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8. Mesh adaptivity in the mortar finite element method

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Introduction

The mortar element method [BMP94, BMP93] becomes an important tool for mesh adaptivity in finite elements. Indeed, completely independent finite element discretizations can be used on the subdomains of a nonconforming partition of the initial domain without overlapping. This solves the contradiction between conformity and regularity and allows for working on a fully adapted mesh with a much smaller number of degrees of freedom.

In the case of the Laplace equation in a polygon Ω

$$\begin{cases} -\Delta u = f & \text{in}\,\Omega, \\ \\ u = 0 & \text{on}\,\partial\Omega. \end{cases} \tag{1}$$

the *a priori* analysis of the discrete problem obtained at each step of adaptivity is performed in [BM00], and optimal estimates are proved. The first results of *a posteriori* analysis [BOV99, Woh99] required saturation assumptions. However, as explained in [BH00], these assumptions can be avoided for some appropriate residual type error indicators, and fully optimal estimates are derived for the Laplace equation, in the sense that the error in the energy norm is equivalent to the Hilbertian sum of error indicators, up to some negligible terms related to the data. We recall these estimates and prove the efficiency of these error indicators thanks to some numerical experiments.

An outline of this paper is as follows. In the second section, we describe the discrete problem. Next we introduce the error indicators and recall from [BH00] the results of *a posteriori* analysis, which consist of a global upper bound for the error and a local upper bound for each indicator. In the third section, we describe our adaptivity algorithm and present some numerical experiments that seem in good coherency with the previous estimates.

The discrete problem and its error indicators

Let $(\mathcal{T}_h^0)_{h^0}$ be a family of "coarse" triangulations of the domain Ω, in the usual sense: each $\mathcal{T}_h^0$ is a finite set of triangles such that $\overline{\Omega}$ is the union of these triangles and the intersection of two different elements of $\mathcal{T}_h^0$, if not empty, is a vertex or a whole edge of both of them. As usual, h^0 denotes the maximal diameter of the elements of $\mathcal{T}_h^0$. We make the further assumption that this family is regular, *i.e.* there exists a positive

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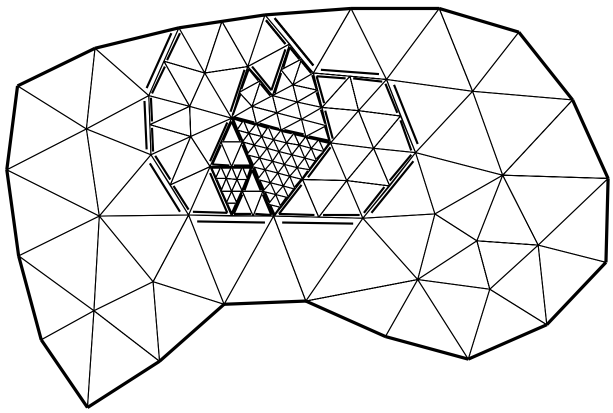

Figure 1: Example of mesh

constant σ such that, for all h^0 and for all K in $\mathcal{T}_h^0$, the ratio of the diameter of K to the diameter of its inscribed circle is smaller than σ.

Starting from this family $(\mathcal{T}_h^0)_{h^0}$, we build iteratively new families of refined triangulations as follows. Assuming that the family $(\mathcal{T}_h^{n-1})_{h^{n-1}}$ is known, for each value of the parameter h^{n-1},
• for arbitrary positive integers ℓ, we cut some elements of $\mathcal{T}_h^{n-1}$ into $2^{2\ell}$ subtriangles by iteratively joining the midpoints of the edges of these elements,
• we denote by $\mathcal{T}_h^{n,k}$ the set of these triangles which have area equal to 2^{-2k} the area of the triangle K of $\mathcal{T}_h^0$ in which they are contained, and by K^n the largest value of k such that $\mathcal{T}_h^{n,k}$ is not empty,
• we denote by $\Omega^{n,k}$ the open subdomain of Ω such that $\overline{\Omega}^{n,k}$ is the union of the triangles of $\mathcal{T}_h^{n,k}$,
• and we call $\mathcal{T}_h^n$ the union of the $\mathcal{T}_h^{n,k}$.
Figure 1 illustrates a triangulation $\mathcal{T}_h^n$ (with $K^n = 3$). The discretization parameter δ is now the pair (n, h^n), where h^n denotes the maximal diameter of the elements of $\mathcal{T}_h^n$.

