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Overlapping Methods with Perfectly Matched Layers for the Solution of the Helmholtz Equation

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INTRODUCTION

In recent years, a considerable effort has been devoted to the the study of preconditioners for scalar and vector Helmholtz equations; see, e.g., [Des91, Gha96, CCEW97, MSRKA97]. In this paper, we build a class of overlapping Schwarz preconditioners for a finite element approximation of a scalar Helmholtz problem with a first–order Sommerfeld condition, in two dimensions. Perfectly Matched Layers and two kinds of boundary conditions are employed to build the local problems.

Let $\Omega \subset R^2$ be a bounded connected polygon. We consider the following Helmholtz problem for the complex–valued function u:

$$P(u) := -\Delta u - k^2 u = f, \quad \text{in } \Omega \tag{1}$$

$$\frac{\partial u}{\partial n} + ik\, u = 0, \quad \text{on } \partial\Omega, \tag{2}$$

where the frequency k is positive and the source f has support contained in Ω.

Equations 1 and 2 can be derived from the full 3D Maxwell's equations for time-harmonic fields and first–order Silver–Muller boundary conditions, when considering waves with the electric and magnetic fields, respectively, parallel and perpendicular to the xy–plane (TM waves). Then, 1 and 2 are the equations for the z-component of the magnetic field.

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Eleventh International Conference on Domain Decomposition Methods
Editors Choi-Hong Lai, Petter E. Bjørstad, Mark Cross and Olof B. Widlund

We then approximate Equations 1 and 2, by introducing a triangulation $\mathcal{T}_h$ of Ω, made of quadrilaterals of maximum diameter h, and considering the standard finite element (FE) space $V_h \subset H^1(\Omega)$, consisting of continuous piecewise bilinear functions. Triangular linear FE spaces can also be employed; see [QV94]. The well–posedness, stability, and accuracy of the corresponding linear system are studied in [IB95]. In particular, we recall that for the stability of the linear problem, the condition $kh < 1$ must be satisfied; see [IB95]. As is well–known, a restriction on kh requires that there are enough discretization points per wavelength. The number of points per wavelength is defined as

$$ppw = 2\pi/kh.$$

SCHWARZ METHODS

In the following, we suppose that the domain Ω is a rectangle and, for simplicity, that the triangulation $\mathcal{T}_h$ is uniform. We want to build a preconditioner for Equations 1 and 2, and consider an overlapping decomposition, built in the following way:

Starting from a decomposition of the rectangle Ω into M nonoverlapping rectangles $\{\Omega_i\}$, we extend each rectangle, and obtain a family of overlapping subdomains $\{\Omega_i'\}$, such that their boundaries do not cut through the elements. Let $\Omega_{i,d}'$ be the extended part of Ω_i and let d be the the overlap,

$$d := \delta h,$$

defined as the thickness of $\Omega_{i,d}'$. Each subdomain can then be decomposed as

$$\overline{\Omega_i'} = \overline{\Omega_i} \cup \overline{\Omega_{i,d}'}.$$

For the generic subproblem on the subdomain Ω_i', we solve a Helmholtz problem with the Perfectly Matched Layers studied in [ZC96]. The Perfectly Matched Layer in Ω_i' is exactly $\Omega_{i,d}'$ and the corresponding operator is defined as

$$P_d(u) := \begin{cases} -\Delta u - k^2 u, & \text{in } \Omega_i \\ -\text{div } (\Lambda \, \mathbf{grad}\, u) - k^2 a_z u, & \text{in } \Omega_{i,d}'. \end{cases} \tag{3}$$

Here $\Lambda = \text{diag}\,\{a_x(x,y), a_y(x,y)\}$ and $a_z(x,y)$ are suitable complex functions, that ensure that incident waves on the boundary between Ω_i and $\Omega_{i,d}'$ are not reflected and that the energy of a wave traveling inside $\Omega_{i,d}'$ is partially absorbed. In the layer that is perpendicular to the x axis, for instance, they are given by

$$a_x^{-1} = a_y = a_z = 1 - i\alpha \left(\frac{y}{d}\right)^m, \tag{4}$$

where α is an absorption coefficient. We refer to [ZC96] for the exact expression of the coefficients a_x, a_y and a_z in the general case, and to [CM97] for a discussion of practical issues of Perfectly Matched Layers.

