11

Interface Preconditioners and Multilevel Extension Operators

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INTRODUCTION

Interface problems arising in nonoverlapping domain decomposition methods can often be viewed as restriction of the original elliptic problem to a discrete trace space. As has been demonstrated in many particular examples (see the references below), this connection can also be used for the construction of interface preconditioners and extension operators. We present an abstract additive Schwarz framework for this deduction, with applications to multilevel schemes.

ABSTRACT FRAMEWORK

We assume familiarity with the theory of stable space splittings and additive Schwarz methods for solving symmetric positive definite (spd) variational problems in a Hilbert space [DW90, Xu92, Osw94a, SBG96]. Let the bilinear forms $a(\cdot,\cdot)$ and $b_j(\cdot,\cdot)$ be bounded and spd on V and V_j, respectively, and $R_j : V_j \to V$ be linear operators. Then we call

$$\{V; a\} \cong \sum_j R_j \{V_j; b_j\} \tag{1}$$

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Eleventh International Conference on Domain Decomposition Methods
Editors Choi-Hong Lai, Petter E. Bjørstad, Mark Cross and Olof B. Widlund

a *stable space splitting*, with stability constants $0 < \mu_1 \le \mu_2 < \infty$, if

$$\mu_1 a(v,v) \le |||v|||^2 := \inf_{v_j \in V_j:\, v=\sum_j R_j v_j} \sum_j b_j(v_j,v_j) \le \mu_2 a(v,v) \quad \forall\, v \in V\ . \tag{2}$$

If $V_j \subset V$ and if R_j represents the natural embeddings then R_j will be omitted. The upper estimate is equivalent to assuming that for any $v \in V$ and any $\epsilon > 0$ we can find a decomposition

$$v = \sum_j R_j v_j\ ,\quad v_j \in V_j \quad : \quad \sum_j b_j(v_j,v_j) \le (1+\epsilon)\mu_2 a(v,v)\ , \tag{3}$$

while the lower bound in (2) is often reduced to the verification of so-called strengthened Cauchy-Schwarz inequalities [Xu92].

Given a continuous linear operator $T : V \to X$, its range $\hat{V} \subset X$, can be converted into a Hilbert space if we introduce the following norm on $\hat{V}$:

$$||\hat{v}||_{\hat{V}} = \inf_{v\in V\,:\,\hat{v}=Tv} ||v||_V \qquad \forall\, \hat{v} \in \hat{V}\ . \tag{4}$$

E.g., if V is a function space with domain Ω, and T represents a trace operator to a subset Γ then this is the common *implicit* definition of the trace space norm. The construction of equivalent *explicit* (or *intrinsic*) norms is one of the major problems in the theory of trace spaces. Given a $\hat{v} \in \hat{V}$, the construction of a $v \in V$ with $\hat{v} = Tv$ and $||v||_V \le C_{ext}||\hat{v}||_{\hat{V}}$ is called *extension problem*. Clearly, one is interested in an as small as possible constant $C_{ext} < \infty$, and an efficient realization of the mapping $E : \hat{v} \to v$. Similar definitions occur in connection with *Schur complement problems* in substructuring methods.

We present the assumptions for a generic construction of a stable space splitting for $\{\hat{V};\hat{a}\}$ if a splitting (1) is available for $\{V;a\}$. Here, $\hat{a}(\cdot,\cdot)$ is a bounded and spd bilinear form on $\hat{V}$. In particular, this implies that

$$\hat{a}(Tv,Tv) \le C_1 a(v,v) \qquad \forall\, v \in V\ , \tag{5}$$

and that for any $\hat{v} \in \hat{V}$

$$\exists\, v \in V \ :\quad \hat{v} = Tv\ ,\ \ a(v,v) \le C_2 \hat{a}(\hat{v},\hat{v})\ . \tag{6}$$

with some absolute constants $0 < C_1, C_2 < \infty$. Suppose that T_j acts on V_j, with range $\hat{V}_j$, and that bounded spd forms $\hat{b}_j$ on $\hat{V}_j$ are introduced (in complete analogy with the introduction of $\hat{V}$ and $\hat{a}$). We assume that all appearing constants are uniform w.r.t. j. In particular,

