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Robust Subspace Correction Methods for Thin Elastic Shells

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Introduction

In this paper we address the issue of developing subspace correction methods for thin elastic shells which are *robust* with respect to the thickness, i.e. they exhibit convergence which is not adversely affected by the small thickness of the shell. This kind of robustness has been tackled so far only in some specific cases, such as zero curvature and linear approximation in the transverse variable (see e.g. [Bre96, Kla96, AFW97, BS98]). In this paper we invoke the so-called *Korn's type inequality in subspaces* [OX96, OX97b, OX97a, OX98a] to present a technique for developing robust iterative subspace correction methods applicable to thin elastic shells of arbitrary geometry discretized by finite elements of any order.

Boundary value problem of elastostatics.

Consider a boundary value problem of linear elastostatics in variational form:

$$\mathbf{u} \in \mathbf{V}: \quad E(\mathbf{u}, \mathbf{v}) = F(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V} \tag{1}$$

where the space of admissible displacements $\mathbf{V}$ is given by

$$\mathbf{V} = \{\mathbf{v} \in H^1(\Omega)^3 : \mathbf{v} = 0 \text{ on } \Gamma \subset \partial\Omega\}, \quad \Omega \subset \mathbf{R}^3$$

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and the elastic energy form E is given by

$$E(\mathbf{u}, \mathbf{v}) = \int_\Omega \varepsilon(\mathbf{u})^T H\, \varepsilon(\mathbf{v})\, d\Omega,$$

where the vector ε is formed by the six components of the strain tensor:

$$\varepsilon(\mathbf{v}) = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{12}, \varepsilon_{23}, \varepsilon_{31})^T, \quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

and H is symmetric positive definite matrix whose elements are piecewise smooth functions. The right-hand side of (1) is given by

$$F(\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\Omega + \int_{\Gamma_0} \mathbf{T} \cdot \mathbf{v}\, d\Gamma_0, \quad \Gamma_0 = \partial\Omega \setminus \Gamma$$

Here and throughout the paper we use boldface for 3-vectors, i.e. $\mathbf{u}= (u_1, u_2, u_3)$ etc.

Thin shell.

The domain of a shell is defined as an image $\Omega = \Phi(\hat{\Omega})$ of a plate-like *reference* domain $\hat{\Omega}$:

$$\hat{\Omega} = \{\ (\xi_1, \xi_2, \eta) : (\xi_1, \xi_2) \in \Omega_\xi \subset R^2, -\frac{t}{2} < \eta < \frac{t}{2}\ \}$$

We assume that the mapping $(x, y, z) = \Phi(\xi_1, \xi_2, \eta)$ is

- continuous and piecewise-C^2
- non-degenerate on $\overline{\hat{\Omega}}$
- linear in η

Numerical solution.

Let us approximate the space of admissible displacements $\mathbf{V}$ by e.g. the space of piecewise-polynomial functions:

$$\mathbf{V} \approx \mathbf{V}^{pq} = \{\ \mathbf{v} \in \mathbf{V} : \mathbf{v} \circ \Phi \in \hat{\mathbf{V}}^{pq}\ \}$$

where $\hat{\mathbf{V}}^{pq}$ is the space of piecewise-polynomial functions $\hat{\mathbf{v}}(\xi_1, \xi_2, \eta)$ which are of degree p in ξ_1, ξ_2 and of degree q in η. By Galerkin projection onto $\mathbf{V}^{pq}$ we obtain the following discretization of (1):

$$\mathbf{u} \approx \mathbf{u}^{pq} \in \mathbf{V}^{pq} : \quad E(\mathbf{u}^{pq}, \mathbf{v}^{pq}) = F(\mathbf{v}^{pq}) \quad \forall \mathbf{v}^{pq} \in \mathbf{V}^{pq} \tag{2}$$

In this paper we consider the solution of the problem (2) by iterative subspace correction methods of additive type (see e.g. [Xu92]).

Additive subspace correction (ASC).

In this section we omit for simplicity the superscripts p and q.

