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Domain Decomposition Methods for the Steady Stokes Equations

EINAR M. RØNQUIST [1]

Introduction

We compare and discuss four different steady Stokes solvers: the Uzawa algorithm, an iterative substructuring method, and two new domain decomposition methods. The methods are tested in the context of high-order spectral element discretizations. The new domain decomposition methods can be categorized as an additive-type preconditioner for the global Stokes system comprising the solution of a coarse Stokes system, the solution of local, non-overlapping pressure systems, and the solution of either local, overlapping viscous systems or a global viscous system. A comparison of the four methods by solving the driven cavity problem in two and three dimensions shows that the new domain decomposition methods applied to the global Stokes system can yield excellent convergence rates. The results for the new methods also indicate that the treatment of the pressure and the velocity degrees-of-freedom in the preconditioner can be quite different; in particular, a non-overlapping treatment suffices for the pressure while an overlapping treatment can be used for the velocity.

Additive Schwarz methods are well established in the context of solving systems of equations derived from the finite element discretization of elliptic problems [DW94, SBG96]. Even though the concept of an overlap is less well-defined for high-order spectral element discretizations, the additive Schwarz methods can also be used in this context. This follows from the fact that the original spectral element system is spectrally equivalent to a finite element system derived from a linear triangulation/tetrahedrazation of the underlying spectral element mesh. Hence, the

1 Nektonics, Inc., 15 Everett Street, Arlington, MA 02474-6901, U.S.A.
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additive Schwarz preconditioners used for standard, low-order finite element systems can also be used as preconditioners for the spectral element systems. Numerical experiments demonstrating this approach were first presented in [Pah93] in the context of solving the Poisson problem, and later analyzed in [Cas97]. Similar techniques have also been used with success in order to solve the consistent pressure Poisson operator [Fis97, FMTar].

Additive Schwarz methods have only recently been used for the full (indefinite) Stokes system in coupled form. In [KPar], a Krylov space method such as the GMRES method is used in combination with an indefinite preconditioner based upon overlapping Schwarz techniques. The additive version of this preconditioner comprises a coarse (indefinite) Stokes system and local (indefinite) Stokes subproblems associated with the overlapping subdomains. The work in [KPar] was done in the context of stable, mixed, linear finite elements.

In the context of high-order spectral element methods, primarily iterative substructuring methods have been considered for the global, coupled Stokes system in the past [Røn96, Røn98, Cas96, PW97]; the techniques reported in [Røn96, Røn98] have also been extended to solve the steady Navier-Stokes equations in three dimensions. Recently, however, the methodology presented in [KPar] has also been extended to spectral elements; see [Pavar].

In this paper, we consider a new overlapping, additive Schwarz approach for the steady Stokes system in three dimensions. In contrast to the work presented in [KPar] in the context of linear finite elements, and to the work in [Pavar] in the context of spectral elements, we abandon the concept of considering the "full" Stokes operator for the local problems associated with the subdomains. Instead, we consider two new domain decomposition methods where the local pressure solutions are computed based upon non-overlapping techniques, while the velocity degrees-of-freedom are preconditioned either based upon overlapping methods for the local problems or by solving a global problem for the viscous system.

The steady Stokes problem

We consider here the solution of the steady Stokes equations in a domain $\Omega \subset R^d$, $d = 2, 3$: Find a velocity $\mathbf{u} = (u_1, .., u_d) \subset H_0^1(\Omega)^d$ and a pressure $p \subset L_0^2(\Omega)$ such that

$$\begin{aligned} \int_\Omega \mu \nabla \mathbf{u} \, : \, \nabla \mathbf{v} \, d\Omega - \int_\Omega p \nabla \cdot \mathbf{v} \, d\Omega &= \int_\Omega \mathbf{f} \cdot \mathbf{v} \, d\Omega \quad \forall \mathbf{v} \in H_0^1(\Omega)^d \, , \\ \int_\Omega q \nabla \cdot \mathbf{u} \, d\Omega &= 0 \quad \forall q \in L_0^2(\Omega) \, . \end{aligned}$$

Here, μ is the viscosity and $\mathbf{f}$ is a body force.