$$\hat{b}_j(T_j v_j, T_j v_j) \le C_3 b_j(v_j,v_j) \qquad \forall\, v_j \in V_j\ , \tag{7}$$

for some C_3, independent of j. Let $E_j : \hat{V}_j \to V_j$ be right inverses of T_j, $T_j E_j = \mathrm{Id}_{\hat{V}_j}$, such that

$$b_j(E_j\hat{v}_j, E_j\hat{v}_j) \le C_4 \hat{b}_j(\hat{v}_j,\hat{v}_j) \qquad \forall\, v_j \in V_j\ , \tag{8}$$

for some C_4, independently of j. The last, and most critical condition is on the kernels of the operators T_j and TR_j: For all j,

$$\mathcal{N}(\mathcal{T}_|) \subset \mathcal{N}(\mathcal{T}\mathcal{R}_|)\ . \tag{9}$$

Theorem 1 *Under the assumptions (5), (6), (7), (8), and (9), the stability of (1) implies the stability of the space splitting*

$$\{\hat{V};\hat{a}\} \cong \sum_j \hat{R}_j\{\hat{V}_j;\hat{b}_j\} \qquad (\ \hat{R}_j = TR_jE_j\ )\ , \tag{10}$$

with stability constants $\hat{\mu}_1, \hat{\mu}_2$ *satisfying*

$$0 < (C_1C_4)^{-1}\mu_1 \le \hat{\mu}_1 \le \hat{\mu}_2 \le C_2C_3\mu_2 < \infty\ . \tag{11}$$

Proof. Let $\hat{v} = \sum_j \hat{R}_j\hat{v}_j$ be any representation of $\hat{v}$. Set $v_j = E_j\hat{v}_j$ and $v = \sum_j R_jv_j$. Obviously,

$$Tv = \sum_j TR_jv_j = \sum_j TR_jE_j\hat{v}_j = \sum_j \hat{R}_j\hat{v}_j = \hat{v}\ ,$$

and by (5), (8), and the lower estimate in (2), we have

$$\begin{aligned}\hat{a}(\hat{v},\hat{v}) &= \hat{a}(Tv,Tv) \le C_1a(v,v) \le C_1\mu_1^{-1}\sum_j b_j(v_j,v_j)\\ &= C_1\mu_1^{-1}\sum_j b_j(E_j\hat{v}_j,E_j\hat{v}_j) \le C_1C_4\mu_1^{-1}\sum_j \hat{b}_j(\hat{v}_j,\hat{v}_j)\ .\end{aligned}$$

This gives the lower stability estimate for (10).

In the other direction, take a v with $\hat{v} = Tv$ satisfying (6), and decompose it optimally in the sense of (3) using the upper stability estimate in (2). Set $u_j = E_jT_jv_j$, $\hat{v}_j = T_jv_j$, and observe that $T_j(u_j - v_j) = (T_jE_j - \mathrm{Id}_{\hat{V}_j})T_jv_j = 0$. Thus, by (9) $0 = TR_j(u_j - v_j) = TR_jE_jT_jv_j - TR_jv_j = \hat{R}_j\hat{v}_j - TR_jv_j$. This implies

$$\hat{v} = Tv = \sum_j TR_jv_j = \sum_j \hat{R}_j\hat{v}_j\ ,$$

and, taking into account also (7),

$$\begin{aligned}\sum_j \hat{b}_j(\hat{v}_j,\hat{v}_j) &= \sum_j \hat{b}_j(T_jv_j,T_jv_j) \le C_3\sum_j b_j(v_j,v_j)\\ &\le (1+\epsilon)C_3\mu_2a(v,v) \le (1+\epsilon)C_2C_3\mu_2\hat{a}(\hat{v},\hat{v})\ .\end{aligned}$$

Letting $\epsilon \to 0$ finishes the proof of Theorem 1. Note that we could have replaced (9) by the slightly weaker condition $\mathcal{R}(\mathrm{Id}_{V_j} - E_jT_j) \subset \mathcal{N}(\mathcal{T}\mathcal{R}_j)$.

