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Preconditioning Operators for Elliptic Problems with Bad Parameters

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INTRODUCTION

In this paper we design preconditioning operators for the system of grid equations approximating the following boundary value problem.

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} + a_0(x)u = f(x), \quad x \in \Omega, \quad u(x) = 0, \quad x \in \Gamma \qquad (1)$$

We suppose that Ω is a bounded and polygonal domain, where Γ does denote its boundary. Let Ω be a union of $n+1$ nonoverlapping subdomains Ω_i, such that

$$\overline{\Omega} = \bigcup_{i=0}^{n} \overline{\Omega}_i\,, \quad \Omega_i \cap \Omega_j = \emptyset, \quad i \neq j,$$

holds. Here we have the polygonal subdomains Ω_i in the interior of Ω. Their boundaries are given by Γ_i, $i = 1, \ldots, n$. The domain Ω_0 is defined to be multiple connected having the boundary $\Gamma \bigcup (\bigcup_{i=1}^{n} \Gamma_i)$. We denote by $H_i = \text{diam}(\Omega_i)$ the diameter of the i-th subdomain, $i = 1, \ldots, n$. We assume small parameters H_i such that

$$0 < H_i \leq 1$$

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Eleventh International Conference on Domain Decomposition Methods
Editors Choi-Hong Lai, Petter E. Bjørstad, Mark Cross and Olof B. Widlund

is valid. Furthermore, for any subdomain Ω_i, if there exists a subdomain Ω_j such that

$$\mathrm{dist}(\Omega_i, \Omega_j) \leq \alpha_1 H_i$$

holds, then the conditions

$$H_j = O(H_i) \quad \text{and} \quad \alpha_2 H_i \leq \mathrm{dist}(\Omega_i, \Omega_j)$$

must be fulfilled, where α_1 and α_2 are constants wich are independent of the parameter $H_i, i = 1, \ldots, n$. This means that for any subdomain Ω_i there is no other subdomain in the neighbourhood determined by $O(H_i)$.

Let us introduce the bilinear form

$$a(u,v) = \int_\Omega \left(\sum_{i,j=1}^{2} a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + a_0(x)uv \right) dx$$

and the linear functional

$$l(v) = \int_\Omega f(x)v dx.$$

We suppose that the coefficients of the problem (1) are such that $a(u,v)$ is a symmetric bilinear form in the Sobolev space $H_0^1(\Omega)$. Let the inequalities

$$\alpha_3 a(u,v) \leq \int_\Omega \epsilon(x)|grad(u)|^2 \leq \alpha_4 a(u,v) \quad \forall u \in H_0^1(\Omega).$$

be fulfilled with positive constants α_3, α_4, which are independent of the parameter ϵ. Here we fix

$$\epsilon(x) = \mathrm{const} = \epsilon_i, \quad \forall x \in \Omega_i,$$

where we have

$$\epsilon_0 = 1, \quad 0 < \epsilon_i \leq 1, \quad i = 1, \ldots, n \,. \tag{2}$$

The linear functional $l(v)$ is continuous in $H_0^1(\Omega)$. The weak formulation of (1) is given as follows. Find $u \in H_0^1(\Omega)$ such that the following is valid for all $v \in H_0^1(\Omega)$

$$a(u,v) = l(v) \,. \tag{3}$$

Let $\Omega^h = \bigcup_{i=0}^{n} \Omega_i^h$ be a quasiuniform triangulation of the domain Ω, which can be characterized by the parameter h.
We denote by W the space of real continuous functions being linear on the triangles of the triangulation Ω^h. Using the finite element method, see e.g. [Mar82], the variational formulation (3) can be transfered to the well known system of linear algebraic equations

$$Au = f \,. \tag{4}$$

The condition number of the matrix A depends on the parameters h, H_i and ϵ_i, and can be large. Our purpose is the design of a preconditioner B for the problem (4), such that the following inequalities are valid for all vectors $u \in R^N$

$$c_1(Bu,u) \leq (Au,u) \leq c_2(Bu,u) \,. \tag{5}$$

Here the symbol N denotes the dimension of the space W, and c_1 and c_2 are positive constants independent of the parameters h, H_i, and ϵ_i. Furthermore, the multiplication of a vector by B^{-1} should be easy to implement numerically causing low costs.
The preconditioning operator B is constructed by using the nonoverlapping and overlapping (but without "overlapping" in the coefficients) domain decomposition methods. The analysis of these methods refers to the well known Neumann-Dirichlet domain decomposition method. However, the suggested methods do not require the exact solution of subproblems with Dirichlet boundary condition.

