12th International Conference on Domain Decomposition Methods
Editors: Tony Chan, Takashi Kako, Hideo Kawarada, Olivier Pironneau,

7. The mortar element method with overlapping subdomains

Yves Achdou[1] , Yvon Maday [2]

Introduction

The mortar element methods were introduced in [BMP94] for non overlapping domain decompositions in order to couple different variational approximations in different subdomains. In the finite element context, one important advantage of the mortar element methods is that it allows for using structured grids in subdomains thus fast solvers [AAH+98]. The resulting methods are nonconforming but still yield optimal approximations. The literature on the mortar element methods is growing numerous see [AMW99] and reference therein.

In this paper, we shall discuss the case of overlapping subdomains, with meshes constructed in an independent manner in each subdomain. As pointed by F. Hecht, J.L. Lions, and O. Pironneau, [LP99, HLP99] such a situation can occur if the domain of computation is a scene constructed by Constructive Solid Geometry as usual in Image Synthesis and Virtual Reality : each object of the scene is described by set operations on primitive shapes like cubes, cylinders, spheres and cones. With VRML (the language of VR), the objects may be described as unions of more elementary objects with primitive shapes, which are never intersected, so it is not possible to construct a global mesh. Each simple object must have its individual mesh. In [LP99, HLP99], many algorithms (including algorithms from control theory) for this situation are proposed, and cover more general cases than overlapping subdomains (domain with holes for example).

We also note that independent of the development of the mortar methods, overlapping domain decomposition with non matching grids has been used for finite difference discretizations in the engineering community : these methods are often referred to as the *chimera methods* see [CH90, SB87],

To our knowledge, mortar methods with overlapping subdomains have been proposed first by Y. Kuznetsov [Kuz97] who focused on iterative solvers with Lagrange multipliers. For two overlapping subdomains, the mortar method has been analyzed by X.C. Cai and M. Dryja and M. Sarkis [CDS99] in two dimensions. They have considered two subdomains, with non matching grids and piecewise linear Lagrange finite elements. In particular, they have considered the case when the overlapping parameter is 0, (two rectangular subdomains for a L shaped domain). They have also proposed iterative solvers and preconditioners for the linear systems arising from the mortar discretization.

In this paper, we generalize their method in two dimensions, with more than two subdomains. We shall see that technical difficulties arise when the boundary of two

[1]laboratoire ASCI, Université Paris Sud and INSA Rennes, 20 Av des Buttes de Coesmes, 35043 Rennes, France
[2]laboratoire ASCI, Université Paris Sud and Laboratoire d'Analyse numérique, Université Paris 6

subdomains cross each other. For simplicity, we consider the Laplace equation and we rule out the case when the overlap may vanish. For such situations, one should mix the method described in [CDS99] and the one below.

This paper contains the results of a more detailed work, see [YM00] where the proofs of the results below are given, and where iterative preconditioned solvers are discussed too.

Description of the method and numerical analysis

First definitions

In all what follows, c or C will stand for various constants, independent from the geometric parameters.

We consider a polygonal domain Ω of $\mathbb{R}^2$ and the model boundary value problem in Ω

$$\begin{array}{rcll} -\Delta u & = & f & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega. \end{array} \tag{1}$$

We consider first a family of overlapping subdomains $(\Omega_k)_{k\in\{1,\ldots,K\}}$ with polygonal shapes covering Ω:

$$\Omega = \bigcup_{k=1}^{K} \Omega_k. \tag{2}$$

We denote by H_k the diameter of Ω_k and H the maximal diameter $H = \max_{1\leq k\leq K} H_k$. We assume that there exists a constant c such that for any k, $1 \leq k \leq K$, $cH \leq H_k \leq H$. We also suppose that there exists a constant τ such that any subdomain Ω_k contains a ball of diameter greater than τH.

For any subdomain Ω_k, we denote by δ_k the minimum distance of overlap between Ω_k and $\cup_{i\neq k}\Omega_i$:

$$\delta_k = \inf_{x\in\Omega_k\setminus\cup_{i\neq k}\Omega_i} \ \inf_{y\in\cup_{i\neq k}\Omega_i\setminus\Omega_k} |x-y|.$$

We also define $\delta \equiv \min_k \delta_k$.