The continuous Stokes problem is discretized using the spectral element method [MP89]. Following this approach, we first decompose the computational domain Ω into K (conforming) spectral elements,

$$\overline{\Omega} = \cup_{k=1}^K \overline{\Omega}_k .$$

Each spectral element Ω_k is a quadrilateral element in R^2, and a hexahedral element

in R^3. Throughout this paper, a single spectral element will also be synonymous with a single subdomain.

On the reference element $\Omega_{\mathtt{ref}} =]-1,1[^d$ associated with each spectral element, the velocity components are approximated as polynomials of degree N in each spatial direction, while the pressure is approximated as a polynomial of degree $N-2$. That is, each spectral element is a $Q_N - Q_{N-2}$ element. More precisely, if we denote the discrete space for the velocity as V and the discrete space for the pressure as W, these spaces are defined as

$$\begin{aligned} V &= \{v : v_{|\overline{\Omega}_k} \circ F_k \in Q_N(\overline{\Omega}_{\mathtt{ref}})^d \} \cap H_0^1(\Omega)^d \ , \\ W &= \{w : w_{|\Omega_k} \circ F_k \in Q_{N-2}(\Omega_{\mathtt{ref}}) \} \cap L_0^2(\Omega) \ . \end{aligned}$$

Here, F_k represents the isoparametric (or affine) mapping from the reference domain $\Omega_{\mathtt{ref}}$ onto Ω_k, that is, $\Omega_k = F_k(\Omega_{\mathtt{ref}})$, $k = 1, ..., K$.

In order to derive a set of algebraic equations, we choose (local) tensor-product bases for our discrete spaces V and W, and we evaluate the integrals in the variational formulation by tensor-product (Gauss-Lobatto-Legendre) quadrature. The resulting set of discrete equations can then be written as

$$\underline{\mathbf{S}}\,\underline{\mathbf{x}} = \underline{\mathbf{g}} \tag{1}$$

where

$$\underline{\mathbf{S}} = \begin{bmatrix} \underline{\mathbf{A}} & -\underline{\mathbf{D}}^T \\ \underline{\mathbf{D}} & \underline{0} \end{bmatrix} \ , \ \underline{\mathbf{x}} = \begin{bmatrix} \underline{\mathbf{u}} \\ \underline{p} \end{bmatrix} \ , \ \underline{\mathbf{g}} = \begin{bmatrix} \underline{\mathbf{f}} \\ \underline{\mathbf{0}} \end{bmatrix} \ .$$

Here, $\underline{\mathbf{A}}$ is the discrete viscous operator (in this case, the vector Laplacian), $\underline{\mathbf{D}}$ is the discrete divergence operator, and its transpose $\underline{\mathbf{D}}^T$ is the discrete gradient operator. The vector $\underline{\mathbf{u}}$ contains the nodal velocity values, $\underline{p}$ represents the nodal pressure values, and the components of $\underline{\mathbf{f}}$ are the nodal forces. Boldface symbols are associated with multi-dimensional quantities, while non-boldface symbols are associated with scalar quantities.

Solution methods

There exist many iterative algorithms for solving the indefinite saddle point problem (1); see for example [MMPR93, Elm96, Cas96, PW97, Pav97, Røn96]. In this section, we will focus our attention on a few iterative methods. In the next section, numerical results will be presented, and the methods will be compared.

