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The FETI Method for Mortar Finite Elements

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INTRODUCTION

The Finite Element Tearing and Interconnecting (FETI) method is a Lagrange multiplier based iterative substructuring method. It was introduced by Farhat and Roux [FR91]; a detailed presentation is given in [FR94], a monograph by the same authors. Originally used to solve second order, self-adjoint elliptic equations, it has later been extended to many other problems, e.g. time-dependent problems, cf. Farhat, Chen, and Mandel [FCM95], plate bending problems, cf. Farhat et al [FCR98, FM98, FMT96], and heterogeneous elasticity problems with composite materials, cf. Farhat and Rixen [RF97, RF99].

In this paper, we present a numerical study of the FETI method for two dimensional self-adjoint elliptic equations, when the underlying finite elements defined on the subdomains of Ω are low order mortar finite elements. We use geometrically nonconforming mortar finite elements of the second generation, for which no continuity conditions are imposed at the vertices.

We have tested three different preconditioners: the Dirichlet preconditioner, which has been used successfully for conforming finite elements (see Farhat, Mandel, and Roux [FMR94]), a block diagonal preconditioner used by Lacour [Lac97a], and

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a new preconditioner of Klawonn and Widlund [KW99]. The last preconditioner performs best, and has scalability properties similar to those of the classical FETI algorithm with the Dirichlet preconditioner in the case of a conforming finite element discretization.

For other work on preconditioners for mortar finite element discretizations, see Achdou, Kuznetsov, and Pironneau [AKP95], Kuznetsov [Kuz95], Achdou, Maday, and Widlund [AMW99], and the references therein.

The rest of the paper is structured as follows. In the next section, we describe the mortar finite element method. In section 3, we present the classical FETI method, and in section 4, our new FETI algorithm for mortars together with the preconditioner introduced in Lacour [Lac97a]. In the last section, we present numerical results for all the different preconditioners.

MORTAR FINITE ELEMENTS

The mortar finite element methods were first introduced by Bernardi, Maday, and Patera in [BMP94], and a three dimensional version was developed by Ben Belgacem and Maday in [BBM97]. They are nonconforming finite elements that allow for a nonconforming decomposition of the computational domain and for the optimal coupling of different variational approximations in different subregions. It has been shown that the global error is bounded by the sum of the local best approximation errors on each subregion. We are working with geometrically nonconforming mortars, i.e. we do not require that the intersection of the boundaries of two different subregions is empty, a vertex, or an entire edge.

Using mortars instead of conforming finite elements has some significant advantages. The mesh generation is more flexible and can be made quite simple on the individual subregions. It is also possible to move different parts of the mesh relative to each other, which is useful for time dependent problems and in design optimization. The mortar methods also allow for local refinement of finite element models in only certain subdomains of the computational domain, and they are well suited for parallel computing.

Let us briefly describe the mortar finite element space V^h, restricting our discussion to the two dimensional case. The computational domain Ω is decomposed into a nonoverlapping polygonal partition $\{\Omega_i\}_{i=\overline{1:N}}$. The restriction of the mortar space to any subregion Ω_i is a conforming P_1 or Q_1 finite element space. Across the interface Γ, i.e. the set of points that belong to the boundaries of at least two subregions, pointwise continuity is not required. We partition Γ into a union of nonoverlapping edges of the subregions $\{\Omega_i\}_{i=\overline{1:N}}$, called nonmortars. The edges not chosen to be nonmortars, are called mortars. Note that the mortars also form a partition of the interface. On the two sides of an edge which coincides with a nonmortar, there are two distinct traces of the mortar functions. We only require that the difference of these two traces is L^2−orthogonal to a space of test functions.