Assumption 1 *We assume that the intersection of two subdomains' boundaries can only be isolated points, called crosspoints. We assume that there exists a constant α, $0 < \alpha \leq \frac{\pi}{2}$ such that the angles (taken not greater than $\frac{\pi}{2}$) between two subdomains boundaries crossing each other are all greater than α. For simplicity, we assume also that a given crosspoint is neither the intersection of more than two subdomains' boundaries, nor the vertex of a subdomain.*

Assumption 2 *We assume that there exists a constant number N_1 such that, for any ball B of diameter H, $B \cap \Omega$ is covered by at most N_1 subdomains.*

This assumption yields two important consequences:

Property 1 *We denote by ω_k the union of the subdomains intersecting Ω_k, and by $\mathcal{I}_k$ the set of the integers i such that $\Omega_i \subset \omega_k$. There exists a constant $n_1(N_1)$ such that, for any k, $1 \leq k \leq K$, cardinal$(\mathcal{I}_k) \leq n_1(N_1)$.*

Figure 1: The spaces Z_k^l and $\widetilde{W}_k^l$: nodal bases

Property 2 *There exists a constant $n_2(N_1)$ such that, the number of subdomains containing a given point in Ω is bounded by $n_2(N_1)$.*

We also make the assumption

Assumption 3 *The number of crosspoints lying on $\partial\Omega_k$ is bounded by a constant N_2.*

On each subdomain Ω_k, we have a family of triangular meshes $\mathcal{T}_{k,h_k}$ whose triangles have maximal diameters h_k. The meshes are constructed in an independent manner. The mesh points on $\partial\Omega_k$ need not match with the mesh points in the overlapping subdomains. We assume that the families $(\mathcal{T}_{k,h_k})_{h_k}$ are shape regular and quasi uniform, see [Cia78]. We agree to simplify the notations by replacing $\mathcal{T}_{k,h_k}$ by $\mathcal{T}_k$.

Assumption 4 *We call $h = \max_k h_k$ and we assume that, for a given positive constant C,*

$$Ch < \delta. \tag{3}$$

Associated with the mesh $\mathcal{T}_k$, we consider the spaces Z_k and X_k of piecewise linear Lagrange finite elements

$$Z_k = \left\{ \begin{array}{l} u_k \text{ is continuous in } \overline{\Omega}_k, \\ \forall t \in \mathcal{T}_k, u_k|_t \text{ is linear} \end{array} \right\}, \quad X_k \equiv \{u_k \in Z_k, u_k = 0 \text{ on } \partial\Omega \cap \partial\Omega_k\}.$$

Each space X_k and Z_k is supplied with its usual nodal basis functions. We define $X = \Pi_{k=0}^K X_k$. The vectors $u = (u_k)_{k\in\{1\ldots K\}}$ of X are collections of functions defined in the subdomains, but no continuity constraints are imposed at the subdomains boundaries. The nodal basis of X can be found by taking the product of the nodal bases of the spaces X_k.

We denote by $(\Gamma_k^l)_{l\in\{1\ldots E_k\}}$ the edges of $\partial\Omega_k$. For an edge Γ_k^l of $\partial\Omega_k$, we denote by Z_k^l the space of functions obtained by taking the trace on Γ_k^l of the functions of Z_k, and by $\mathcal{T}_k^l$ the trace of the mesh $\mathcal{T}_k$ on Γ_k^l. The space Z_k^l is the space of piecewise linear Lagrange finite elements on $\mathcal{T}_k^l$.

The matching condition

In order to discretize (1), we need to define a subspace Y of X by imposing weak continuity constraints at the subdomains boundaries $\partial\Omega_k$, $1 \leq k \leq K$.

For an edge Γ_k^l of $\partial\Omega_k\backslash\partial\Omega$ we denote by $\widetilde{W}_k^l$ the subspace of Z_k^l of the functions whose restrictions to the extreme elements of $\mathcal{T}_k^l$ are constant. Such spaces are used as mortar spaces for the non overlapping case (see [BMP94]). Here, we will have to additionally modify them locally, near the crosspoints.