Method U: The Uzawa algorithm

A classical solution method, the Uzawa algorithm, consists of first decoupling the Stokes saddle point problem into a positive semi-definite form for the pressure,

$$\underline{\mathbf{D}}\,\underline{\mathbf{A}}^{-1}\underline{\mathbf{D}}^T\,\underline{p} = -\underline{\mathbf{D}}\,\underline{\mathbf{f}} \ . \tag{2}$$

Eliminating the hydrostatic mode, the pressure system can be solved iteratively via a nested conjugate gradient (CG) iteration. Note that each (outer) pressure iteration

involves the inversion of the discrete viscous operator, $\underline{\mathbf{A}}$. Since this operator is symmetric, positive definite, the viscous system can also be solved via an (inner) conjugate gradient iteration.

Once the pressure has been computed, the velocity can be found by simply solving the viscous system

$$\underline{\mathbf{A}}\,\mathbf{u} = \underline{\mathbf{D}}^T \underline{p} + \underline{\mathbf{f}} \; . \tag{3}$$

An excellent preconditioner for the pressure system (2) is the mass matrix, $\underline{\tilde{M}}$, associated with the pressure degrees-of-freedom. Denoting the Uzawa pressure operator as $\underline{U} = \underline{\mathbf{D}}\,\underline{\mathbf{A}}^{-1}\underline{\mathbf{D}}^T$, it can be shown that the condition number

$$\kappa(\underline{\tilde{M}}^{-1}\underline{U}) \leq c_1/\beta_h^2 \; ,$$

where c_1 is a constant and β_h is the inf-sup parameter [MPR92, PW97].

In order to solve the viscous system (the system (3), as well as the solution of the viscous system associated with each outer pressure iteration), we can choose from a plethora of available preconditioners. For example, for *low-order* finite elements, an overlapping Schwarz method [DW94, SBG96] yields a condition number

$$\kappa(\underline{\mathbf{B}}^{-1}\underline{\mathbf{A}}) \leq c_a\,(1 + H/\delta) \; . \tag{4}$$

Here, $\underline{\mathbf{B}}$ is the additive Schwarz preconditioner corresponding to the viscous operator (in this case, the vector Laplacian), c_a is a constant, H is the diameter of the subdomains, and δ is the overlap. As mentioned in the Introduction, we can also recover the result given in (4) for spectral elements; see [Pah93, Cas97].

Method S: An iterative substructuring algorithm

In this section, we briefly review an iterative substructuring method (i.e., a *non-overlapping* preconditioner) originally proposed in [Røn96]. This method has been extended to solve a variety of steady, incompressible flow problems, including problems described by the steady, three-dimensional Navier-Stokes equations with free surface boundary conditions; see [Røn98]. For comparison reasons, we present this method (here, for the Stokes system only) in a slightly different manner than in [Røn96] and [Røn98]. First, the preconditioned Stokes system is written as

$$\underline{\mathbf{Q}}^{-1}\underline{\mathbf{S}}\,\mathbf{x} = \underline{\mathbf{Q}}^{-1}\underline{\mathbf{g}} \; . \tag{5}$$

This preconditioned system is solved using a Krylov space method such as the Generalized Conjugate Residual (GCR) method. The preconditioner can be expressed as

$$\underline{\mathbf{Q}}^{-1} = \underline{\mathbf{Q}}_0^{-1} + \sum_{k=1}^{K} \underline{\mathbf{Q}}_{k,v}^{-1} + \sum_{k=1}^{K} \underline{\mathbf{Q}}_{k,p}^{-1} + \underline{\mathbf{Q}}_{\Gamma}^{-1} \tag{6}$$

where

$$\underline{\mathbf{Q}}_0^{-1} \;=\; \underline{\mathbf{R}}_0^T\; \underline{\tilde{\mathbf{S}}}_0^{-1}\, \underline{\mathbf{R}}_0 \; ,$$

$$\underline{\mathbf{Q}}_k^{-1} = \underline{\mathbf{R}}_k^T \, \underline{\tilde{\mathbf{S}}}_k^{-1} \, \underline{\mathbf{R}}_k \quad , k = 1, ..., K \ ,$$

