On Schwarz Alternating Methods for the Incompressible Navier-Stokes Equations in N Dimensions

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INTRODUCTION

This paper considers four Schwarz Alternating Methods for the N-dimensional, steady, viscous, incompressible Navier Stokes equations, $N \leq 4$. It is shown that the Schwarz sequences converge to the true solution provided the Reynolds number is sufficiently small. This appears to be the first attempt to prove convergence of Schwarz Alternating Methods for the Navier Stokes equations.

The Schwarz Alternating Method was devised by H. A. Schwarz more than one hundred years ago to solve linear boundary value problems. It has garnered interest recently because of its potential as a very efficient algorithm for parallel computers. In Tai and Espedal [TE98], Tai and Xu [TX98], and Dryja and Hackbusch [DH97], they show convergence of Schwarz methods for some nonlinear problems. In Lui [Luiar], proofs of convergence of Schwarz Alternating Methods for some 2nd-order nonlinear elliptic PDEs were given. In this sequel, we prove convergence of four Schwarz methods for the N-dimensional, steady, incompressible, viscous Navier Stokes equations, $N \leq 4$. Many authors have demonstrated numerically the effectiveness of Schwarz methods in solving fluid problems. See, for example, the proceedings of the annual domain

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decomposition conferences, beginning with [GGMP88]. We mention in particular [DDG+92], [CGK+98] and [KKS98]. This paper follows closely the framework developed in the fundamental paper of Lions [Lio88] where the convergence of the Schwarz method for the Stokes equations is proved. While we concentrate mostly on the two-subdomain case, we show an additive Schwarz version which converges for the multiple-subdomain case.

Let Ω be a bounded domain in $\mathbf{R}^N$ with a smooth boundary. Suppose $\Omega = \Omega_1 \cup \Omega_2$, where the subdomains Ω_i have smooth boundaries and are overlapping. Let $H_0^1(\Omega)^N$ denote the Cartesian product of N Sobolev spaces $H_0^1(\Omega)$ (consisting of all functions whose first derivatives are in $L^2(\Omega)$ and whose value vanishes on $\partial\Omega$ in the trace sense) and define $L^2(\Omega)^N$ as the Cartesian product of N copies of $L^2(\Omega)$. The space $H^{1/2}(\partial\Omega)^N$ is defined similarly. Let $V = \{u \in H_0^1(\Omega)^N,\ \text{div}\ u = 0\}$ and $V_i = \{u \in H_0^1(\Omega_i)^N,\ \text{div}\ u = 0\}$, $i = 1, 2$. Let u_i denote the i-th component of the vector u. Denote the inner product in $H_0^1(\Omega)^N$ by $[u, v] = \sum_{i=1}^{N} \int_\Omega \nabla u_i \cdot \nabla v_i$ and let $\|u\|_1 = [u, u]^{1/2}$. Let $\triangle_i^{-1}$ be the inverse of the Laplacian operator considered as an operator from the dual space $H^{-1}(\Omega_i)^N$ onto $H_0^1(\Omega_i)^N$, $i = 1, 2$. We take overlapping to mean that $H_0^1(\Omega)^N = H_0^1(\Omega_1)^N + H_0^1(\Omega_2)^N$. In this paper, a function in $H_0^1(\Omega_i)^N$ is considered as a function defined on the whole domain by extension by zero.

When the two subdomains are overlapping, the work of Lions [Lio88] shows that $V = V_1 + V_2$. Let P_i denote the orthogonal (with respect to the inner product $[\cdot, \cdot]$) projection from V onto V_i, $i = 1, 2$. It is well known that

$$d \equiv \max\left(\|(I - P_2)(I - P_1)\|_2,\ \|(I - P_1)(I - P_2)\|_2\right) < 1.$$

See Lions [Lio88] and Bramble et. al. [BPWX91]. Throughout this paper, C will denote a positive constant which may not be the same at different occurrences.

In the next section, we give a statement of the problem including an estimate for the nonlinear term. Following that, we sketch a proof of convergence for a nonlinear Schwarz sequence where each subdomain problem is a nonlinear one. In the next three sections, we develop three variations of the nonlinear Schwarz sequence. These sequences are more practical in that only linear subdomain problems are encountered and that in two of these, the subdomain problems are independent so that they can be solved concurrently. In the final section, we discuss the case of many subdomains and non-homogeneous boundary conditions.