We consider an edge Γ_k^l of $\partial\Omega_k\backslash\partial\Omega$. Let $(j_i)_{i\in\{1\cdots n_k^l\}}$ be the family of the indices such that $|\Gamma_k^l\cap\Omega_{j_i}|>0$ and $j_i\neq k$. Note that, from assumption 3 , n_k^l is bounded by a constant C. For $i\in\{1\cdots n_k^l\}$, we define $\Gamma_k^{l,i}=\Gamma_k^l\cap\Omega_{j_i}$. From (2),

$$\Gamma_k^l=\bigcup_{i=1}^{n_k^l}\Gamma_k^{l,i}.$$

Call $p_k^l(x)$ the piecewise constant function defined on Γ_k^l by

$$p_k^l(x)=\sum_{i=1}^{n_k^l}1_{\Gamma_k^{l,i}}(x). \tag{4}$$

From (2) and property 2, there exists a constant C such that $1\le p_k^l\le C$.

Given W_k^l a space of test functions defined on Γ_k^l, the first possible matching condition on Γ_k^l will be of the form

$$\forall w\in W_k^l,\quad \int_{\Gamma_k^l}\frac{1}{p_k^l(x)+1}\left(u_k(x)-\frac{1}{p_k^l(x)}\sum_{i=1}^{n_k^l}1_{\Gamma_k^{l,i}}(x)u_{j_i}(x)\right)w(x)dx=0. \tag{5}$$

Basically, the space W_k^l will be a subspace of $\widetilde{W_k^l}$, and the spaces will differ essentially due to the presence of crosspoints.

There remains now to define the space W_k^l. Suppose that for $i\in\{1,\dots,n_k^l\}$, $\Gamma_k^l\cap\partial\Omega_{j_i}\neq\emptyset$. Then, from assumption 1, we know that the intersections do not take place at a vertex of $\partial\Omega_{j_i}$ and let $\Gamma_{j_i}^{l'}$ be the edge of $\partial\Omega_{j_i}$ such that $\Gamma_k^l\cap\Gamma_{j_i}^{l'}$ is a point denoted by x . If no special attention is taken for the choices of W_k^l and $W_{j_i}^{l'}$, then the matching condition on Γ_k^l and $\Gamma_{j_i}^{l'}$ will strongly couple the degrees of freedom of u_k and u_{j_i} near the crosspoint x, and there might be cases when these conditions are too restrictive *i.e.* the functions u_k and u_{j_i} must be constant even zero near x. To avoid such a situation, and also in order to get a solver with good parallel properties, we have to relax the weak continuity condition near x.

We call $(x_m)_{m\in\{1,\dots M_k^l\}}$ the nodes of $\mathcal{T}_k^l$ different from the endpoints of Γ_k^l, and $(\phi_m)_{m\in\{1,\dots M_k^l\}}$ (resp. $(\psi_m)_{m\in\{0,\dots M_k^l+1\}}$) the nodal basis functions of $\widetilde{W_k^l}$ (resp. of Z_k^l). Note that $\phi_m=\psi_m$ for $2\le m\le M_k^l-1$.

We select the nodes for which the support of the corresponding basis function of X_k does not intersect $\Gamma_{j_i}^{l'}$: we obtain the set of nodes $(x_m)_{m\in\{1,\dots m_1\}\cup\{m_2,\dots M_k^l\}}$. We call $\widetilde{\phi}_{m_1}$ the continuous function vanishing outside (x_{m_1-1},x_{m_2}), linear on (x_{m_1-1},x_{m_1}) and on (x_{m_1},x_{m_2}), and such that $\widetilde{\phi}_{m_1}(x_{m_1})=1$. Likewise, $\widetilde{\phi}_{m_2}$ is the continuous function vanishing outside (x_{m_1},x_{m_2+1}), linear on (x_{m_1},x_{m_2}) and on (x_{m_2},x_{m_2+1}), and such that $\widetilde{\phi}_{m_2}(x_{m_2})=1$. The space $W_k^{l,x}$ is defined by

$$W_k^{l,x}\equiv span(\phi_1,\dots,\phi_{m_1-1},\widetilde{\phi}_{m_1},\widetilde{\phi}_{m_2},\phi_{m_2+1},\dots\phi_{M_k^l}). \tag{6}$$