$$\underline{\mathbf{Q}}_{k,v}^{-1} = \begin{bmatrix} \underline{\mathbf{I}} & \underline{0} \\ \underline{\mathbf{0}} & \underline{0} \end{bmatrix} \underline{\mathbf{Q}}_k^{-1} \ ,$$

$$\underline{\mathbf{Q}}_{k,p}^{-1} = \begin{bmatrix} \underline{\mathbf{0}} & \underline{0} \\ \underline{\mathbf{0}} & \underline{\mathrm{I}} \end{bmatrix} \underline{\mathbf{Q}}_k^{-1} \ ,$$

$$\underline{\mathbf{Q}}_\Gamma^{-1} = \underline{\mathbf{R}}_\Gamma^T \, diag(\underline{\mathbf{A}})^{-1} \, \underline{\mathbf{R}}_\Gamma \, [\underline{\mathbf{I}} \ + \underline{\mathbf{D}}^T (\sum_{k=1}^{K} \underline{\mathbf{Q}}_{k,p}^{-1})] \ .$$

Briefly explained, the preconditioner consists of a sum of three contributions: (i) a global, coarse Stokes solution ($\underline{\mathbf{Q}}_0^{-1}$); (ii) local Stokes solutions associated with the K subdomains ($\underline{\mathbf{Q}}_k^{-1}$, $k = 1, ..., K$); (iii) an approximation to the velocity along the interface Γ between the subdomains ($\underline{\mathbf{Q}}_\Gamma^{-1}$). The operators $\underline{\mathbf{R}}_0$, $\underline{\mathbf{R}}_k$, $k = 1, .., K$ and $\underline{\mathbf{R}}_\Gamma$ denote the restriction operators associated with the coarse problem, the local subdomain problems and the interface problem, respectively; the transpose of these operators denote the corresponding prolongation (or extension) operators.

The coarse solution is obtained by restricting the residual from the original $Q_N - Q_{N-2}$ spectral element mesh to a low-order $Q_2 - P_1$ finite element mesh. In what follows, the coarse mesh has the same number of elements as the fine mesh (i.e., the coarse mesh is induced by the fine mesh). The operator $\underline{\tilde{\mathbf{S}}}_0$ represents the coarse Stokes operator which is inverted by a direct solver. The indefinite Stokes system is made solvable by insisting that the average (global) pressure be zero.

Each local Stokes system assumes homogeneous velocity boundary conditions. Hence, each local Stokes operator, $\underline{\tilde{\mathbf{S}}}_k$, is also indefinite, and the average (local) pressure is assumed to be zero. Each local Stokes system is solved by *explicitly* constructing and factoring the symmetric Uzawa pressure operator *locally* within each element; only back substitution is thus necessary in order to obtain the pressure within each subdomain. Once the local pressure solution has been computed, each local velocity can be found by solving a *local* system of the form (3); see [Røn96] for more details.

Finally, the velocity along the subdomain interfaces is found simply by inverting the diagonal of the viscous operator, $\underline{\mathbf{A}}$ (which is trivial). The associated extension operator $\underline{\mathbf{R}}_\Gamma^T$ represents an extension by zero to all the interior subdomain nodes; again, for more details, see [Røn96].

Method H1: A hybrid Schwarz/substructuring algorithm

In this and the next section, we present two variants of a preconditioner which we will refer to as a hybrid Schwarz/substructuring method. Note that the word "hybrid" is here used to emphasize the fact that part of the preconditioner is of a non-overlapping type, and part of the preconditioner is of an overlapping type. Using the notation from

the previous section, the preconditioner can be expressed as

$$\underline{\mathbf{Q}}^{-1} = \underline{\mathbf{Q}}_0^{-1} + \sum_{k=1}^{K} \underline{\hat{\mathbf{Q}}}_{k,v}^{-1} + \sum_{k=1}^{K} \underline{\mathbf{Q}}_{k,p}^{-1} \tag{7}$$