Assume that the space $\mathbf{V}$ is decomposed into the sum of subspaces:

$$\mathbf{V} = \mathbf{V}_1 + \ldots + \mathbf{V}_n$$

Following the subspace decomposition technique (see e.g. [Xu92, Osw94]) we can transform the problem (2) into an abstract operator equation

$$T\mathbf{u} = \mathbf{g} \tag{3}$$

where

$$T = T_1 + \ldots + T_n, \quad \mathbf{g} = \mathbf{g}_1 + \ldots + \mathbf{g}_n$$

$$T_i : \mathbf{V} \to \mathbf{V}_i : \quad E(T_i\mathbf{v}, \mathbf{w}) = E(\mathbf{v}, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{V}_i$$

$$\mathbf{g}_i \in \mathbf{V}_i : \quad E(\mathbf{g}_i, \mathbf{w}) = F(\mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{V}_i$$

The equation (3) is then solved using a suitable iterative algorithm:

$$\mathbf{u}^{k+1} = \mathbf{u}^k + \tau_k(T\mathbf{u}^k - \mathbf{g})$$

Each iteration involves the calculation of $T\mathbf{u}^k$ which, according to the definition of the operator T, amounts to solving problems in subspaces. Thus, the problem in the whole space $\mathbf{V}$ is reduced to solving problems in the subspaces $\mathbf{V}_i$.

Subspace splitting and convergence of ASC.

There are several approaches to analysing the convergence of the above iterative scheme of which we opt for the one suggested by Oswald which is based on the concept of a stable subspace splitting [Osw94].

If a norm $||\cdot||_{(i)}$ is introduced in each subspace $\mathbf{V}_i$ then the set of pairs $\{\mathbf{V}_i, ||\cdot||_{(i)}\}_{i=1,n}$ is called a splitting of the pair $\{\mathbf{V}, ||\cdot||\}$, which is formally expressed as

$$\{\mathbf{V}, ||\cdot||\} = \sum_{i=1}^{n} \{\mathbf{V}_i, ||\cdot||_{(i)}\} \tag{4}$$

The so-called additive Schwarz norm $[\cdot]$ for (4) is defined as

$$[\mathbf{v}] = \inf \left\{ \left(\sum_{i=1}^{n} ||\mathbf{v}_i||_{(i)}^2 \right)^{\frac{1}{2}} : \ \mathbf{v}_i \in \mathbf{V}_i, \ \sum_{i=1}^{n} \mathbf{v}_i = \mathbf{v} \right\}$$

and the following two values are called characteristic numbers of (4):

$$a_0 = \inf_{\mathbf{v} \in \mathbf{V}} \frac{[\mathbf{v}]}{||\mathbf{v}||}, \quad a^0 = \sup_{\mathbf{v} \in \mathbf{V}} \frac{[\mathbf{v}]}{||\mathbf{v}||}$$

and the ratio a^0/a_0 is called the condition number of this splitting.

If the energy norm is taken for the norms in $\mathbf{V}$ and $\mathbf{V}_i$, i.e. $\|\cdot\| = \|\cdot\|_{(i)} = \|\cdot\|_E \equiv E(\cdot,\cdot)^{1/2}$ then the condition number of the operator T is equal to the condition number of the corresponding splitting (see e.g. [Osw94]).

Stable splitting and robust convergence.

The splitting of the space $\mathbf{V}^{pq}$ is called stable if its condition number is uniformly bound with respect to the number of subspaces, the discretization parameters p and q and the parameters of the problem we are concerned with (in the case in hand – thickness of the shell).

If the splitting is stable, then the condition number of the operator T is also uniformly bound with respect to the parameters mentioned above, and accordingly the convergence of the subspace correction method is robust with respect to those parameters.