The space $W_k^{l,x}$ is displayed on Figure 2. For what follows, we also define the space

$$\begin{aligned}X_k^{l,x}&\equiv\{u\in X_k^l, u=0 \text{ at the endpoints of } \Gamma_k^l \text{ and } x_{m_1+1},\dots,x_{m_2-1}\}\\&=span(\psi_1,\dots,\psi_{m_1},\psi_{m_2},\dots\psi_{M_k^l}).\end{aligned} \tag{7}$$

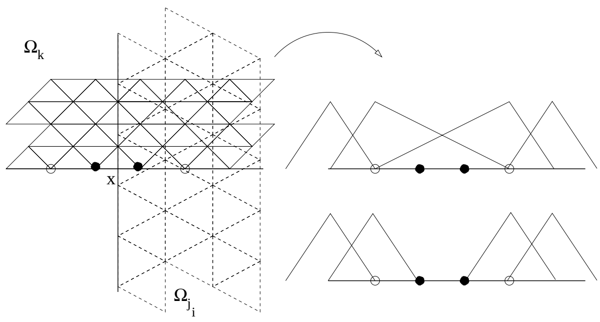

Figure 2: The spaces $W_k^{l,x}$ and $X_k^{l,x}$ (only two subdomains have been represented)

Definition 1 *For the crosspoint x, we define the zone of influence of x on Γ_k^l as the interval (x_{m_1-1}, x_{m_2+1}). We also define the zone of influence of a vertex x of Ω_k on Γ_k^l as the union of the two elements of $\mathcal{T}_k^l$ next to x. From Assumption 1, the zone of influence of a crosspoint has a size smaller than Ch.*

Assumption 5 *The zones of influence of two crosspoints on Γ_k^l are disjoint. Moreover, the zones of influence on Γ_k^l of a crosspoint and a vertex of Ω_k are disjoint.*

Finally, we define $\mathcal{X}_k^l$ the set of crosspoints on Γ_k^l and we set

$$W_k^l \equiv \bigcap_{x \in \mathcal{X}_k^l} W_k^{l,x} \tag{8}$$

and, likewise

$$X_k^l \equiv \bigcap_{x \in \mathcal{X}_k^l} X_k^{l,x}, \tag{9}$$

and Y is the subspace of X defined by

$$Y \equiv \{u \in X; \forall k \in \{1 \dots K\}, \forall l \in \{1 \dots E_k\}, u \text{ satisfies (5) } \}\,. \tag{10}$$

for W_k^l defined by (8) and (6).

Remark 1 *The functions in W_k^l will resemble those of $\widetilde{W_k^l}$ except at a few nodes near crosspoints. Furthermore, from assumption 5, these exceptional regions around crosspoints are disjoint.*

Remark 2 *The spaces W_k^l and X_k^l have the same dimension.*

Let $\mathcal{V}_k$ be the set of the nodes containing

1. the vertices of $\partial\Omega_k$.
2. all the other nodes of $\mathcal{T}_k$ on $\partial\Omega_k$ such that the support of the corresponding nodal basis function of X_k intersects another subdomain's boundary.

Lemma 1 *For a given crosspoint x on Γ_k^l, let $(x_m)_{m\in\{1,\dots m_1\}\cup\{m_2,\dots M_k^l\}}$ be the nodes of $\mathcal{T}_k^l$ involved in the above construction of $W_k^{l,x}$. Let δ^{--}, δ^-, δ^+ and δ^{++} be defined by $\delta^{--} = \frac{x_{m_1}-x_{m_1-1}}{x_{m_2}-x_{m_1}}$, $\delta^- = \frac{x_{m_1+1}-x_{m_1}}{x_{m_2}-x_{m_1}} < 1$, $\delta^+ = \frac{x_{m_2}-x_{m_2-1}}{x_{m_2}-x_{m_1}} < 1$ and $\delta^{++} = \frac{x_{m_2+1}-x_{m_2}}{x_{m_2}-x_{m_1}}$. Assume that there exists a constant c such that for all crosspoint x,*

$$\begin{array}{l} \frac{3}{2}\delta^- + \delta^{--} - (\delta^+)^2 \ge c, \\ \frac{3}{2}\delta^+ + \delta^{++} - (\delta^-)^2 \ge c, \end{array} \tag{11}$$

then there exists a constant C independent of h such that

$$\inf_{u\in W_k^l} \sup_{0\neq w\in X_k^l} \frac{\int_{\Gamma_k^l} \frac{1}{p_k^l(x)+1} u(x)w(x)dx}{\|w\|_{L^2(\Gamma_k^l)}} \ge C\|u\|_{L^2(\Gamma_k^l)}. \tag{12}$$