We see that we no longer have an interface system for the velocity degrees-of-freedom. Instead, we solve K (local) *overlapping* systems for the velocity,

$$\underline{\hat{\mathbf{Q}}}_{k,v}^{-1} = \underline{\hat{\mathbf{R}}}_{k,v}^{T} \underline{\hat{\mathbf{A}}}_k^{-1} \underline{\hat{\mathbf{R}}}_{k,v} \left[\underline{\mathbf{I}} + \underline{\mathbf{D}}^T \left(\sum_{k=1}^{K} \underline{\mathbf{Q}}_{k,p}^{-1} \right) \right] \quad k = 1, ..., K. \tag{8}$$

Here, $\underline{\hat{\mathbf{R}}}_{k,v}$ is an operator which restricts the residual of the momentum equations to the nodes associated with $\overline{\Omega}_k$ *plus* one layer of nodes outside this subdomain; we denote this extended subdomain as $\overline{\Omega}_k'$. Hence, for this preconditioner, we consider the minimum-overlapping case for the velocity *only*. Next, an approximation of the discrete viscous operator is constructed based upon a triangularization/tetrahedrazation of the Gauss-Lobatto-Legendre nodes associated with the overlapping subdomain $\overline{\Omega}_k'$; linear finite elements are used within each triangle/tetrahedron. The resulting local systems $\underline{\hat{\mathbf{A}}}_k$, $k = 1, .., K$ are then assembled and factored. Hence, only back substitution is required in order to obtain the local velocities during each preconditioning step. (Note that it is sufficient to only assemble and factor the *scalar* Laplacian associated with each extended subdomain.)

The restricted residual for the local velocity problem consists of two contributions: the original residual from the momentum equations plus the gradient operator $\underline{\mathbf{D}}^T$ acting on the local (discontinuous) pressure solutions; see (8). A natural way to motivate this approach is to consider (3) with $\underline{\mathbf{f}}$ playing the role of the original momentum residual, and the pressure $\underline{p}$ playing the role of the local pressure solutions.

An important observation here is the fact that we have abandoned the framework of carrying along the "full" Stokes operator when going from the original, global operator to the operators associated with the local subproblems. Here, a non-overlapping method is used in order to obtain the local pressure solutions, while an overlapping method is used in order to compute the local velocities.

Method H2: A hybrid Schwarz/substructuring algorithm

The next variant of the hybrid Schwarz/substructuring preconditioner can be expressed as

$$\underline{\mathbf{Q}}^{-1} = \underline{\mathbf{Q}}_0^{-1} + \underline{\mathbf{Q}}_v^{-1} + \sum_{k=1}^{K} \underline{\mathbf{Q}}_{k,p}^{-1} \tag{9}$$

where

$$\underline{\mathbf{Q}}_v^{-1} = \underline{\mathbf{A}}^{-1} \left[\underline{\mathbf{I}} + \underline{\mathbf{D}}^T \left(\sum_{k=1}^{K} \underline{\mathbf{Q}}_{k,p}^{-1} \right) \right] . \tag{10}$$

In this case, we abandon the process of computing local velocity solutions associated with the subdomains. Instead, we imagine a process in which we extend each overlapping subdomain to cover the entire domain. One way to look at this variant is as an extreme case of an overlapping method. By comparing Method H1 and H2, we can compare the case of minimum and "maximum" overlap. Any reasonably constructed variant between these two cases could be expected to have a performance (in terms of conditioning) which is between the performance of Method H1 and Method H2.

Similar to the Uzawa method, any good preconditioner for the viscous operator (in this case, the vector Laplacian) can be used to invert $\underline{\mathbf{A}}$.

Numerical Results

In this section, we compare the convergence behavior for the different Stokes solvers by solving the driven cavity problem. The computational domain Ω is the unit square in two dimensions, and the unit cube in three dimensions. On one of the faces, a unit velocity is imposed. On all the other faces, homogeneous velocity boundary conditions are imposed. The extension from homogeneous velocity boundary conditions to non-homogeneous boundary conditions is straight-forward; the previous discussion does not change. All the results were performed in double precision on an SGI Indigo 2 workstation.