A well-known example of the technique for developing subspace correction methods which are robust w.r.t discretization parameters is the overlapping domain decomposition technique. A less known example is the *effective dimensional reduction algorithm* (EDRA) [OX95]. This algorithm is based on the following *semi-discretization* of the space $\mathbf{V}$:

$$\mathbf{V}^q = \{\ \mathbf{v} \in \mathbf{V} : \mathbf{v} \circ \Phi = \sum_{i=0}^{q} \mathbf{v}_i(\xi_1, \xi_2) Q_i(2\eta/t)\ \}$$

where Q_i are the eigenfunction of the eigenvalue problem

$$\int_{-1}^{1} \frac{dQ_i}{d\eta}\frac{dQ}{d\eta}\, d\eta = \lambda_i \int_{-1}^{1} Q_i Q\, d\eta \quad \forall Q : \text{ polyn. of deg. } q$$

The semi-discretized problem is solved using the additive subspace correction method with the subspaces given by

$$\mathbf{V}^q_{3i+j} = \{\ (v_1, v_2, v_3) \in \mathbf{V}^q : v_j \circ \Phi = w(\xi_1, \xi_2) Q_i(2\eta/t),\ v_k = 0,\ k \neq j\ \}, \qquad j = 1, 2, 3;\ i = 0, \ldots, q$$

The EDRA is robust with respect to to the semi-discretization parameter q, but, just as the overlapping DDM, it is not robust with respect to the thickness *in general*, i.e. unless special decompositions are used, such as those presented later on in this paper.

Classical Korn's inequality: diagnosis

The convergence of iterative methods for thin elastic structures deteriorates because of the poor coerciveness of the elastic energy form E as diagnosed by the classical Korn's inequality (see e.g. [Fic72, Hor95]):

$$Ct^2 \|\mathbf{v}\|_1^2 \leq E(\mathbf{v}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}$$

where $\|\cdot\|_1$ is the Sobolev norm, i.e.

$$\|\mathbf{v}\|_1^2 = \int_\Omega (|\mathbf{v}|^2 + \sum_{i=1}^{3} \left|\frac{\partial \mathbf{v}}{\partial x_i}\right|^2) d\Omega$$

Korn's inequality in subspaces: treatment

In the *Korn's type inequality in subspaces* recently introduced by the authors (see [OX96, OX97b, OX97a, OX98a]) the constant in front of the Sobolev norm is expressed in terms of the angle between a given subspace and the subspace $\mathbf{V}_0$ of the displacements which are linear in the transverse direction:

$$\forall \mathbf{U} \subset \mathbf{V} \quad C\beta^2_{\mathrm{U},\mathrm{V}_0}\|\mathbf{v}\|_1^2 \leq E(\mathbf{v},\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{U} \tag{5}$$

where

$$\mathbf{V}_0 = \{ \ \mathbf{v} \in \mathbf{V} : \mathbf{v} \circ \Phi = \mathbf{v}_0(\xi_1,\xi_2) + \eta \mathbf{v}_1(\xi_1,\xi_2) \ \}$$

and

$$\beta_{\mathrm{U},\mathrm{W}} = \min_{\mathbf{u}\in \mathrm{U}} \min_{\mathbf{w}\in \mathrm{W}} \frac{\|\mathbf{u}-\mathbf{w}\|_1}{\|\mathbf{u}\|_1}$$

This paves the way to overcome the difficulty: taking proper care of the displacements which are linear in the transverse direction.

More economic alternatives for $\mathbf{V}_0$

Assume for simplicity $\Phi = I$ and $\mathbf{V} = H_0^1(\Omega)^3$. The subspace $\mathbf{V}_0$ in (5) can be reduced to

$$\mathbf{V}_0 = \{ \ (\eta u(\xi_1,\xi_2), \eta v(\xi_1,\xi_2), w(\xi_1,\xi_2)), \ u,v,w \in H_0^1(\Omega_\xi) \ \} \tag{6}$$

or to

$$\mathbf{V}_0 = \{ \ (\eta w(\xi_1,\xi_2), \eta w(\xi_1,\xi_2), -w(\xi_1,\xi_2)), \ w \in H_0^2(\Omega_\xi) \ \}$$

Robust ASC for thin shells.

First approach.