NONOVERLAPPING DOMAIN DECOMPOSITION

The construction of the preconditioner for the system (4) is performed by means of the Additive Schwarz Method, see e.g. [Lio88],[MN85],[MN88]. To design the preconditioning operator B, we use [Nep91a],[Nep92] decomposing the space W into a sum of subspaces as follows

$$W = W_0 + W_1$$

We divide the nodes of the triangulation Ω^h into two groups, those which lie inside of $\Omega_i^h, i = 1, \ldots, n$ and those which lie in $\overline{\Omega}_0^h$. The subspace W_0 does correspond to the first set. Let us introduce the following sets

$$S = \bigcup_{i=1}^{n} \partial\Omega_i^h,$$

$$W_0 = \left\{u^h \in W | \quad u^h(x) = 0, x \in \overline{\Omega}_0^h \right\},$$

$$W_{0,i} = \{u^h \in W_0 | \quad u^h(x) = 0, \quad x \overline{\in} \Omega_i^h\}, \quad i = 1, 2, \ldots, n.$$

It is clear that W_0 represents the direct sum of the orthogonal subspaces $W_{0,i}$ with respect to the scalar product in $H_0^1(\Omega)$

$$W_0 = W_{0,1} \oplus \ldots \oplus W_{0,n} \ .$$

The subspace W_1 corresponds to the second group of nodes in Ω^h and can be defined as follows. Let the set V be the trace space of the functions given by W on S, i.e. we have

$$V = \{\varphi^h | \quad \varphi^h(x) = u^h(x), \quad x \in S, \quad u^h \in W\}.$$

To define the subspace W_1, we need a norm preserving extension operator of functions given on S into Ω^h. The corresponding construction is based on the following trace lemma.

Lemma 1 *Let Ω be a bounded domain with piecewisely smooth boundary Γ satisfying the Lipschitz condition. Let*

$$\mathrm{diam}(\Omega) = H \ .$$

And let Ω^h be a quasiuniform triangulation of Ω. We denote

$$\begin{aligned} \|\varphi\|^2_{H^{\frac{1}{2}}(\Gamma)} &= H\|\varphi\|^2_{L^2(\Gamma)} + |\varphi|^2_{H^{\frac{1}{2}}(\Gamma)}, \\ \|\varphi\|^2_{L^2(\Gamma)} &= \int_\Gamma \varphi^2(x)dx, \\ |\varphi|^2_{H^{\frac{1}{2}}(\Gamma)} &= \int_\Gamma \int_\Gamma \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^2}\ dxdy\ . \end{aligned}$$

Then, there exists a positive constant c_1, which is independent of the parameters h, H, such that

$$\|\varphi^h\|_{H^{\frac{1}{2}}(\Gamma)} \leq c_1\|u^h\|_{H^1(\Omega)}$$

and

$$|\varphi^h|_{H^{\frac{1}{2}}(\Gamma)} \leq c_1|u^h|_{H^1(\Omega)}$$

hold for any function $u^h \in W$, where $\varphi^h \in V$ is the trace of u^h on the boundary Γ. Vice versa, there exists a positive constant c_2, which is independent of h and H, such that for any function $\varphi^h \in V$ we have the function $u^h \in W$ with

$$\begin{aligned} u^h(x) &= \varphi^h(x), \quad x \in \Gamma\ , \\ \|u^h\|_{H^1} &\leq c_2\|\varphi^h\|_{H^{\frac{1}{2}}(\Gamma)}, \\ |u^h|_{H^1} &\leq c_2|\varphi^h|_{H^{\frac{1}{2}}(\Gamma)}. \end{aligned}$$

To define the subspace W_1, let us use the explicit extension operator

$$t^h : V \to W, \tag{6}$$

which was suggested for second order elliptic problems with smooth coefficients, such that for all $\varphi^h \in V$

$$\|u^h\|_{H^1(\Omega)} = \|t^h\varphi^h\|_{H^1(\Omega)} \leq c_3\|\varphi^h\|_{H^{\frac{1}{2}}(S)}$$

holds, where the corresponding norm is given by

$$\|\varphi\|^2_{H(S)} = \sum_{i=1}^{n} \|\varphi\|^2_{H^{\frac{1}{2}}(\Gamma_i)}\ .$$

For defining and implementing the numerical algorithm see [MN93],[Nep86],[Nep91c]. Now, we can define the subspace W_1 as follows

$$\begin{aligned} W_1 = \{u^h|\quad u^h(x) &= (t^h\varphi^h)(x), \quad x \in \Omega_i, \quad i = 1, \ldots, n, \quad \varphi^h(x) = v^h(x), \quad x \in S, \\ u^h(x) &= v^h(x), \quad x \in \Omega^h_0, \quad v^h \in W\}. \end{aligned}$$

Obviously we have

$$W = W_0 + W_1\ ,$$

and this decomposition of the space W is stable in the following sense.