Corollary 2 *Under the assumptions of Theorem 1, given an arbitrary* $\hat{v} \in \hat{V}$, *let* $\hat{v}_j \in \hat{V}_j$ *be such that the analog of (3) holds:*

$$\hat{v} = \sum_j \hat{R}_j\hat{v}_j\ :\quad \sum_j \hat{b}_j(\hat{v}_j,\hat{v}_j) \le C_5\hat{a}(\hat{v},\hat{v})\ . \tag{12}$$

Then a suitable extension $u = E\hat{v} \in V$ *of* $\hat{v}$ *satisfying* $\hat{v} = Tu$ *is given by*

$$u = \sum_j R_jE_j\hat{v}_j\ :\quad a(u,u) \le C_4C_5\hat{a}(\hat{v},\hat{v}) \le C_1C_4C_5 \inf_{v\in V\,:\,\hat{v}=Tv} a(v,v)\ . \tag{13}$$

If $\hat{v}_j = \hat{Q}_j\hat{v}$ *for some linear operators* $\hat{Q}_j\ :\ \hat{V} \to \hat{V}_j$ *then the extension operator* $E = \sum_j R_jE_j\hat{Q}_j\ :\ \hat{V} \to V$ *is linear and bounded.*

A particular instance, where this abstract framework for constructing additive Schwarz preconditioners and extension operators can be applied, are *multilevel splittings* based on a hierarchy of spaces

$$V_0 \xrightarrow{I_1} V_1 \xrightarrow{I_2} \ldots \xrightarrow{I_J} V_J \xrightarrow{I_{J+1}} \ldots , \tag{14}$$

where $I_j : V_{j-1} \to V_j$ are given prolongation operators acting between subsequent spaces of the hierarchy (again, nesting of $\{V_j\}$ is not assumed). Setting $V = V_J$ and assuming that $a = a_J$ is a bounded and spd bilinear form on V_J, the associated multilevel splitting is given by

$$\{V_J; a_J\} \cong \sum_{j=0}^{J} R_j^J \{V_j; b_j\} , \qquad R_j^k = I_k \ldots I_{j+1} , \quad 0 \le j < k . \tag{15}$$

Suppose that (15) is stable, uniformly w.r.t. J. Assume that the sequence T_j is given, and that $T = T_J$. Furthermore, let (5), (6) hold with $a, \hat{a}$ replaced by $a_J, \hat{a}_J$ and with constants independent of J. We keep the requirements (7) and (8), while (9) is replaced by the existence of another set of linear operators

$$\hat{I}_j : \hat{V}_{j-1} \to \hat{V}_j : \quad T_j I_j = \hat{I}_j T_{j-1} , \quad j \ge 1 . \tag{16}$$

This new requirement implies (9) since

$$T_J R_j^J = (T_J I_J) I_{J-1} \ldots I_{j+1} = \hat{I}_J (T_{J-1} I_{J-1}) \ldots I_{j+1} = \ldots = \hat{I}_J \ldots \hat{I}_{j+1} T_j .$$

Thus, Theorem 1 and Corollary 2 are applicable, leading to splittings and extension operators for $\{\hat{V}_J; \hat{a}_J\}$ and J-independent constants in the estimates. Moreover, by the same reasoning we obtain the recursion formula

$$\hat{R}_j^J = T_J R_j^J E_j = \hat{I}_J \ldots \hat{I}_{j+1} T_j E_j = \hat{I}_J \ldots \hat{I}_{j+1} , \quad 0 \le j < J < \infty .$$

This is important for the efficient implementation of additive and multiplicative Schwarz methods associated with $\{\hat{V}_j\}$.

APPLICATION

Applications in the spirit of Theorem 1 for preconditioning Schur complement problems in substructuring methods for finite element discretizations of second order elliptic boundary value problems are well-known (see [Osw94b], sections 2.3 and 7.1 of [XZ99], [Kho98b] for some special cases and further references). Thess [The98] has recently worked on applications to fourth order plate and shell problems. The Stokes problem has been treated in [Kho98a].