We obtain different preconditioners, by choosing different boundary conditions for the local problems. We consider Dirichlet conditions (*Algorithm 1L*) and Sommerfeld conditions (*Algorithm 2L*). We define the following linear iterations. We start with an initial vector u^0. A full iteration step is performed through M fractional steps, where $u^{n+\frac{j}{M}}$ is the solution of the following problem on the subdomain Ω_j', $j = 1, \cdots, M$:

- **Algorithm 1L** (Dirichlet + Layers)

$$\begin{cases} P_d\left(u_j^{n+\frac{j}{M}} - u^{n+\frac{j-1}{M}}\right) = f - P\left(u^{n+\frac{j-1}{M}}\right), & \text{in } \Omega_j', \\ u_j^{n+\frac{j}{M}} = u_{out}^{n+\frac{j-1}{M}}, & \text{on } \partial\Omega_j'. \end{cases} \tag{5}$$

- **Algorithm 2L** (Sommerfeld + Layers)

$$\begin{cases} P_d\left(u_j^{n+\frac{j}{M}} - u^{n+\frac{j-1}{M}}\right) = f - P\left(u^{n+\frac{j-1}{M}}\right), & \text{in } \Omega_j', \\ \frac{\partial u_j^{n+\frac{j}{M}}}{\partial n_{int}} - ik\, u_j^{n+\frac{j}{M}} = -\frac{\partial u_{out}^{n+\frac{j-1}{M}}}{\partial n_{out}} - ik\, u_{out}^{n+\frac{j-1}{M}}, & \text{on } \partial\Omega_j'. \end{cases} \tag{6}$$

Here n_{int} and n_{out} are the outward and inward normal vectors to $\partial\Omega_j'$, respectively. The function $u_{out}^{n+\frac{j-1}{M}}$ is the iterate at step $n+\frac{j-1}{M}$, defined in $\Omega \setminus \Omega_j'$.

In the definition of the fractional steps 5 and 6, we have chosen to solve the local problems in sequence and obtained multiplicative algorithms, but additive algorithms can also be considered. These basic iterations can be employed to build preconditioners to be combined with a Krylov–type accelerator. A coarse solver can also be added in a standard way, by using the FE discretization of Equations 1 and 2 on a coarse mesh $\mathcal{T}_H$, $H > h$. See [SBG96] for a general discussion of these issues. We also refer to [QV99], for other examples of iteration schemes, similar to Algorithms 1L and 2L, where the continuity of suitable traces across the boundaries of the subdomains is enforced. We note that the corresponding algorithms with no absorption ($\alpha = 0$) have already been studied: see Algorithms 1 and 2 in [CCEW97]. We also remark that Perfectly Matched Layers are only employed for the local solvers.

NUMERICAL RESULTS

In this section, we compare the performance of the two algorithms introduced in the previous section, when varying the overlap, the number of subregions and the diameter of the coarse mesh. We will only present results for multiplicative preconditioners, since we have observed that they give far better performances than the corresponding additive ones, for this particular problem.

We have taken $\Omega = (0,1)^2$ in all our experiments. In the following, n and nc denote the number of discretization nodes in each direction, for the fine and the coarse mesh, respectively, and $nsub \times nsub$ is the number of subdomains. We also define the *wavelap* as the fraction of a wavelength that is covered by the overlap:

$$wlp = \frac{\delta}{ppw};$$

see [CCEW97].

In our numerical results, we have employed GMRES acceleration and right preconditioning, with restart equal to 40, a maximum number of iterations equal to 70, and a reduction of the relative residual of the preconditioned system, by a factor of 10^{-6}. For the absorption in the Perfectly Matched Layers, we have chosen $m = 2$, in 4.

For our two methods, an optimal range of values of the absorption coefficient α can be found, which is fairly insensitive to the frequency, the number of points per wavelength, the number of subregions and the diameter of the coarse triangulation. The supporting results are not shown here. We also note that our methods is pretty robust with respect to variations of the absorption coefficient, if $\alpha > 1$ for Algorithm 1L, and $\alpha > 0.25$ for Algorithm 2L. In our experiments, we have chosen $\alpha = 2.0$ for Algorithm 1L and $\alpha = 0.75$ for Algorithm 2L.

The first set of tables shows the dependence on *ppw* (or n, equivalently), the overlap and the number of subregions, for a fixed value of the frequency and no coarse space; see Table 1 for $ppw = 10.1$ and Table 2 for $ppw = 13.5$. The tables show results for Algorithm 1L with $\alpha = 2.0$, Algorithm 2L with $\alpha = 0.75$ and Algorithm 2L with $\alpha = 0$.

Table 1 Number of GMRES iterations, versus δ (wavelap) and $nsub$; $n = 121$, $ppw = 10.1$, $k = 75$, $nc = 0$; first rows for Algorithm 1L with $\alpha = 2.0$, second rows for Algorithm 2L with $\alpha = 0.75$ and third rows for Algorithm 2L with $\alpha = 0$.

δ	1	2	3	4	5	6
wavelap	0.10	0.20	0.30	0.39	0.49	0.59
nsub = 4	>70	37	29	27	26	25
	21	20	18	18	17	17
	24	19	19	20	18	21
nsub = 5	>70	44	32	29	28	26
	25	22	21	20	19	18
	28	22	23	23	23	26
nsub = 8	>70	61	43	36	33	31
	35	31	28	27	26	23
	41	34	38	48	44	>70

As expected, without a coarse space and for a fixed δ, the number of iterations increases with the number of subregions. By comparing the results for Algorithm 2L, one can see that the increase is larger if no absorption is present. We also remark that Sommerfeld boundary conditions for the local problems ensure faster convergence; see also [CCEW97]. In this case, results for $\alpha > 0$ are somewhat better than those for $\alpha = 0$ for a few subregions, and considerably better for many subregions. In particular, some absorption ensures a steady decrease of the number of iterations when the overlap is increased.