More formally, if γ is a nonmortar side, let $V^h(\gamma)$ be the continuous piecewise polynomial space which is the restriction of V^h to γ. Let $\tilde{\gamma}$ be the union of the parts of the mortars opposite γ. Then $w_h \in V^h$ is a mortar function if its restrictions, $w_\gamma = w_{|_\gamma}$ and $w_{\tilde{\gamma}} = w_{|_{\tilde{\gamma}}}$, satisfy the following L^2-orthogonality condition for every nonmortar side γ:

$$\int_\gamma (w_\gamma - w_{\bar\gamma})\psi ds = 0, \quad \forall \psi \in \Psi^h(\gamma). \tag{1}$$

Here, $\Psi^h(\gamma)$ is the space of the test functions consisting of piecewise linear functions from $V^h(\gamma)$ which are constant in the first and last mesh intervals.

THE FETI METHOD

In this section, we review the original FETI method of Farhat and Roux for elliptic problems discretized by conforming finite elements. We consider P_1 or Q_1 finite elements with a typical mesh size h.

We first partition the finite element mesh along mesh lines into N non–overlapping subdomains $\Omega_i \subset \Omega, i = 1, \dots, N$, such that the subdomain boundary nodes match across the interface. For all subdomains $\Omega_i, i = 1, \dots, N$, we construct local stiffness matrices K_i and local load vectors f_i. Denote by K the block–diagonal stiffness matrix with the K_i on the diagonal and by f the vector $[f_1, \dots, f_N]$. Analogously, we denote by u_i the vector of nodal values on Ω_i and by u the vector $[u_1, \dots, u_N]$.

Let $B = [B_1, B_2, \dots, B_N]$ be a matrix which measures the jump of a given vector $u = [u_1, \dots, u_N]$; $Bu = 0$ means that the values of the degrees of freedom, at nodes which belong to at least two subdomains, coincide.

Let us consider the following minimization problem with constraints:

$$J(u) := \frac{1}{2} u^t K u - f^t u \to \min \quad \text{subject to } Bu = 0. \tag{2}$$

By introducing Lagrange multipliers λ for the constraint $Bu = 0$, we obtain the saddle point problem

$$\begin{array}{rcrcl} Ku & + & B^t\lambda & = & f, \\ Bu & & & = & 0. \end{array} \tag{3}$$

Let R be a given matrix that spans the nullspace of K, i.e. range$R$ = ker$K$. The solution of the first equation in (3) exists if and only if $f - B^t\lambda \in$ range$K$. Then, we have

$$u = K^\dagger(f - B^t\lambda) + R\alpha,$$

where $K^\dagger$ is the pseudoinverse of K which provides a solution orthogonal to the nullspace of K and α has to be determined.

To formulate the FETI method, we need the following notation:

$$G := BR, F := BK^\dagger B^t,\ d := BK^\dagger f,\ P := I - G(G^tG)^{-1}G^t,\ e := R^t f.$$

Note that P is an l_2–orthogonal projector onto kerG^t =: V. Elimination of the primal variables u in (3) gives

$$BK^\dagger B^t\lambda = BK^\dagger f + BR\alpha, \tag{4}$$

which leads to

$$\begin{array}{rcl} PF\lambda & = & Pd, \\ G^t\lambda & = & e. \end{array}$$

The FETI algorithm is the solution of this dual problem with a preconditioned projected conjugate gradient (PCG) method, where all the increments $\lambda_k - \lambda_0$ are in V.

Given an initial approximation λ_0 with $G^t \lambda_0 = e$, we have to solve

$$PF\lambda = Pd, \quad \lambda \in \lambda_0 + V. \tag{5}$$

One possible preconditioner is of the form

$$M := B \begin{bmatrix} O & O \\ O & S \end{bmatrix} B^t, \tag{6}$$

where S is the Schur complement of K obtained by eliminating the interior degrees of freedom. In the application of M, at each iteration step, N independent Dirichlet problems have to be solved; M is known as the Dirichlet preconditioner. Note that the Schur complement never has to be computed explicitly, since only the action of S on a vector is needed.

It has been shown by Mandel and Tezaur [MT96] that the condition number κ of $PMPF$ satisfies

$$\kappa(PMPF) \leq C \left(1 + \log \frac{H}{h}\right)^3,$$

where C is a positive constant independent of h, H.