Let u be a function in $L^2(\Gamma_k^l)$. As a consequence of lemma 1, and if (11) is satisfied, the problem : find $u_k^l \in Z_k^l$ such that :

$$\begin{array}{l} u_k^l \text{ is given at the nodes of } \Gamma_k^l \cap \mathcal{V}_k, \\ \forall w_k^l \in W_k^l, \quad \displaystyle\int_{\Gamma_k^l} \frac{1}{p_k^l(x)+1} u_k^l(x) w_k^l(x)dx = \int_{\Gamma_k^l} \frac{1}{p_k^l(x)+1} u(x) w_k^l(x)dx \end{array} \tag{13}$$

is well posed. Furthermore, if we impose that $u_k^l = 0$ at the nodes in $\Gamma_k^l \cap \mathcal{V}_k$, then we have

$$\|u_k^l\|_{L^2(\Gamma_k^l)} \le C\|u\|_{L^2(\Gamma_k^l)}. \tag{14}$$

Likewise, let x_i be a given node in $\Gamma_k^l \cap \mathcal{V}_k$. Under the same technical assumptions, the solution of the problem: find $\widetilde{\psi}_i \in Z_k^l$ such that

$$\begin{array}{l} \widetilde{\psi}_i(x_i) = 1, \\ \widetilde{\psi}_i = 0 \text{ at the other nodes of } \Gamma_k^l \cap \mathcal{V}_k, \\ \forall w_k^l \in W_k^l, \quad \displaystyle\int_{\Gamma_k^l} \frac{1}{p_k^l(x)+1} \widetilde{\psi}_i(x) w_k^l(x)dx = 0, \end{array} \tag{15}$$

satisfies

$$\|\widetilde{\psi}_i\|_{L^2(\Gamma_k^l)} \le Ch^{\frac{1}{2}}. \tag{16}$$

The discrete problem

From now on, we shall assume that the conditions (11) are satisfied.

Let $\sigma(x) = \sum_{k=1}^K 1_{\Omega_k}(x)$. From Property 2, σ is bounded from above by a constant, and $\sigma \ge 1$. Consider the discrete problem : find $u \in Y$ such that for all $v \in Y$,

$$\sum_{k=1}^K \int_{\Omega_k} \frac{1}{\sigma} \nabla u_k \cdot \nabla v_k = \sum_{k=1}^K \int_{\Omega_k} \frac{1}{\sigma} f v_k. \tag{17}$$

Call a the symmetric bilinear form on Y :

$$a(u,v) = \sum_{k=1}^{K} \int_{\Omega_k} \frac{1}{\sigma} \nabla u_k \cdot \nabla v_k. \tag{18}$$

Now, we wish to obtain an estimate on the ellipticity constant, under typical but not necessarily optimal assumptions.

Assumption 6 *Let Ω_k be a subdomain. We assume that for a positive constant C, for each $i \neq k \in \mathcal{I}_k$, there exists an edge Γ_i^e and a sub-interval γ_i of Γ_i^e such that*

- $\gamma_i \subset \Omega_k$.
- $|\gamma_i| > CH$.
- γ_i *is the union of elements of* $\mathcal{T}_i^e$.

Lemma 2 *Under the assumptions 1 to 6 and (11), there exists a constant C_e independent on the mesh parameters such that*

$$\forall u \in Y, \quad a(u,u) \geq C_e \sum_{k=1}^{K} \int_{\Omega_k} \left(|\nabla u_k|^2 + u_k^2\right). \tag{19}$$

If only assumptions 1 to 5 and (11) are satisfied, we have (19), but we only know that there exists a constant C independent on the mesh parameters such that

$$C_e \leq C \frac{1}{\max_l (1 + \log \frac{H}{h_l})}. \tag{20}$$

Error analysis

By the Berger-Scott-Strang lemma, see [BSS72, SF73], we know that the error of the method is the sum of a consistency error plus a best approximation error: calling u^* be the weak solution of (1),

$$\|u - u^*\|_* \leq \frac{1}{C_e} \left(\inf_{v \in Y} |u^* - v|_* + \sup_{0 \neq v \in Y} \frac{|a(u^*, v) - \sum_{k=1}^{K} \int_{\Omega_k} \frac{1}{\sigma} f v_k|}{|v|_*} \right). \tag{21}$$

where C_e is the ellipticity constant. The first term in the right hand side of (21) is a best approximation error while the second one is a consistency error due to non conformity.