By comparing Tables 1 and 2, one can see that, for a fixed number of subregions, a constant value of the wavelap gives comparable numbers of iterations. This shows the importance of this parameter in the analysis of overlapping methods for Helmholtz problems. This has already been pointed out in [CCEW97, MSRKA97] for other Schwarz algorithms.

Tables 3 and 4 show the results when a coarse space is added. For a fixed value of ppw and $nsub$, they show the number of iterations when varying the wavelap and the size of the coarse space, for different values of the frequency. Results are given for Algorithm 1L with $\alpha = 2.0$, Algorithm 2L with $\alpha = 0.75$ and Algorithm 2L with $\alpha = 0$.

We observe an initial deterioration of the performances when a very coarse space is added, but note a considerable improvement, when the number of coarse points per wavelength ($cppw$) is sufficiently large (greater than or equal to 4).

As for Tables 1 and 2, we remark that for Algorithm 2L, some absorption ensures better performances. We also remark that, for a fixed value of the wavelap and the number of coarse points per wavelength, the number of iterations increases with the frequency.

CONCLUDING REMARKS

We do not show any results for Algorithm 1L with $\alpha = 0$. This case was considered in [CCEW97] and it generally performs very poorly. From the numerical results, we can deduce that, in general, adding Perfectly Matched Layers to the local problems, improves the

Table 2 Number of GMRES iterations, versus δ (wavelap) and $nsub$; $n = 161$, $ppw = 13.5$, $k = 75$, $nc = 0$; first rows for Algorithm 1L with $\alpha = 2.0$, second rows for Algorithm 2L with $\alpha = 0.75$ and third rows for Algorithm 2L with $\alpha = 0$.

δ	1	2	3	4	5	6
wavelap	0.07	0.15	0.22	0.30	0.37	0.44
nsub = 4	>70	48	33	28	28	27
	26	21	20	19	19	19
	56	21	20	21	22	22
nsub = 5	>70	64	40	32	30	28
	29	24	22	22	21	20
	>70	25	24	25	27	27
nsub = 8	>70	>70	53	43	39	36
	42	33	32	31	30	29
	>70	38	37	42	65	65

performance of Schwarz methods for Helmholtz equations.

The key parameters of the algorithms are the wavelap for the one–level algorithms and the wavelap, the number of coarse points per wavelengths, and the frequency, for the two–level algorithms.

The methods developed in this paper can be easily generalized to the full three–dimensional Maxwell's equations, using the theory of PMLs developed in [ZC96] for the three–dimensional case and results are forthcoming.

ACKNOWLEDGMENTS

The author is grateful to Olof Widlund for his help and enlightening discussions of my work and to Jean David Benamou for suggesting the problem.

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Table 3 Number of GMRES iterations, versus δ (wavelap) and nc; $ppw = 20$, $nsub = 8$, $n = 121$, $k = 38$; first rows for Algorithm 1L with $\alpha = 2.0$, second rows for Algorithm 2L with $\alpha = 0.75$ and third rows for Algorithm 2L with $\alpha = 0$.

δ	1	2	3	4	5	6
wavelap	0.05	0.10	0.15	0.20	0.25	0.30
nc = 0	>70	>70	>70	58	51	49
cppw = 0.0	68	55	43	39	41	47
	>70	>70	52	47	52	>70
nc = 16	>70	>70	61	44	42	40
cppw = 2.6	>70	63	44	46	52	68
	>70	>70	51	55	>70	>70
nc = 24	30	31	28	27	25	24
cppw = 4.0	51	28	24	22	22	22
	>70	>70	27	24	25	53
nc = 32	25	25	24	23	23	21
cppw = 5.3	43	21	18	19	19	21
	>70	43	18	18	27	33

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Table 4 Number of GMRES iterations, versus δ (wavelap) and nc; $ppw = 20$, $nsub = 8$, $n = 161$, $k = 50.6$; first rows for Algorithm 1L with $\alpha = 2.0$, second rows for Algorithm 2L with $\alpha = 0.75$ and third rows for Algorithm 2L with $\alpha = 0$.

δ	1	2	3	4	5	6
wavelap	0.05	0.10	0.15	0.20	0.25	0.30
nc = 0	>70	>70	>70	58	48	45
cppw = 0.0	69	46	39	37	35	37
	>70	>70	47	43	47	>70
nc = 16	>70	>70	>70	>70	>70	>70
cppw = 2.6	>70	>70	>70	>70	>70	>70
	-	-	-	-	-	-
nc = 24	>70	63	52	41	40	44
cppw = 4.0	59	47	37	48	52	40
	>70	-	49	60	>70	>70
nc = 32	69	36	30	33	33	31
cppw = 5.3	46	30	36	30	25	24
	>70	>70	43	31	28	>70