We illustrate the application of the above abstract framework by deriving an optimal complexity extension procedure for linear finite element boundary data on locally refined meshes (for similar results on preconditioning, see [KP98]). For simplicity, let Ω be a bounded polyhedral domain in $\mathbf{R^d}$, $d \ge 2$, equipped with a nested sequence of regular, quasi-uniform simplicial partitions $\mathcal{T}_j$ of element diameter $\approx 2^{-j}$, $j \ge 0$. Let

$\tilde{V}_j$ be the space of linear finite elements w.r.t. $\mathcal{T}_|$, the standard basis function in $\tilde{V}_j$ associated with a vertex $P_{j,i}$ of $\mathcal{T}_|$ is denoted by $N_{j,i}$. *Nested refinement* is modelled by selecting a finite increasing sequence

$$\emptyset = \Omega'_0 \subset \Omega'_1 \subset \ldots \Omega'_J \subset \Omega'_{J+1} = \Omega \ , \tag{17}$$

where each Ω'_j, $1 \leq j \leq J$, is or empty or the closure of the union of some simplices from $\mathcal{T}_{|-\infty}$. The sequence $\Omega_j = \mathrm{clos}(\Omega \backslash \Omega'_j)$ is decreasing and, roughly speaking, represents the simplices of $\mathcal{T}_{|-\infty}$ which are refined at level j. The spaces V_j are spanned by all basis functions $N_{j,i}$ for which $P_{j,i}$ belongs to the interior of Ω_j, and as the finite element space V corresponding to this refinement process we take $V = \sum_{j=0}^{J} V_j$ (this construction is discussed in [Osw94a]section 4.2.3, be aware of some index errors there!). It is easy to see that an algebraic basis $\mathcal{N}$ for V is given by all $N_{j,i} \in V_j \backslash V_{j+1}$, $j \leq J$. Although V is not a traditional finite element space associated with an triangulation, its approximation power on each domain Ω_j is almost as good as that of $\tilde{V}_j$ (and less good in Ω'_j). Along the lines of [Osw94a]section 4.2.3 it follows that

$$\{V; (\cdot,\cdot)_{H^1(\Omega)}\} \cong \sum_{j=0}^{J} \sum_{i\,:\,N_{j,i} \in V_j} \{V_{j,i}, 2^{2j}(\cdot,\cdot)_{L_2(\Omega)}\} \ , \tag{18}$$

where $V_{j,i}$ are the one-dimensional spaces spanned by individual $N_{j,i}$, and with bounds for the stability constants that are independent on J and $\{\Omega'_j\}$. The stability of the splitting (18) expresses nothing but the optimality of BPX multilevel preconditioning in the nested refinement case, and can be derived from the corresponding result in the uniform refinement case (see [Osw94a]Theorem 19) by quasi-interpolant techniques.

Let us now consider the construction of a discrete harmonic extension operator $E \,:\, \hat{V} \equiv V|_\Gamma \to V$, where Γ is the boundary of Ω (note that everything extends to polyhedral manifolds Γ consisting of $(d-1)$-dimensional faces of simplices in $\mathcal{T}_\prime$). Obviously, $\hat{V} = \sum_{j=0}^{J} \hat{V}_j$, where $\hat{V}_j \equiv V_j|_\Gamma \subset \hat{\tilde{V}}_j \equiv \tilde{V}_j|_\Gamma$. The sequence $\{\hat{V}_j\}$ could have been derived by a nested refinement procedure on Γ similar to the one described above (consider the triangulations $\hat{\mathcal{T}}_j$ induced from $\mathcal{T}_|$ on Γ, and use $\hat{\Omega}'_j = \Omega'_j \cap \Gamma$ for the increasing sequence (17)). A basis in $\hat{V}_j$ is given by all non-vanishing $\hat{N}_{j,i} = N_{j,i}|_\Gamma$, where $N_{j,i} \in V_j$. Accordingly, a basis $\hat{\mathcal{N}}$ in $\hat{V}$ is given by all $\hat{N}_{j,i} \in \hat{V}_j \backslash \hat{V}_{j+1}$, $j \leq J$.