Lemma 3 Consistency error. *Let u^* be the weak solution of (1). Assume that $u^*|_{\Omega_k}$ belongs to $H^{\sigma_k}(\Omega_k)$, with $\sigma_k > \frac{3}{2}$. Then the consistency error is bounded by*

$$C(1 + \max_k \log \frac{H}{h_k}) \left(\sum_{k=1}^{K} \max_{i \in \mathcal{I}_k} \left(1 + \sqrt{\frac{h_i}{h_k}}\right)^2 h_k^{2(\sigma_k - 1)} |u^*|^2_{H^{\sigma_k}(\Omega_k)} \right)^{\frac{1}{2}}.$$

Lemma 4 Best approximation error. *Let $v^* \in H^1(\Omega)$ be such that for $1 \le k \le K$, $v^*|_{\Omega_k} \in H^{\sigma_k}(\Omega_k)$ with $2 \ge \sigma_k > 1$. Then there exists $v \in Y$ such that*

$$\sum_{k=1}^{K} \frac{1}{h_k} \|v_k^* - v_k\|_{L^2(\Omega_k)} + |v_k^* - v_k|_{H^1(\Omega_k)} \le C \sum_{k=1}^{K} h_k^{\sigma_k - 1} |v_k^*|_{H^{\sigma_k}(\Omega_k)}. \tag{22}$$

Then the error estimate is given by the following theorem :

Theorem 1 *Assume that the solution u^* of (1) is such that for $1 \le k \le K$, $u^*|_{\Omega_k} \in H^{\sigma_k}(\Omega_k)$ with $2 \ge \sigma_k > \frac{3}{2}$. Then there exists a constant C such that, if $u \in Y$ is the solution of (17)*

$$\begin{aligned}&\textstyle\sum_{k=1}^{K} \|u_k^* - u_k\|_{H^1(\Omega_k)} \\ &\le \frac{C}{C_e}(1 + \max_k \log \frac{H}{h_k}) \left(\sum_{k=1}^{K} \max_{i \in \mathcal{I}_k} \left(1 + \sqrt{\frac{h_i}{h_k}}\right)^2 h_k^{2(\sigma_k - 1)} |u^*|^2_{H^{\sigma_k}(\Omega_k)} \right)^{\frac{1}{2}},\end{aligned} \tag{23}$$

where C_e is the ellipticity constant.

Remark 3 *It seems possible but not easy to improve the consistency error estimate and get rid of some logarithmic factors. It will be the topic of a future research.*

A strengthened matching condition

We give below an example of stronger matching conditions in the neighborhood of crosspoints.

With the notations introduced in § 7, it is possible to strengthen the previous matching condition by supplementing the previous test function space $W_k^{l,0} \equiv W_k^l$ with Q supplementary spaces $(W_k^{l,q})_{1 \le q \le Q}$ (to be defined below) such that $dim(W_k^l) + \sum_{q=1}^{Q} dim(W_k^{l,q}) \le dim(\widetilde{W_k^l})$. Typically, each new space will correspond to a crosspoint on Γ_k^l. We define the direct sum : $\overline{W_k^l} = \bigoplus_{q=0}^{Q} W_k^{l,q}$, and we introduce a family of coefficients $\lambda_{0i} = 1$ for $1 \le i \le n_k^l$ and $\lambda_{qi} \in \{0, 1\}$ for $1 \le q \le Q$ and $1 \le i \le n_k^l$ (these coefficients will be defined below) and we call p_q the function defined on Γ_k^l by