Theorem 1 *[OX98b] Denote* $\mathbf{V}_0^p = \mathbf{V}_0 \cap \mathbf{V}^{pq}$. *If the constant* a_0 *for*

$$\{\mathbf{V}^{pq}, \|\cdot\|_1\} = \sum_{i=1}^{n} \{\mathbf{V}_i^{pq}, \|\cdot\|_1\}$$

is independent of n, p, q, t *and*

$$E(\sum_{i=1}^{n} \mathbf{v}_i) \leq C \sum_{i=1}^{n} E(\mathbf{v}_i)$$

then

$$\{\mathbf{V}^{pq}, \|\cdot\|_E\} = \{\mathbf{V}_0^p, \|\cdot\|_E\} + \sum_{i=1}^{n} \{\mathbf{V}_i^{pq}, \|\cdot\|_E\}$$

is stable and, hence, the corresponding ASC is robust with respect to n, p, q *and* t.

The conditions of this theorem are satisfied by most overlapping DDMs with $\mathbf{V}_0^p$ added to the usual subspace decomposition.

Second approach.

Theorem 2 *[OX98b] If the space* $\mathbf{V}^{pq}$ *can be decoupled into the sum*

$$\mathbf{V}^{pq} = \mathbf{V}_0^p + \mathbf{V}_0^{pq}: \quad \beta(\mathbf{V}_0^p, \mathbf{V}_0^{pq}) \geq \beta_0 > 0$$

and the splitting

$$\{\mathbf{V}_0^{pq}, \|\cdot\|_1\} = \sum_{i=1}^{n} \{\mathbf{V}_i^{pq}, \|\cdot\|_1\}$$

is stable then

$$\{\mathbf{V}^{pq}, \|\cdot\|_E\} = \{\mathbf{V}_0^p, \|\cdot\|_E\} + \sum_{i=1}^{n} \{\mathbf{V}_i^{pq}, \|\cdot\|_E\}$$

is stable and, hence, the corresponding ASC is robust with respect to n, p, q *and* t.

The conditions of this theorem are satisfied by EDRA with $\mathbf{V}_1, \ldots, \mathbf{V}_6$ replaced by $\mathbf{V}_0^p$ or, in the case of a plate, with $\mathbf{V}_3, \mathbf{V}_4, \mathbf{V}_5$ replaced by the subspace $\mathbf{V}_0^p$ given by (6).

Numerical example.

Here we present numerical results for the shell shown in Fig. 1. The elastic moduli are: $E = 4 \cdot 10^{10}$ and $\nu = 0.2$ for the outer layer and $E = 10^{10}$ and $\nu = 0.4$ for the inner. The left horizontal edge is clamped (i.e $\mathbf{u} = 0$) and the right edge is simply supported (i.e. $u_3 = 0$) and subject to a unit traction force in the direction of the x-axis (i.e. $T_1 = 1$ and $T_2 = 0$). The front edge is simply supported (i.e. $u_2 = 0$). The discretised problem is solved using the modified EDRA described briefly in the previous section.

Tables 1 and 2 show the number of the iterations of the subspace correction method executed until the relative residual falls below 10^{-4} for $t/R = 0.01$ and various values of

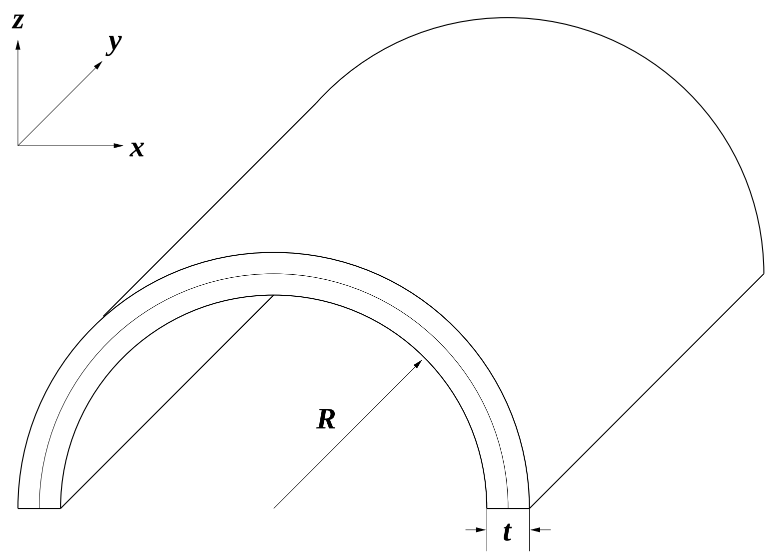

Figure 1 Two-layered thin elastic shell.