For the Uzawa algorithm, a nested conjugate gradient iteration is used. The numerical results reported below indicate the number of *outer* CG iterations required in order to reduce the initial residual in (2) with five orders of magnitude.

For the three domain decomposition solvers, the numerical results indicate the number of GCR iterations necessary in order to reduce the initial residual with five orders of magnitude. For these solvers, the associated coarse grid is based upon $Q_2 - P_1$ finite elements.

In Table 1 and Table 2, we report the two-dimensional results. The total number of degrees-of-freedom, $\mathcal{N}_{d.o.f.}$, is also indicated together with the total number of degrees-of-freedom on the coarse grid, $\mathcal{M}_{d.o.f.}$. Table 1 shows how the number of iterations varies when varying the order N of each spectral element. Here, $K = 4 \times 4 = 16$ equal spectral elements (or subdomains) are used. The results in Table 1 are also plotted in Figure 1.

The results in Table 1 show that the convergence rate for Method U and Method H2 depends very weakly upon the order of each spectral element, in particular for higher values of N. The results are consistent with the results in [MMPR93] for the Uzawa algorithm, where it is found that the inf-sup parameter depends only weakly upon N in two dimensions. Thus, these results indicate that Method H2 also has a very weak dependence upon N.

The results for Method H1 are similar to Method S for $N \leq 10$. However, for $N > 10$, the number of iterations for Method S appears to grow linearly with N, while the growth rate is sublinear for Method H1. Note that Method H1 corresponds to a minimum overlap for the velocity degrees-of-freedom, and that the ratio $H/\delta \sim \mathcal{O}(N^2)$ as $N \Rightarrow \infty$. Hence, based upon the estimate (4), one might expect the iteration count to grow linearly with N for Method H1. It appears that this estimate is somewhat pessimistic, at least for $N \leq 20$.

The results in Table 2 are obtained by keeping the order of the elements fixed to $N = 10$, while using 2 different decompositions, $K = 4 \times 4 = 16$ and $K = 8 \times 8 = 64$. As expected, the convergence rate does not depend upon the number of subdomains, K.

In Table 3, we report the corresponding three-dimensional results. The decomposition we have used corresponds to $K = 4 \times 4 \times 4 = 64$. These results are also plotted in Figure 2. We do not yet have results for Method H1 in R^3. Because of the rapidly increasing problem size with N in R^3, we have not been able to use very high values of N. Hence, we have probably not seen the asymptotic behavior of the methods in R^3; see [MMPR93]. However, even these limited results indicate that the number of iterations does not grow faster than linearly with the polynomial degree N for any of the methods (at least for $N \leq 8$).

TABLE 1

N	U	H2	H1	S	$\mathcal{N}_{d.o.f.}$	$\mathcal{M}_{d.o.f.}$
4	10	14	23	22	593	145
6	11	18	28	27	1,457	145
8	11	20	33	32	2,705	145
10	11	22	36	38	4,337	145
12	12	23	38	43	6,353	145
14	12	23	40	50	8,753	145
16	13	24	44	57	11,537	145
18	14	24	49	61	14,705	145
20	14	25	53	69	18,257	145

TABLE 2

K	U	H2	H1	S	$\mathcal{N}_{d.o.f.}$	$\mathcal{M}_{d.o.f.}$
16	11	22	36	38	4,337	145
64	11	22	35	40	17,665	641

TABLE 3

N	U	H2	S	$\mathcal{N}_{d.o.f.}$	$\mathcal{M}_{d.o.f.}$
3	15	17	26	4,504	1,284
4	15	22	33	11,852	1,284
5	16	25	39	24,672	1,284
6	18	29	46	44,500	1,284
7	19	32	52	72,872	1,284
8	21	35	60	111,324	1,284

