note that the dual logarithm ld is not present in Theorem 3. However, as the total number of coefficients in J^h_N grows like $N(\log N)^{d-1}$, both types of approximation finally contain a logarithmic factor which depends on the spatial dimension d. Proofs and further details are given in Kupka [Kup97].

SPARSE GRID SPECTRAL METHODS

Fourier Galerkin spectral methods for the PDE $Lu = f$, where $f \in Y \subset L^2(\boldsymbol{T}^d)$, L is a (linear or non-linear) differential operator, and u is the solution of the problem, can be defined in the variational form: $(Lu^N - f, v) = 0$ for all $v \in S_N$. Here, $u^N \in S_N \subset X$ is the numerical approximation to u and X is some Sobolev space such that $u \in X$ (cf. Canuto et al. [CHQZ88]). The resulting equations yield a finite number of Fourier coefficients $\hat{u}_N$ which define the (numerical) approximation to u in the subspace S_N. In Fourier collocation (pseudospectral) methods both L and the inner product (e, v) are approximated via the interpolation of u and f on a grid G_N. Derivatives of u (contained in L) are approximated by derivatives of interpolation polynomials. A *sparse grid spectral method* is obtained by replacing the operators P^f_N and I^f_N used for the traditional (full grid) spectral methods by their counterparts P^h_N (resp. P^s_n) and I^s_n. As a consequence, in sparse grid Fourier Galerkin methods, the coefficients for Fourier approximations are taken from the (even) hyperbolic (resp. step) hyperbolic cross and, likewise, functions in sparse grid Fourier collocation methods are interpolated by Fourier polynomials defined through a sparse grid G^s_n. Differential operators and inner products can then be approximated accordingly.

Using the error estimates of Theorems 3 and 4 together with the standard approach for obtaining convergence estimates for spectral methods (cf. Canuto et al.[CHQZ88]), one can apply such technique to sparse grid spectral methods. Examples include the stationary Helmholtz equation, a problem $Lu = f$ with a coercive operator L, the transient Helmholtz and the heat equation, as well as the linear advection equation (see Kupka [Kup97]). Basically, each term which does not involve the operator I_n^s requires the replacement of $||w||_{m,2}$ with $||w||_{m,2,2}$, whereas otherwise a logarithmic factor $\mathrm{ld}N^{d-1}$ has to be inserted (here, N is the maximum number of grid points per coordinate direction and w must belong to some appropriate smoothness class). As an example, for the Helmholtz equation $-\triangle u + \lambda u = f$ with $\lambda > 0$ on $\boldsymbol{T}^2$, the error estimate for Fourier collocation on a full grid is $||u - u_c^N||_{1,2} \leq CN^{1-m}(||u||_{m,2} + ||f||_{m-1,2})$ in comparison with $||u - u_c^N||_{1,2} \leq C'N^{1-m}(||u||_{m,2,2} + (\mathrm{ld}N)||f||_{m-1,2,2})$ for Fourier collocation on a sparse grid (f and thus u are assumed to be sufficiently smooth). Note that — as usual — $\dim S_N = O(N \log N)$ for the sparse grid case, but $O(N^2)$ for the full grid. In accordance with intuition, a von Neumann analysis of fully discrete approximations to the heat and advection equation reveals that restrictions on the time step are dominated by the most anisotropic grids "contained" in a sparse grid. This can reduce the overall efficiency of sparse grid spectral methods for some explicit solvers for time dependent problems (Kupka [Kup97]).

An efficient implementation of sparse grid spectral methods requires adequate data structures. In Fortran90 such a structure can be realized as a user defined data type containing a pointer to a one-dimensional array of another user defined data type which in turn has a pointer to an array of real values (floating point numbers) as its only component. This structure may be iterated for the case $d > 2$. Tests have shown that usual manipulations such as copying of variables, addition, and of course Fourier transformation, are slower by a factor of 3 to 5 for such a sparse grid structure with several 100 to a few 1000 points in comparison to standard arrays of rank d of the same total number of points. Although more elegant, a recursive sparse grid data structure is likely to suffer from a much lower computational efficiency due to an enhanced internal overhead. The (discrete) Fourier transformation can be done by the algorithm of Hallatschek [Hal92] which consists of a one dimensional (fast) Fourier transformation followed by a transformation into a hierarchical basis. The algorithm is performed sequentially for each spatial dimension. A final sequence of re-transformations is necessary to obtain Fourier coefficients in the standard basis $\{e^{i\mathbf{kx}}\}$. The algorithm can be run backwards to obtain function values at grid points (see Kupka [Kup97] for a discussion, also for an implementation of a fast Fourier transformation on sparse grids for the case $d = 2$).