Let $T \,:\, V \to \hat{V}$, $T_{j,i} \,:\, V_{j,i} \to \hat{V}_{j,i}$ coincide with the restriction of the trace operator $|_\Gamma$ to the corresponding spaces (if $P_{j,i} \notin \Gamma$ then $\hat{V}_{j,i}$ is the null space and $T_{j,i}$ the null operator). Since the operators $R_{j,i}$ represent the natural embeddings for $V_{j,i} \subset V$, condition (9) is automatically fulfilled. Define extension operators $E_{j,i} \,:\, \hat{V}_{j,i} \to V_{j,i}$ by simply setting $E_{j,i}(c\hat{N}_{j,i}) = cN_{j,i}$. Combining the $E_{j,i}$ with j fixed leads to the usual level-j extension-by-zero operator $E_j \,:\, \hat{V}_j \to V_j$. Obviously, the assumptions (7), (8) are satisfied if we set

$$b_{j,i}(u,v) = 2^{2j}(u,v)_{L_2(\Omega)} \ , \qquad \hat{b}_{j,i}(\hat{u},\hat{v}) = 2^j(\hat{u},\hat{v})_{L_2(\Gamma)} \ .$$

From the first part of Theorem 1 it follows that

$$\{\hat{V}; (\cdot,\cdot)_{\hat{V}}\} \cong \sum_{j=0}^{J} \sum_{i\,:\,\hat{N}_{j,i} \in \hat{V}_j} \{\hat{V}_{j,i}, 2^j(\cdot,\cdot)_{L_2(\Gamma)}\} \tag{19}$$

is also a stable splitting.

The next steps are as follows. First, it can be shown that

$$\|\hat{v}\|_{\hat{V}} \approx \|\hat{v}\|_{H^{1/2}(\Gamma)} \equiv \inf_{u\in H^1(\Omega)\,:\,v=Tu} \|u\|_{H^1(\Omega)} \qquad \forall \hat{v} \in \hat{V} \; . \tag{20}$$

This relates the $\|\cdot\|_{\hat{V}}$ norm defined in (4) to the norm in the trace space $H^{1/2}(\Gamma)$ for $H^1(\Omega)$ functions (intrinsic $H^{1/2}(\Gamma)$ norms are described, e.g., in [Kho98b, Xu92]). By definition of $\|\cdot\|_{\hat{V}}$, we obviously have $\|\hat{v}\|_{H^{1/2}(\Gamma)} \le \|\hat{v}\|_{\hat{V}}$. The proof of the opposite estimate uses the same techniques as were applied in [Osw94a]section 4.2.3 to derive (18) from the standard BPX-splitting for the whole space $\{H^1(\Omega);(\cdot,\cdot)_{H^1}\}$. The necessary quasi-interpolant operators will be described below, the details are omitted.

Secondly, in order to define an extension operator $E : \hat{V} \to V$ along the lines of Corollary 2, we need a realization of (12). Using the L_2-stability of the finite element bases (this fact will be used below without further mentioning), the desired representation follows if we construct a sequence of linear, uniformly L_2-bounded projections $\hat{P}_j : L_2(\Gamma) \to \hat{\hat{V}}_j$ such that $\hat{Q}_j = \hat{P}_j - \hat{P}_{j-1}$ maps $\hat{V}$ into $\hat{V}_j$, $0 \le j \le J$ (set $\hat{P}_{-1} = 0$). We use specific quasi-interpolants: Set

$$\hat{P}_j \hat{v}(x) = \sum_{i\,:\,P_{j,i}\in\Gamma} (\hat{v}, \hat{\lambda}_{j,i})_{L_2(\Gamma)} \hat{N}_{j,i}(x) \; , \tag{21}$$

where the functions $\hat{\lambda}_{j,i}$ are piecewise linear (but not continuous!) on the $(d-1)$-dimensional simplices of $\hat{\mathcal{T}}_j$ and have support $\hat{\Lambda}_{j,i}$ consisting of a few simplices attached to $P_{j,i}$. More precisely, we define

$$\hat{\Lambda}_{j,i} = \begin{cases} \mathrm{supp}\,\hat{\mathrm{N}}_{\mathrm{j,i}} & \text{if } \mathrm{supp}\,\hat{\mathrm{N}}_{\mathrm{j,i}} \subset \hat{\Omega}_{\mathrm{j+1}} \\ \mathrm{supp}\,\hat{\mathrm{N}}_{\mathrm{j,i}} \cap \Omega'_{\mathrm{j+1}} & \text{otherwise} \end{cases} \; . \tag{22}$$