This result is similar to estimates for other non–overlapping domain decomposition methods; see Dryja, Smith, and Widlund [DSW94] for iterative substructuring methods, Dryja and Widlund [DW95] for Neumann-Neumann algorithms, and Mandel [Man93] and Mandel and Brezina [MB96] for balancing algorithms.

THE FETI METHOD FOR MORTARS

The FETI algorithm can also be applied when mortar finite elements are considered on Ω. The price we have to pay for the inherent flexibility of the mortar finite elements is related to the fact that the Lagrange multiplier matrix B is more complicated in this case compared to that arising in the classical FETI method with conforming finite elements. This is due to the fact that we no longer have matching nodes across the interface.

Let us briefly describe the construction of the matrix B. From the mortar conditions (1), we see that the interior nodes of the nonmortar sides are not associated with genuine degrees of freedom in the finite element space V^h. Let w be a mortar finite element function and γ a nonmortar. Let w_γ^1 be the vector constructed from the values of w at the interior nodes of γ, and w_γ^2 be the vector constructed from the values of w at all the nodes on the edges on the interface opposite to γ. We assume that the intersection of γ and the support of the corresponding nodal basis functions is not empty. Then w_γ^1 is uniquely determined by w_γ^2, and the mortar conditions (1) for γ can be written in matrix form as

$$M_\gamma w_\gamma^1 - N_\gamma w_\gamma^2 \;=\; 0,$$

where M_γ is a tridiagonal matrix and N_γ is a banded matrix. The matrix B will have one block, B_γ, for each nonmortar side, which consists of the columns of the corresponding matrices M_γ and N_γ and zeros in the other columns. The matrix K is again a block–diagonal matrix $diag_{i=1}^N K_i$, where the local stiffness matrices $K_i, i = 1, \ldots, N$, are obtained from the finite element discretizations on each subdomain Ω_i. As in the case of conforming finite elements, we now have to solve the problem

$$\begin{array}{lclcl} Ku & + & B^t\lambda & = & f, \\ Bu & & & = & 0. \end{array} \tag{7}$$

Note that, in contrast to the conforming finite element case, the K_i can now be build from different discretizations on each subdomain Ω_i. For a more detailed discussion of mortar finite elements with Lagrange multipliers, see Ben Belgacem [BB94] or Braess, Dahmen, and Wieners [BDW98]. To obtain a dual problem as in (5), we can now proceed completely analogously to sect. 14. Due to space limitations, we will not repeat these steps here.

For the dual problem (5) obtained from the mortar system (7), we use the preconditioner

$$\widehat{M} \;:=\; (BB^t)^{-1} B \begin{bmatrix} O & O \\ O & S \end{bmatrix} B^t (BB^t)^{-1}, \tag{8}$$

introduced by Klawonn and Widlund in [KW99]. It will also be shown in [KW99] that $\widehat{M}$ is an almost optimal preconditioner in the conforming finite element case, in the sense that

$$\kappa(P\widehat{M}PF) \;\leq\; C\left(1 + \log\frac{H}{h}\right)^2.$$

Another preconditioner $\overline{M}$ has been suggested by Lacour in [Lac97a, Lac97b]. It can be obtained from $\widehat{M}$ by taking only the block diagonal part of BB^t, instead of the whole matrix BB^t, i.e.

$$\overline{M} := (diag B_\gamma B_\gamma^t)^{-1} B \begin{bmatrix} O & O \\ O & S \end{bmatrix} B^t (diag B_\gamma B_\gamma^t)^{-1}, \tag{9}$$

where $diag B_\gamma B_\gamma^t$ has an entry on the block–diagonal for each nonmortar γ.

NUMERICAL RESULTS

In this section, we first present computational results for the preconditioners discussed in the previous sections in the case of mortar finite elements. As a model problem we have considered the Poisson equation on the unit square $\Omega = [0,1]^2$ with zero Dirichlet boundary conditions.