$$p_q(x) = \sum_{i=1}^{n_k^l} \lambda_{qi} 1_{\Gamma_k^{l,i}}(x). \tag{24}$$

Then the strengthened matching condition reads

$$\forall w \in W_k^{l,0}, \quad \int_{\Gamma_k^l} \frac{1}{p_k^l(x) + 1} \left(u_k(x) - \frac{1}{p_0(x)} \sum_{i=1}^{n_k^l} 1_{\Gamma_k^{l,i}}(x) u_{j_i}(x) \right) w(x) dx = 0, \tag{25}$$

$$\forall q \in \{1, \ldots, Q\}, \ \forall w \in W_k^{l,q}, \quad \int_{\Gamma_k^l} \left(u_k(x) - \frac{1}{p_q(x)} \sum_{i=1}^{n_k^l} \lambda_{qi} 1_{\Gamma_k^{l,i}}(x) u_{j_i}(x) \right) w(x) dx = 0. \tag{26}$$

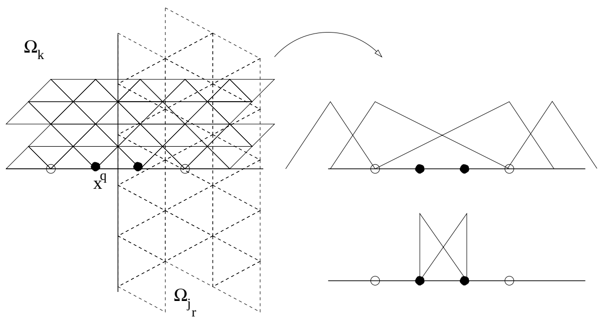

Figure 3: The spaces $W_k^{l,0}$ and $W_k^{l,q}$ (only two subdomains have been represented). In the case presented here, the dimension of $W_k^{l,q}$ is two.

Remark 4 *Conditions (25, 26) are stronger than (5), since* $W_k^{l,0} = W_k^l$.

We have to specify the spaces $W_k^{l,q}$, for $q \geq 1$. Call $(x^q)_{1 \leq q \leq Q}$ the crosspoints $x^q \in \mathcal{X}_k^l$. For a crosspoint x^q, (assume that $\{x^q\} = \Gamma_k^l \cap \Gamma_{j_r}^{l'}$), see Figure 3, we call $\{x_{m_1+1}, \ldots, x_{m_2-1}\}$ the nodes of $\mathcal{T}_k^l$ for which the support of the corresponding basis function of X_k intersects the edge $\Gamma_{j_r}^{l'}$.

We call $\widetilde{\phi}_{m_1+1}$ the piecewise linear and continuous (except at x_{m_1+1}) function, vanishing outside $[x_{m_1+1}, x_{m_1+2})$, linear on $[x_{m_1+1}, x_{m_1+2})$, and equal to 1 at x_{m_1+1} and 0 at x_{m_1+2}. Likewise, we call $\widetilde{\phi}_{m_2-1}$ the piecewise linear and continuous except at x_{m_2-1} function, vanishing outside $(x_{m_2-2}, x_{m_2-1}]$, linear on $(x_{m_2-2}, x_{m_2-1}]$, and equal to 1 at x_{m_2-1} and 0 at x_{m_2-2}.

We define $W_k^{l,q} \equiv span(\widetilde{\phi}_{m_1+1}, \phi_{m_1+2} \ldots, \phi_{m_2-2}, \widetilde{\phi}_{m_2-1})$. The spaces $W_k^{l,0}$ and $W_k^{l,q}$ are displayed on Figure 3. Note that with this choice of $W_k^{l,q}$, the supports of the functions in $W_k^{l,q}$ do not intersect the supports of the functions of X_k^{l,x_q}.

We have obviously

$$dim(\widetilde{W_k^l}) = dim(\bigoplus_{q=0}^{Q} W_k^{l,q}).$$

Now we need to define the coefficients λ_{qi}. We set $\lambda_{0i} = 1$, for all $1 \leq i \leq n_k^l$. For $k \geq 1$, assume that $\{x_q\} = \Gamma_k^l \cap \Gamma_{j_r}^{l'}$. Then we set $\lambda_{qr} = 0$ and $\lambda_{qi} = 1$, for all $1 \leq i \leq n_k^l$, $i \neq r$.