To ensure the projection property w.r.t. $\hat{V}_j$, we define

$$\hat{\lambda}_{j,i}(x) = \Big(\int_{\hat{\Lambda}_{j,i}} \hat{N}_{j,i}\, dy\Big)^{-1} \hat{\psi}_{j,i}(x) \; , \quad x \in \hat{\Lambda}_{j,i} \; , \tag{23}$$

where the piecewise linear function $\hat{\psi}_{j,i}$ has value d at $P_{j,i}$, and -1 at all other vertices of simplices attached to $P_{j,i}$. The reader can easily verify that

$$(\hat{N}_{j,l}, \hat{\lambda}_{j,i})_{L_2(\Gamma)} = \delta_{i,l} \; , \qquad |(\hat{v}, \hat{\lambda}_{j,i})_{L_2(\Gamma)}|^2 \le C 2^{j(d-1)} \|\hat{v}\|^2_{L_2(\hat{\Lambda}_{j,i})} \; ,$$

which ensures the projection property w.r.t. $\hat{\hat{V}}_j$ and the uniform L_2-boundedness of the $\hat{P}_j$, respectively. Finally, if $\hat{v} \in \hat{V}$ then $\hat{v}$ is a linear finite element function w.r.t. $\mathcal{T}_{|-\infty}$ outside $\hat{\Omega}_j$. Thus, by our choice of the $\hat{\Lambda}_{j,i}$, both $\hat{P}_j$ and $\hat{P}_{j-1}$ reproduce $\hat{v}$ on $\hat{\Omega}'_j$ exactly. This gives $\hat{Q}_j\hat{v} = 0$ on this set and since $\hat{Q}_j\hat{v} \in \hat{\hat{V}}_j$ we arrive at $\hat{Q}_j\hat{v} \in \hat{V}_j$ for any $\hat{v} \in \hat{V}$.

Let $\hat{Q}_{j,i}$ ($\hat{Q}_j = \sum_i \hat{Q}_{j,i}$) be the resulting linear mappings from $\hat{V}$ into the one-dimensional $\hat{V}_{j,i}$ corresponding to the basis functions $\hat{N}_{j,i}$. Using the properties of $\hat{P}_j, \hat{Q}_j$ in conjunction with (19), (20), from Corollary 2 we deduce

Corollary 3 *Under the above assumptions on the refinement process, the multilevel extension operator* $E = \sum_{j=0}^{J} E_j \hat{Q}_j = \sum_{j=0}^{J} \sum_i E_{J,i} \hat{Q}_{j,i} \; : \; \hat{V} \to V$ *satisfies*

$$\|E\hat{v}\|_{H^1(\Omega)} \le C \|\hat{v}\|_{H^{1/2}(\Gamma)} \qquad \forall \, \hat{v} \in \hat{V} \,, \tag{24}$$

where the constant C *is independent of the particular refinement process.*

We briefly describe how to efficiently implement this extension operator. First of all, we need to fix vector representations for finite element functions in V and $\hat{V}$. In both cases, we could use the bases $\mathcal{N}$, $\hat{\mathcal{N}}$ mentioned above or (allowing for some non-uniqueness in the representation) slightly larger generating systems

$$\mathcal{R} = \{\mathcal{N}_{j,i} \; : \; \mathcal{N}_{j,i} \in V_j \,, \, j \le J\} \,, \quad \hat{\mathcal{R}} = \{\hat{\mathcal{N}}_{j,i} \; : \; \hat{\mathcal{N}}_{j,i} \in \hat{V}_j \,, \, j \le J\} \,.$$

It can be shown that $\#\mathcal{R} \le C\#\mathcal{N}$, $\#\hat{\mathcal{R}} \le \#\hat{\mathcal{N}}$, with C independent on the refinement process. The operator E naturally maps any $\hat{v}$ into a coefficient vector w.r.t. a subset of $\mathcal{R}$ of the same size as $\hat{\mathcal{R}}$, and can be implemented in $O(\#\hat{\mathcal{R}})$ operations. Thus, except for the conversion of $E\hat{v}$ into a representation w.r.t. $\mathcal{N}$, storage and amount of work associated with E are proportional to $\dim \hat{V}$.