In the geometrically nonconforming case, cf. sect. 14, the domain Ω has been partitioned into $N \in \{16, 32, 64, 128\}$ rectangles of average diameter H, cf. Figure 14.

We use Q_1 elements of mesh size h. The local meshes on the subdomains do not match across the interface and the mortar conditions are enforced by using Lagrange

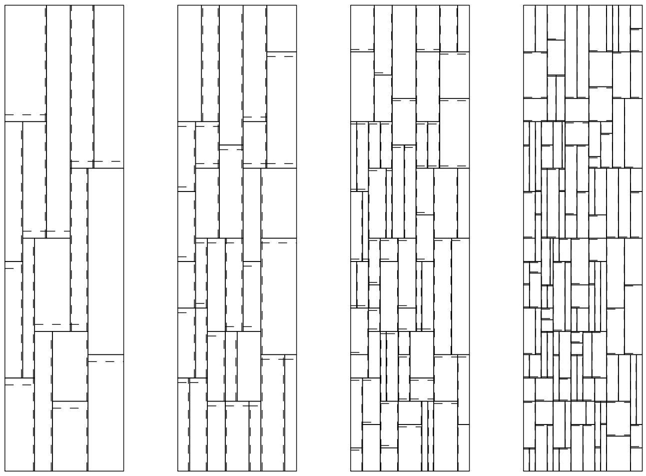

Figure 1 Geometrically non-conforming decompositions of the unit square into $16, 32, 64$, and 128 subdomains.

Table 1 Geometrically non–conforming domain decomposition; Non-matching grids across the interface.

Number of Subdomains	H/h	Precond $\widehat{M}$	Precond $\overline{M}$	Precond M
16/32/64/128	4	11/12/15/16	20/24/32/33	108/233/487/1107
16/32/64/128	8	13/14/16/18	22/25/33/37	290/438/1071/1413
16/32/64/128	16	14/15/18/20	23/27/35/40	406/620/1725/1761
16/32/64/128	32	15/16/20/22	24/27/39/42	486/692/2130/–

multipliers. The stopping criterion used in the PCG routine is the relative reduction of the initial dual residual by 10^{-6}.

In Table 1, we report the iteration counts for the new preconditioner $\widehat{M}$, cf. (8), the preconditioner $\overline{M}$, cf. (9), and the Dirichlet preconditioner M, cf. (6).

As a comparison, we also present iteration counts for the geometrically conforming case using the preconditioners $\widehat{M}$ and M, cf. Table 2. Here, Ω is partitioned in a conforming fashion into 16, 36, 64, and 121 squares.

Table 2 Geometrically conforming domain decomposition; Matching grids across the interface.

Number Subdomains	H/h	Precond $\widehat{M}$	Precond M
16/36/64/121	4	7/9/9/9	18/23/25/25
16/36/64/121	8	9/10/10/10	19/24/25/25
16/36/64/121	16	10/11/11/12	20/26/27/27
16/36/64/121	32	11/13/13/13	21/28/28/28

Note that in order to ensure matching grids across the interface, in the geometrically conforming case the number of nodes per edge of subdomain (i.e. H/h) is the same for every edge, while in the nonconforming case it can be much more general.

In both cases, geometrically conforming and nonconforming, the new preconditioner $\widehat{M}$ yields a lower number of iterations than the other two preconditioners. In the geometrically nonconforming case, the Dirichlet preconditioner M does not seem to yield a numerically scalable method.

In general, $\widehat{M}$ needs some extra work in comparison to the other two preconditioners. In each iteration step, when we multiply a vector by the preconditioner, we have to solve two systems with the matrix BB^t. In the conforming case, BB^t is very close to twice the identity matrix, and therefore little extra work is required. In the mortar case, BB^t is an almost block diagonal matrix, with each block corresponding to a nonmortar γ, and of size equal the number of interior nodes on γ. A more detailed numerical study, which also takes the complexity of the different preconditioners into account is given in [SK98].

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