Discussion

We now discuss the computational complexity for the Stokes solvers. From the numerical results for the driven cavity problem, we notice that the minimum number

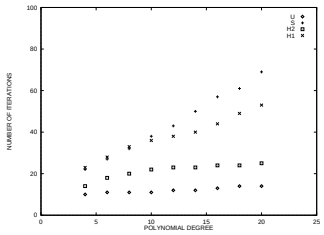

Figure 1 Two-dimensional convergence results

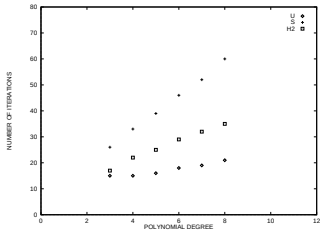

Figure 2 Three-dimensional convergence results

of iterations is achieved for the Uzawa algorithm, both in two and three dimensions. On the other hand, each CG pressure iteration in the Uzawa algorithm involves the inversion of the multi-dimensional viscous operator, as well as global matrix-vector products involving the gradient and divergence operator. The dominant work per iteration is associated with solving the viscous system; the work associated with preconditioning the Uzawa pressure operator is small.

For all the domain decomposition methods, each GCR iteration involves two matrix-vector products with the full Stokes operator. Note that, in contrast to the Uzawa algorithm, the viscous operator here enters as part of the matrix-vector *product*, and not as an operator that needs to be *inverted*. The preconditioning step involves the solution of a coarse Stokes system (effected by a back substitution), as well as the solution of local Uzawa pressure operators (effected by a set of K back substitutions). In addition, for Method S and Method H1, a set of K local viscous systems are solved (effected by a set of $d \cdot K$ back substitutions). For Method S, this corresponds to local, interior solves, while for Method H1, a minimum overlap is used. For Method H2, however, a global viscous system is solved in each preconditioning step.

The number of iterations for Method H2 is almost twice the number of iterations for Method U for larger values of N. Hence, the cost of Method H2 is about 4 times the cost of Method U. Both methods can be regarded as examples of nested iterative methods, with Method U representing a nested CG/CG iteration, and Method H2 representing a nested GCR/CG iteration.

The above comparison of Method U and Method H2 definitely favors the Uzawa

method for a standard Stokes problem. However, there are several reasons why Method H2 is interesting. First of all, the results for Method H2 (as well as for Method H1) demonstrates that it is, in fact, possible to separate the treatment of the pressure and the velocity when constructing the Stokes preconditioner. This observation gives rise to an added flexibility.

Second, Method H2 demonstrates that close to optimal convergence rates may be achieved using high-order spectral elements. Even though no theory yet exists for this method, it is doubtful that the convergence rate can be expected to be better than the convergence rate for Method U (i.e., the dependence on the inf-sup parameter may be the limiting factor).

If the number of inner CG iterations in the Uzawa algorithm is higher than about ten, Method S and Method H1 will have the lowest computational cost of all the Stokes solvers tested here; this will, indeed, be the typical case. The computational cost associated with Method S and Method H1 is roughly the same.

In the context of using low-order finite element methods, the results presented in this paper, combined with those in [KPar], offer strong evidence that low-order methods, with a separate velocity and pressure treatment, can yield excellent convergence rates. More numerical results are necessary in order to determine whether a separate treatment of the velocity and the pressure (i.e., an overlapping approach for the velocity and a non-overlapping method for the pressure) is better than associating the "full" Stokes operator on each extended subdomain; see [KPar, KP98].

Finally, we mention that for all of the domain decomposition methods we have considered here, the coarse grid is *essential* in order to obtain the correct solution. This is due to the fact that the local pressure solutions are based upon a *non-overlapping* approach, and a zero mean is enforced in each subdomain. Thus, only the coarse system will provide the correct pressure levels in each subdomain.

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