To substantiate this claim, we outline a computational scheme for $\hat{Q}_j \hat{v}$. The basic idea is to switch to yet another representation of finite element functions, this time w.r.t. the standard basis of discontinuous piecewise linear functions w.r.t. the given nested refinement structure. Set $\hat{\mathcal{B}} = \cup_{j=0}^{J} \hat{\mathcal{B}}_j$, where $\hat{\mathcal{B}}_j$ consists of all piecewise linear functions $\hat{\psi}_{\Delta,P}$ with support in a simplex $\Delta \in \hat{\mathcal{T}}_j$ belonging to $\hat{\Omega}_j^*$ and having value 1 at the vertex P of Δ and vanishing at all other vertices. Minimally, a suitable $\hat{\Omega}_j^*$ should contain $\hat{\Omega}_j$ and possibly an additional layer of $\hat{\mathcal{T}}_{j-1}$ simplices containing the supports of all $\hat{N}_{j,i}$ whose supports intersect with $\hat{\Omega}_{j+1}$ (the reason for this will become clear below). Obviously, the size of $\hat{\mathcal{B}}$ is still within a constant multiple of $\dim \hat{V}$. We also need the subsets $\hat{\mathcal{B}}_j' \subset \hat{\mathcal{B}}_j$ of all those functions $\hat{\psi}_{\Delta,P}$ for which $\Delta \in \hat{\Omega}_j^* \backslash \hat{\Omega}_{j+1}^*$ (again, $\hat{\Omega}_{J+1}^* = \emptyset$, and $\Omega_0^* = \Omega$).

The unique coefficients of $\hat{v} \in \hat{V}$ w.r.t. $\hat{\mathcal{B}}' = \cup_{j=0}^{J} \hat{\mathcal{B}}_j'$ can be computed fast, from either the $\hat{\mathcal{R}}$ or the $\hat{\mathcal{N}}$ representation, by recursion in $j = 0, 1, \ldots, J$. After this, by recursion in $j = J, J-1, \ldots, 0$, all scalar products $(\hat{v}, \hat{\psi}_{\Delta,P})_{L_2(\Gamma)}$ associated with functions $\hat{\psi}_{\Delta,P} \in \hat{\mathcal{B}}_j$ can be computed in an amount of operations which stays proportional to $\#\hat{\mathcal{B}}$. Indeed, for a level j simplex $\Delta \in \hat{\Omega}_j^* \backslash \hat{\Omega}_{j+1}^*$ this reduces to a multiplication with a local Gram matrix, while for $\Delta \subset \hat{\Omega}_{j+1}^*$ the refinability of the basis functions $\hat{\psi}_\Delta$ is used: each such function can be expressed as a linear combination of a fixed finite number of functions $\hat{\psi}_{\Delta',P'}$ associated with $\Delta' \in \hat{\mathcal{T}}_{j+1}$ in $\hat{\Omega}_{j+1}^*$ for which the corresponding integrals have been computed before. Finally, by formula (21) this gives the coefficient vector of $\hat{Q}_j \hat{v} = \hat{P}_j \hat{v} - \hat{P}_{j-1} \hat{v}$ in the standard basis of $\hat{V}$ if we express the $\hat{N}_{j,i}$ as the corresponding finite linear combination of $\hat{\psi}_\Delta$ and recall that due to $\hat{P}_j \hat{v} = \hat{P}_{j-1} \hat{v}$ outside $\hat{\Omega}_j$ only terms with $\mathrm{supp}\,\hat{\mathrm{N}}_{\mathrm{j,i}} \subset \hat{\Omega}_{\mathrm{j}}$ and $\hat{N}_{j-1,i} \not\equiv 0$ on $\hat{\Omega}_j$ are to be considered (here, the special definition of $\hat{\Omega}_j^*$ is used). Concatenating all these vectors results in the coefficients of $E\hat{v}$ w.r.t. $\mathcal{R}$ (coefficients corresponding to functions $N_{j,i} \in \mathcal{R}$ with $N_{j,i}|_\Gamma \equiv 0$ are set to 0). More details can be found in an extended version of this note at `http://cm.bell-labs.com/who/poswald`.

CONCLUSION

We have presented an abstract framework for deriving multilevel preconditioners and extension operators, such as needed for interface problems and other Schur complement problems arising in non-overlapping domain decomposition methods for symmetric elliptic boundary value problems. The basic assumption is the availibility of a suitable multilevel decomposition for the global variational problem. Details are given for linear finite element discretizations on partitions arising from nested local refinement.

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