NAVIER-STOKES EQUATIONS

The N-dimensional, steady, viscous, incompressible Navier Stokes equations in non-dimensional form is

$$\begin{aligned} (u \cdot \nabla)u &= -\nabla p + \frac{1}{R}\triangle u + f \text{ on } \Omega \\ \text{div}\ u &= 0 \text{ on } \Omega \end{aligned}$$

with the boundary conditions

$$u = g \text{ on } \partial\Omega,$$

where $u \in H^1(\Omega)^N$, g is the given velocity on the boundary with $||g||_{H^{1/2}(\partial\Omega)^N} \leq 1$, f is the forcing term in $L^2(\Omega)^N$, p is the pressure and R is the Reynolds number. We assume that g satisfies the compatibility condition $\int_{\partial\Omega} g \cdot n = 0$ where n denotes the (outward) unit normal. We use u to denote the unique solution to this equation.

We shall need the following lemma. A similar version states the continuity of a certain trilinear form containing the convective term of the Navier Stokes equations. See, for instance, Temam [Tem95] on page 12.

Lemma 1 *Let $U = \{u \in H^1(\Omega)^N,\ div\ u = 0\}$. For $N \leq 4$, define the function $F : U \times U \to H_0^1(\Omega)^N$ by $F(u,v) = \triangle^{-1}(u \cdot \nabla)v$. Then $||F(u,v)||_1 \leq C||u||_1\ ||v||_1$.*

In light of this lemma, we restrict to the case $N \leq 4$ in the remainder of this paper. Initially, *we only discuss the case $g \equiv 0$* which has a clearer exposition. The general case will be addressed in the last section. In the next section, we define a nonlinear Schwarz sequence.

NONLINEAR SCHWARZ SEQUENCE

Let $u^{(0)} \in V$. For $n = 0, 1, 2, \cdots$, define the nonlinear Schwarz sequence as

$$\begin{aligned} (u^{(n+\frac{1}{2})} \cdot \nabla)u^{(n+\frac{1}{2})} &= -\nabla p^{(n+\frac{1}{2})} + \frac{1}{R}\triangle u^{(n+\frac{1}{2})} + f \text{ on } \Omega_1 \\ \text{div } u^{(n+\frac{1}{2})} &= 0 \text{ on } \Omega_1 \\ u^{(n+\frac{1}{2})} &= u^{(n)} \text{ on } \partial\Omega_1 \end{aligned}$$

and

$$\begin{aligned} (u^{(n+1)} \cdot \nabla)u^{(n+1)} &= -\nabla p^{(n+1)} + \frac{1}{R}\triangle u^{(n+1)} + f \text{ on } \Omega_2 \\ \text{div } u^{(n+1)} &= 0 \text{ on } \Omega_2 \\ u^{(n+1)} &= u^{(n+\frac{1}{2})} \text{ on } \partial\Omega_2. \end{aligned}$$

Here, $u^{(n+\frac{1}{2})}$ is considered as a function in V by defining it to be $u^{(n)}$ on $\Omega \setminus \Omega_1$ and $u^{(n+1)}$ is defined as $u^{(n+\frac{1}{2})}$ on $\Omega \setminus \Omega_2$. The first thing to check is that the compatibility conditions

$$\int_{\partial\Omega_1} u^{(n)} \cdot n = 0 = \int_{\partial\Omega_2} u^{(n+\frac{1}{2})} \cdot n$$

are satisfied. We claim that they are satisfied for all n provided

$$\int_{\Gamma_i} u^{(0)} \cdot n = 0, \ i = 1, 2 \tag{1}$$

hold, where $\Gamma_i = \partial\Omega_i \cap \Omega_{3-i}$.