SOME NUMERICAL EXPERIMENTS AND CONCLUSIONS

Numerical experiments presented by Kupka [Kup97] have exhibited the following properties of Fourier sparse grid spectral methods. First, for insufficient or low resolution, sparse grid Fourier collocation is subject to aliasing errors and spurious solutions in the same way as a full grid method. On the other hand, spectral convergence can easily be verified for infinitely smooth solutions, just as the finite

Table 1 The relative error in various L^p norms — approximated on a dense full grid (l_p norms) — for full grid approximations u_c^N (on G_N^f, top) and sparse grid ones (on G_n^s, bottom) to the solution u of a Helmholtz problem described in the text. The quadratic convergence as a function of N is evident. However, considering the total number of grid points the convergence is only linear for the full grid method, but still nearly quadratic in the sparse grid case.

N	grid points	$l_1(u_c^N - u)$	$l_2(u_c^N - u)$	$l_\infty(u_c^N - u)$
8	256	3.69902e-3	5.79361e-3	2.67764e-2
16	1024	9.21957e-4	1.42771e-3	7.09712e-3
32	4096	2.30752e-4	3.57583e-4	1.82264e-3
64	16384	5.78924e-5	8.98336e-5	4.61524e-4
n	grid points	$l_1(u_c^N - u)$	$l_2(u_c^N - u)$	$l_\infty(u_c^N - u)$
6	256	3.47985e-3	3.86827e-3	7.96669e-3
7	576	8.69380e-4	9.62935e-4	2.03200e-3
8	1280	2.17657e-4	2.41199e-4	5.13192e-4
9	2816	5.44621e-5	6.03750e-5	1.28952e-4

convergence order for solutions with a finite number of partial derivatives in all spatial directions. However, whereas a sparse grid method for a numerical solver with a fixed approximation order (finite differences, etc.) is more efficient than its full grid counterpart as long as the solution is sufficiently smooth, sparse grid spectral methods are more useful, if the solution has particular smoothness properties: consider the polynomial $g(z) = \pi^4/90 - \pi^2 z^2/12 + \pi z^3/12 - z^4/48$ on $[0, 2\pi]$. It is easily verified that $g(z) \in H_p^3(\boldsymbol{T})$, but $g(z) \notin H_p^4(\boldsymbol{T})$. Taking $f(x,y) = -g''(x)g(y) - g(x)g''(y) + \lambda g(x)g(y)$ it can immediately be seen that $u(x,y) = g(x)g(y)$ is the solution of $-\triangle u + \lambda u = f$ on $\boldsymbol{T}^2$. Also, $u \in H_p^3(\boldsymbol{T}^2)$ and $u \in \dot{S}^{3,2}$, but $u \notin H_p^4(\boldsymbol{T}^2)$. The resulting higher efficiency of sparse grid Fourier collocation in comparison with Fourier collocation on full grids is illustrated in Table 1.

With sparse grid Fourier collocation for the heat equation even non-smooth initial conditions can be handled reasonably well. The same does not hold true for the linear advection equation: the initial condition $u(t = 0, x, y) = u_0(x, y)$ with $u_0 = 1$ in a rectangular subdomain of $\boldsymbol{T}^2$ and $u_0 = 0$ elsewhere, causes the sparse grid method to fail while the full grid one only suffers from oscillations. Filtering, smoothing, or adaptivity thus may help to make sparse grid spectral methods more efficient building blocks for a domain decomposition solver than their full grid counterparts (which is expected from their "universality", i.e. when functions from different smoothness classes are approximated, cf. Temlyakov [Tem93]). Although due the more expensive programming model the sparse grid spectral methods have a break-even point which is lower for memory than for speed, neither is prohibitive even for moderate accuracy requirements.

The work of Pöplau and Sprengel mentioned in the introduction makes a similar study of Chebychev (and other) spectral methods both possible and worthwhile.

Likewise, the analysis of other sparse grid methods can benefit from the approximation theoretical results developed for sparse grid spectral methods. Whether turbulence simulations in fluid mechanics also benefit from the *hierarchical bases* inherent to sparse grid methods (priv. communication with M. Griebel during DD11, cf. the sparse grid FFT discussed above) is a question that requires a more careful study.

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