Table 1 Uniform convergence with respect to p ($q = 20$).

Polynomial degree p	1	2	3	4
Number of iterations	13	16	16	16

p and q. Table 3 shows the number of the iterations of the subspace correction method under the same stopping criterion for $p = 2$, $q = 20$ and various values of t/R. These results clearly demonstrate that the convergence of the subspace correction method based on the splitting given in the previous section is *uniform* with respect to the thickness parameter t and the discretisation parameters p and q. Thus, the subspace correction method presented here is robust with respect to all these parameters.

REFERENCES

[AFW97] Arnold D. N., Falk R. S., and Winther R. (1997) Preconditioning discrete approximations of the Reissner-Mindlin plate model. *Math. Model. Numer. Analysis* 31: 517–557.

[Bre96] Brenner S. (1996) Multigrid methods for parameter dependent problems. *Math. Model. Numer. Analysis* 30: 265–297.

[BS98] Bramble J. H. and Sun T. (1998) A negative-norm least squares method for Reissner-Mindlin plates. *Math. comp.* 67: 901–916.

[Fic72] Fichera G. (1972) Existence theorems in elasticity. In Flügge S. and Truesdell C. (eds) *Handbuch der Physik, Vol 6a/2*, pages 347–389. Springer Verlag.

[Hor95] Horgan C. O. (1995) Korn's inequalities and their applications in continuum mechanics. *SIAM Review* 37(4): 491–511.

Table 2 Uniform convergence with respect to q ($p = 2$).

Polynomial degree q	15	20	25	30
Number of iterations	16	16	16	16

Table 3 Uniform convergence with respect to the thickness parameter d

Thickness ratio t/R	0.08	0.04	0.02	0.01
Number of iterations	18	17	16	16

[Kla96] Klawonn A. (1996) Preconditioners for indefinite problems. Technical Report 8/96-N, Universität Münster.

[Osw94] Oswald P. (1994) *Multilevel finite element approximation.* Teubner, Stuttgart.

[OX95] Ovtchinnikov E. E. and Xanthis L. S. (1995) Effective dimensional reduction for elliptic problems. *C. R. Acad. Sci. Paris, Série I* 320: 879–884.

[OX96] Ovtchinnikov E. E. and Xanthis L. S. (1996) A new Korn's type inequality for thin domains and its application to iterative methods. *Comput. Meths. Appl. Mech. Engrg.* 138: 299–315.

[OX97a] Ovtchinnikov E. E. and Xanthis L. S. (1997) The Korn's type inequality in subspaces and thin elastic structures. *Proc. R. Soc. Lond. A* 453: 2003–2016.

[OX97b] Ovtchinnikov E. E. and Xanthis L. S. (1997) A new Korn's type inequality for thin elastic structures. *C. R. Acad. Sci. Paris, Série I* 324: 577–583.

[OX98a] Ovtchinnikov E. E. and Xanthis L. S. (1998) The discrete Korn's type inequality in subspaces and thin elastic structures. *Comput. Meths. Appl. Mech. Engrg.* 160: 23–37.

[OX98b] Ovtchinnikov E. E. and Xanthis L. S. (1998) Iterative subspace correction methods for thin elastic structures and Korn's type inequality in subspaces. *Proc. R. Soc. Lond. A* 454: 2023–2039.

[Xu92] Xu J. (1992) Iterative methods by subspace decomposition and subspace correction. *SIAM Rev.* 34: 581–613.