The claim is proved by induction. For $n = 0$,

$$\int_{\partial\Omega_1} u^{(0)} \cdot n = \int_{\Gamma_1} u^{(0)} \cdot n = 0$$

by (1). Since div $u^{(\frac{1}{2})} = 0$ on $\Omega_1 \cap \Omega_2$,

$$0 = \int_{\Gamma_1} u^{(\frac{1}{2})} \cdot n + \int_{\Gamma_2} u^{(\frac{1}{2})} \cdot n = \int_{\Gamma_1} u^{(0)} \cdot n + \int_{\Gamma_2} u^{(\frac{1}{2})} \cdot n = \int_{\Gamma_2} u^{(\frac{1}{2})} \cdot n.$$

The induction step is proved by a similar calculation.

The main result is that this Schwarz sequence converges to the true solution provided the Reynolds number is sufficiently small.

Theorem 1 *Assuming $u^{(0)}$ satisfies (1) and R is sufficiently small (depending on $\|u^{(0)} - u\|_1$), the nonlinear Schwarz sequence converges geometrically to the true solution u in the norm $\|\cdot\|_1$.*

The proof can be divided into three steps. In the first step, we derive the error equations

$$\begin{aligned} e^{(n+\frac{1}{2})} &= (I - P_1)e^{(n)} + RF_1(e^{(n+\frac{1}{2})}), \\ e^{(n+1)} &= (I - P_2)e^{(n+\frac{1}{2})} + RF_2(e^{(n+1)}), \end{aligned}$$

where $e^{(n+\frac{1}{2})} = u^{(n+\frac{1}{2})} - u$ and $e^{(n)} = u^{(n)} - u$ are the error terms and F_i are certain quadratic nonlinear terms. These nonlinear terms can be bounded using Lemma 1. In the second step, we use induction to show that provided R is smaller than some known constant, $\|e^{(n+\frac{1}{2})}\|_1$, $\|e^{(n)}\|_1 \le M$ for all n, where $M = \|e^{(0)}\|_1$. In the final step, we employ these estimates in the error equations to prove geometric convergence of the Schwarz sequence. A sufficient condition for convergence is that

$$R < \frac{\sqrt{2 - d^2} - d}{C(2\|u\|_1 + M)},$$

where C is a known constant. Full details of the proof will be reported elsewhere.

LINEAR SCHWARZ SEQUENCE

In the previous section, each Schwarz iteration requires the solution of a nonlinear PDE in each subdomain. From a practical point of view, we of course prefer to solve linear problems whenever possible. Now, we demonstrate a version of the Schwarz method where only linear PDEs need to be solved. Convergence still holds but at a possibly smaller R and that the rate of convergence may be a little slower.

Let $u^{(0)} \in V$ and assume it satisfies (1). For $n = 0, 1, 2, \cdots$, define the linear Schwarz sequence as

$$\begin{aligned} (u^{(n)} \cdot \nabla)u^{(n+\frac{1}{2})} &= -\nabla p^{(n+\frac{1}{2})} + \frac{1}{R}\triangle u^{(n+\frac{1}{2})} + f \text{ on } \Omega_1 \\ \text{div } u^{(n+\frac{1}{2})} &= 0 \text{ on } \Omega_1 \\ u^{(n+\frac{1}{2})} &= u^{(n)} \text{ on } \partial\Omega_1 \end{aligned}$$

and

$$(u^{(n+\frac{1}{2})} \cdot \nabla)u^{(n+1)} = -\nabla p^{(n+1)} + \frac{1}{R}\triangle u^{(n+1)} + f \text{ on } \Omega_2$$

$$\begin{aligned} \text{div } u^{(n+1)} &= 0 \text{ on } \Omega_2 \\ u^{(n+1)} &= u^{(n+\frac{1}{2})} \text{ on } \partial\Omega_2. \end{aligned}$$

Notice that the above equations are linear and that the compatibility conditions are satisfied.

The main result is that this Schwarz sequence converges to the true solution provided the Reynolds number is sufficiently small.

Theorem 2 *Assuming $u^{(0)}$ satisfies (1) and R is sufficiently small (depending on $||u^{(0)} - u||_1$), the linear Schwarz sequence converges to the true solution u in the norm $|| \cdot ||_1$.*

The proof is similar to before with geometric convergence if

$$R < \frac{1-d}{C(M+2)}.$$

PARALLEL SCHWARZ SEQUENCE

For the two previous Schwarz methods, the iterates must be computed sequentially. In this section, we suggest a Schwarz sequence where the two subdomain problems are independent and thus they can be solved simultaneously. This sequence also converges provided the Reynolds number is sufficiently small.