Lemma 2 *There exists a positive constant c_4, which is independent of the parameters h, H_i and ϵ_i, such that for any function $u^h \in W$ there exist functions $u_i^h \in W_i, i = 0, 1$, such that we have*

$$u_0^h + u_1^h = u^h ,$$
$$a(u_0^h, u_0^h) + a(u_1^h, u_1^h) \leq c_4 a(u^h, u^h) .$$

Let C_i , $i = 0, 1, \ldots, n$ be the preconditioning operators in the finite element subspaces $H_0^1(\Omega_i)$. Hence, we have the following inequalities for all $u^h \in W \cap H_0^1(\Omega_i)$

$$c_5 \|u^h\|^2_{H^1(\Omega_i)} \leq (C_i u, u) \leq c_6 \|u^h\|^2_{H^1(\Omega_i)} , \qquad (7)$$

where the constants c_5, c_6 are independent of the parameters h and H_i. For example, these operators C_i can be constructed using the fictitious space lemmata in [Nep91c],[Nep91b],[Nep92],[Nep95],[Xu96]. We extend the operator C_i outside of Ω_i by zero and denote by C_i^+ the pseudo-inverse operator belonging to this extension. We introduce the following operator

$$B_{\text{nov}}^{-1} = tC_0^+ t^* + \frac{1}{\epsilon_1} C_1^+ + \cdots + \frac{1}{\epsilon_n} C_n^+ .$$

Here the operator t^* is the adjoint to t. The following theorem holds.

Theorem 1 *There exist positive constants c_7, c_8, which are independent of the parameters h, H_i and ϵ_i, such that the following inequalities are fulfilled for all $u \in R^N$*

$$c_1 (B_{\text{nov}} u, u) \leq (Au, u) \leq c_2 (B_{\text{nov}} u, u) .$$

OVERLAPPING DOMAIN DECOMPOSITION

The goal of this section is the design of the preconditioning operators for the problem (4) without using the extension operator t given in (6).
Let C be the preconditioning operator in the finite element space W, such that for all functions $u^h \in W$ we have

$$c_1 \|u^h\|^2_{H^1(\Omega)} \leq (Cu, u) \leq c_2 \|u^h\|^2_{H^1(\Omega)} ,$$

where the constants c_1, c_2 are independent of h. We denote the preconditioner B_{ov}^{-1} as follows

$$B_{\text{ov}}^{-1} = C^{-1} + \frac{1}{\epsilon_1} C_1^+ + \cdots + \frac{1}{\epsilon_n} C_n^+ .$$

Here the pseudoinverses C_i^+ are given by (7). The following theorem holds.

Theorem 2 *There exist positive constants c_3, c_4, which are independent of the parameters h, H_i and ϵ_i, such that the inequalities*

$$c_3 (B_{\text{ov}} u, u) \leq (Au, u) \leq c_4 (B_{\text{ov}} u, u)$$

are fulfilled for all $u \in R^N$.

Proof:
In the case of $\epsilon_i = 1\,, i = 1, \ldots, n$, using Theorem 1 there exist constants c_5, c_6, which are independent of h and H_i, such that

$$c_5(C^{-1}u, u) \leq tC_0^+t^* + C_1^+ + \cdots + C_n^+ \leq c_6(C^{-1}u, u)$$

holds for all $u \in R^N$. From (2) we get

$$0 \leq (C_i^+u, u) \leq \frac{1}{\epsilon_i} \leq (C_i^+u, u) \quad \forall u \in R^N .$$

Hence, we have

$$\begin{aligned}
(B_{\text{nov}}^{-1}u, u) &= tC_0^+t^* + \frac{1}{\epsilon_1}C_1^+ + \cdots + \frac{1}{\epsilon_n}C_n^+ \\
&\leq tC_0^+t^* + C_1^+ + \cdots + C_n^+ + \frac{1}{\epsilon_1}C_1^+ + \cdots + \frac{1}{\epsilon_n}C_n^+ \\
&\leq \max\{c_6, 1\}((C^{-1} + \frac{1}{\epsilon_1}C_1^+ + \cdots + \frac{1}{\epsilon_n}C_n^+)u, u) = \max\{c_6, 1\}(B_{\text{ov}}^{-1}u, u) \\
&\leq \max\{c_6, 1\}\max\{\frac{1}{c_5}, 1\}(tC_0^+t^* + C_1^+ + \cdots + C_n^+ + \frac{1}{\epsilon_1}C_1^+ + \cdots + \frac{1}{\epsilon_n}C_n^+)u, u) \\
&\leq 2\max\{c_6, 1\}\max\{\frac{1}{c_5}, 1\}(B_{\text{nov}}^{-1}u, u) \, .
\end{aligned}$$

Remark The above Theorem 2 can be proved directly without using the extension operator t.

Acknowledgment The author wishes to thank U. Langer and J. Schoeberl from the Johannes–Kepler–University in Linz, Austria for promoting and helpfully discussing the topic of this paper.

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