Then Y is the subspace of X defined by

$$Y \equiv \{u \in X; \forall k \in \{1 \ldots K\}, \forall l \in \{1 \ldots E_k\}, u \text{ satisfies (25), (26). } \}. \tag{27}$$

Remark 5 *Let* $u = (u_k) \in Y$. *Then it is very clear from (25), (26) that all the nodal values of* u_k *located on* $\partial\Omega_k$ *except at the vertices of* $\partial\Omega_k$ *can be found from the d.o.f. in the adjacent subdomains and from the d.o.f. located at the vertices of* $\partial\Omega_k$. *With this matching condition, all the nodal values located on* $\partial\Omega_k$ *except at the vertices of* $\partial\Omega_k$ *are slave nodal values.*

Remark 6 *Finding the slave nodal values can be achieved in two steps :*

1. *find the unknown located at the black nodes on Figure 3, by taking the test functions in the spaces $W_k^{l,q}, q > 0$. This corresponds to solving a small linear system with a mass matrix for each crosspoint on Γ_k^l.*
2. *find the remaining nodal values (located on $\Gamma_k^l \backslash \mathcal{V}_k$) by solving a problem of the type (13). We have seen above that this problem is well posed under conditions (11).*

It can be proved that Theorem 1 also holds for these strengthened matching conditions.

References

[AAH+98]Y. Achdou, G. Abdulaiev, J.C. Hontand, Y.Kuznetsov, O. Pironneau, and C.Prud'homme. Non matching grids for fluids. In J. Mandel, C Farhat, and X.C. Cai, editors, *Domain Decomposition Methods 10*, pages 377–383. AMS, 1998. Proceedings from the Tenth International Conference, June 1997, Boulder, Colorado.

[AMW99]Y. Achdou, Y. Maday, and O. Widlund. Substructuring preconditioners for the mortar method in dimension two. *SIAM J. of Numerical Analysis.*, 32(2):551–580, 1999.

[BMP94]Christine Bernardi, Yvon Maday, and Anthony T. Patera. A new non conforming approach to domain decomposition: The mortar element method. In Haim Brezis and Jacques-Louis Lions, editors, *Collège de France Seminar.* Pitman, 1994. This paper appeared as a technical report about five years earlier.

[BSS72]M. Berger, R. Scott, and G. Strang. Approximate boundary conditions in the finite element method. In *Symposia Mathematica.* x, 1972.

[CDS99]X.-C. Cai, M. Dryja, and M. V. Sarkis. Overlapping nonmatching grid mortar element methods for elliptic problems. *SIAM J. Numer. Anal.*, 36:581–606, 1999.

[CH90]G. Chesshire and W. Henshaw. Composite overlapping meshes for the solution of partial differential equations. *J. Comp. Phy*, 90:1–64, 1990.

[Cia78]Philippe G. Ciarlet. *The Finite Element Method for Elliptic Problems.* North-Holland, Amsterdam, 1978.

[HLP99]F. Hecht, J.L. Lions, and O. Pironneau. Domain decomposition algorithm for computed aided design. In A. Sequeira et al, editor, *Applied Nonlinear Analysis*, pages 185–198. Kluwer Academic-Plenum Publishers, New York, 1999.

[Kuz97]Yuri. A. Kuznetsov. Overlapping domain decomposition with non matching grids. In Petter E. Bjørstad, Magne Espedal, and David Keyes, editors, *Domain Decomposition Methods in Sciences and Engineering.* J. Wiley, 1997. Proceedings from the Ninth International Conference, June 1996, Bergen, Norway.

[LP99]J.L. Lions and O. Pironneau. Domain decomposition methods for cad. *C.R. Acad. Sci. Paris*, 328:73–80, 1999.

[SB87]J. Steger and J. Benek. On the use of composite grid schemes in computational aerodynamics. *Comp. Meth. Appl. Mech. Eng.*, 64:301–320, 1987.

[SF73]Gilbert Strang and George J. Fix. *An Analysis of the Finite Element Method.* Prentice-Hall, Englewood Cliffs, N.J., 1973.

[YM00]Y.Achdou and Y. Maday. The mortar element method with overlapping subdomains. Technical report, Universite Pierre et Marie Curie, 2000. submitted.