Let $u^{(0)} \in V$ and assume that it satisfies (1). Define $u^{(-\frac{1}{2})} = u^{(-\frac{3}{2})} = u^{(0)}$. For $n = 0, 1, 2, \cdots$, define the parallel Schwarz sequence as

$$\begin{aligned} (u^{(n)} \cdot \nabla)u^{(n+\frac{1}{2})} &= -\nabla p^{(n+\frac{1}{2})} + \frac{1}{R}\triangle u^{(n+\frac{1}{2})} + f \text{ on } \Omega_1 \\ \text{div } u^{(n+\frac{1}{2})} &= 0 \text{ on } \Omega_1 \\ u^{(n+\frac{1}{2})} &= u^{(n)} \text{ on } \partial\Omega_1 \end{aligned}$$

and

$$\begin{aligned} (u^{(n-\frac{1}{2})} \cdot \nabla)u^{(n+1)} &= -\nabla p^{(n+1)} + \frac{1}{R}\triangle u^{(n+1)} + f \text{ on } \Omega_2 \\ \text{div } u^{(n+1)} &= 0 \text{ on } \Omega_2 \\ u^{(n+1)} &= u^{(n-\frac{1}{2})} \text{ on } \partial\Omega_2. \end{aligned}$$

We define $u^{(n+1)}$ as $u^{(n-\frac{1}{2})}$ on $\Omega \setminus \Omega_2$. Notice that the above equations are linear and can be solved independently. We again can check that the compatibility conditions are satisfied.

Theorem 3 *Assuming $u^{(0)}$ satisfies (1) and R is sufficiently small (depending on $||u^{(0)} - u||_1$), the parallel Schwarz sequence converges to the true solution u in the norm $|| \cdot ||_1$.*

For this sequence, we obtain geometric convergence if

$$R < \frac{1-d}{C(M+2)}.$$

ADDITIVE SCHWARZ SEQUENCE

In this section, we demonstrate the convergence of the Additive Schwarz sequence, originally introduced in [DW87] for linear elliptic equations and has been used successfully in practice.

Let $u^{(0)} \in V$. For $n = 0, 1, 2, \cdots$, define

$$-\frac{1}{R}\triangle d^{(n+\frac{1}{2})} + \nabla p^{(n+\frac{1}{2})} = f - (u^{(n)} \cdot \nabla)u^{(n)} - \nabla p^{(n)} + \frac{1}{R}\triangle u^{(n)} \text{ on } \Omega_1$$

and

$$-\frac{1}{R}\triangle d^{(n+1)} + \nabla p^{(n+1)} = f - (u^{(n)} \cdot \nabla)u^{(n)} - \nabla p^{(n)} + \frac{1}{R}\triangle u^{(n)} \text{ on } \Omega_2,$$

where $d^{(n+\frac{1}{2})} \in V_1$ and $d^{(n+1)} \in V_2$. The additive Schwarz sequence is defined as $u^{(n+1)} = u^{(n)} + \omega(d^{(n+\frac{1}{2})} + d^{(n+1)})$ where ω is a relaxation parameter. Roughly speaking, $d^{(n+\frac{1}{2})}$ and $d^{(n+1)}$ are corrections to the iterate $u^{(n)}$ in the subdomains Ω_1 and Ω_2, respectively and the right-hand sides of the above equations defining the corrections are the residuals of $u^{(n)}$ in the subdomains. Ignoring the nonlinear terms, the above sequence is precisely the additive Schwarz sequence for the Stokes problem. Notice that the above Stokes equations can be solved independently and no additional assumption on $u^{(0)}$ is required.

Theorem 4 *Assuming $0 < \omega < 1/2$ and R sufficiently small, the additive Schwarz sequence converges geometrically to the true solution u in the norm $\|\cdot\|_1$.*

Proof: We only sketch the proof, recording the key equations. From the defining equations, we have

$$\begin{aligned} d^{(n+\frac{1}{2})} &= -P_1 e^{(n)} + RF_1(e^{(n)}) \\ d^{(n+1)} &= -P_2 e^{(n)} + RF_2(e^{(n)}) \end{aligned}$$

from which we obtain

$$e^{(n+1)} = (I - \omega\,(P_1 + P_2))e^{(n)} + \omega R(F_1(e^{(n)}) + F_2(e^{(n)})),$$

where F_1 and F_2 are the same nonlinearities as above. When $0 < \omega < 1/2$, $\|I - \omega\,(P_1 + P_2)\|_1 < 1$. Applying the estimates for F_i and the condition

$$R < \frac{1 - \|I - \omega\,(P_1 + P_2)\|_1}{\omega C(\|u^{(0)} - u\|_1 + 2\|u\|_1)},$$

we can show that $\|e^{(n)}\|_1 \leq \|e^{(0)}\|_1$ for all n. This condition is also sufficient to guarantee geometric convergence of the additive Schwarz sequence.

DISCUSSION AND CONCLUSION

In this paper, we show the convergence of four Schwarz Alternating Methods for the N-dimensional, steady, incompressible Navier Stokes equations, provided the Reynolds number is sufficiently small and $N \leq 4$. Our results are only valid for the two-subdomain case. The result for finitely many subdomains is again that the Schwarz sequence converges provided R is sufficiently small. Except for the additive Schwarz scheme, we cannot give an explicit bound for R. The difficulty is in obtaining an explicit expression for the spectral radius of a certain matrix. We elaborate on this point below. To simplify the analysis, assume that any three distinct subdomains have an empty intersection.

First, we need the following lemma which was shown by Lions [Lio88] for the two subdomain case. The general case is true by induction.

Lemma 2 *Suppose $H_0^1(\Omega)^N = H_0^1(\Omega_1)^N + \cdots + H_0^1(\Omega_m)^N$. Let V and V_i be subspaces of $H_0^1(\Omega)^N$ and $H_0^1(\Omega_i)^N$, respectively, consisting of divergence-free functions. Then $V = V_1 + \cdots + V_m$.*

For m subdomains, we can obtain analogous error equations. For instance,

$$e^{(n+1)} = (I - P_m) \cdots (I - P_1) e^{(n)} + \cdots,$$

where P_i is the orthogonal projection (with respect to $[\cdot, \cdot]$) of V onto V_i and $[+ \cdots]$ is a sum of m quadratic terms. Armed with Lemma 2, it is well known ([Lio88], [BPWX91]) that

$$d = ||(I - P_m) \cdots (I - P_1)||_1 < 1$$

for overlapping subdomains. In a similar way as before, we can show that the Schwarz sequence is bounded and provided $d^{1/m} + o(\epsilon) < 1$ with $\epsilon = RC(M+1)$ sufficiently small, the Schwarz sequence converges. Because it does not seem possible to write down an explicit expression for the $o(\epsilon)$ term, we cannot give an explicit bound on R.

For the additive Schwarz method, it converges if

$$R < \frac{1 - ||I - \omega\,(P_1 + \cdots + P_m)||_1}{m\omega C(||u^{(0)} - u||_1 + 2||u||_1)},$$

where $0 < \omega < 1/K$, where K is the minimum number of colors needed to color the subdomains in such a way that overlapping subdomains are assigned different colors.

Non-homogeneous boundary conditions can also be treated by making a change of variable. Define $w = u - g$, where g is the Dirichlet data on $\partial\Omega$ which is extended to be a divergence-free function in $H^1(\Omega)^N$. Then the Navier Stokes equations become

$$(w \cdot \nabla) w + (g \cdot \nabla) w + (w \cdot \nabla) g = -\nabla p + \frac{1}{R} \triangle w + G$$

for $w \in V$ and

$$G = \frac{1}{R} \triangle g + f - (g \cdot \nabla) g.$$

The analysis goes through as before.

One interesting question is to investigate in what sense does the sequence of pressure converge. Other future work include extending our results to the time dependent case and the consideration of Schwarz methods on non-overlapping subdomains.

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