Next, at each step n, we define the skeleton

$$\mathcal{S}^n = \bigcup_{k=0}^{K^n} \partial\Omega^{n,k} \setminus \partial\Omega, \tag{2}$$

and, as standard in the mortar method [BMP94], we fix a decomposition of it into disjoint (open) mortars

$$\overline{\mathcal{S}}^n = \bigcup_{m=1}^{M^n} \overline{\gamma}^m \quad \text{and} \qquad \gamma^m \cap \gamma^{m'} = \emptyset, \quad 1 \le m < m' \le M^n. \tag{3}$$

We make the final assumption that each $\overline{\gamma}^m$, $1 \le m \le M^n$, is a whole edge of a triangle of one of the triangulations $\mathcal{T}_h^{n,k}$, located on one side of γ^m, and that, on the other side, it is the union of edges of triangles in $\mathcal{T}_h^{n,k_1} \cup \dots \cup \mathcal{T}_h^{n,k_p}$, where all k_i are $> k$. We agree to denote by $k(m)$, $k_1(m), \dots, k_p(m)$, the corresponding exponents k, $k_1, \dots, k_p$, and by $p(m)$ the number p. We call $\mathcal{E}^m$ the set of connected components of $\overline{\gamma}^m \cap \partial\Omega^{n,k_i(m)}$, $1 \le i \le p(m)$. For simplicity, we assume from now on that there exists a constant λ independent of δ such that

$$\forall n, \quad \sup_{1 \le m \le M^n} \sup_{1 \le i \le p(m)} k_i(m) - k(m) \le \lambda. \tag{4}$$

We fix an integer $\ell \ge 2$ and, with each value of δ, we associate the local discrete spaces, for $0 \le k \le K^n$,

$$X^{n,k} = \big\{ v_k \in \mathcal{C}^0(\overline{\Omega}^{n,k});\ \forall K \in \mathcal{T}_h^{n,k},\ v_{k\,|K} \in \mathcal{P}_\ell(K) \big\}, \tag{5}$$

where $\mathcal{P}_\ell(K)$ stands for the space of restrictions to K of polynomials with total degree $\le \ell$.

Remark 1 *We refer to* [BH00] *for the analysis of the case of piecewise affine functions ($\ell = 1$), where some restrictions on the decomposition are needed.*

Let now γ^m, $1 \le m \le M^n$, be one of the mortars. With each e in $\mathcal{E}^m$, we associate the space $\widetilde{W}^m(e)$ of continuous functions on e such that their restrictions to each edge $e' = \overline{\gamma}^m \cap \partial K$ for all K in $\mathcal{T}_h^{n,k_i(m)}$ belongs to $\mathcal{P}_{\ell-1}(e')$ if e' contains an endpoint of e, to $\mathcal{P}_\ell(e')$ if not.

The discrete space $\mathbb{X}_\delta$ is now defined in the usual way, see [BMP93]. It is the space of functions v_δ such that

- their restrictions to each $\Omega^{n,k}$, $0 \le k \le K^n$, belong to $X^{n,k}$,
- they vanish on $\partial\Omega$,
- the following matching condition holds on any γ^m, $1 \le m \le M^n$,

$$\forall e \in \mathcal{E}^m,\ \forall \chi \in \widetilde{W}^m(e), \quad \int_e [v_\delta](\tau)\chi(\tau)\,d\tau = 0, \tag{6}$$

where $[v_\delta]$ denotes the jump of v_δ through e.

Remark 2 *As proposed in the first version of the mortar method* [BMP94], *some further matching conditions can be added, more precisely the functions in $\mathbb{X}_\delta$ can be enforced to be continuous at the endpoints of all γ^m. These conditions are satisfied in the numerical experiments of this paper, but they are not necessary for the analysis.*

For fixed data f in $L^2(\Omega)$, the discrete problem now reads:

Find u_δ in $\mathbb{X}_\delta$ such that

$$\forall v_\delta \in \mathbb{X}_\delta, \quad a_\delta(u_\delta, v_\delta) = \int_\Omega f(\mathbf{x}) v_\delta(\mathbf{x})\,d\mathbf{x}, \tag{7}$$

where the bilinear form $a_\delta(\cdot,\cdot)$ is defined by

$$a_\delta(u_\delta, v_\delta) = \sum_{k=0}^{K^n} \int_{\Omega^{n,k}} \mathbf{grad}\, u_\delta \cdot \mathbf{grad}\, v_\delta\,d\mathbf{x}. \tag{8}$$

Thanks to the matching conditions (6), it is readily checked that this problem has a unique solution. Moreover, the following *a priori* error estimate can be proved as an extension of [BM00, Thm 2.8]: if the solution u of problem (1) is such that each $u_{|\Omega^{n,k}}$, $0 \le k \le K^n$, belongs to $H^{s_k}(\Omega^{n,k})$, $s_k > 1$,

$$\|u - u_\delta\|_{H^1_\delta(\Omega)} \le c \Big(\sum_{k=0}^{K^n} (2^{-k}\, h^0)^{2(s_k-1)}\, \|u\|^2_{H^{s_k}(\Omega^{n,k})} \Big)^{\frac{1}{2}}, \tag{9}$$

where the mesh-dependent norm $\|\cdot\|_{H^1_\delta(\Omega)}$ is defined by

$$\|v\|_{H^1_\delta(\Omega)} = \Big(\sum_{k=0}^{K^n} \|v\|^2_{H^1(\Omega^{n,k})} \Big)^{\frac{1}{2}}. \tag{10}$$

Remark 3 *As explained in* [BB99]*, the matching conditions* (6) *can be enforced thanks to to the introduction of a Lagrange multiplier. In this case, problem* (7) *is equivalent to a saddle-point problem. The corresponding global matrix is symmetric, so that solving it is not expensive.*

However, we are more specifically interested with *a posteriori* estimates. As usual, we fix an approximation f_δ of the function f in the space

$$\mathbb{Z}_\delta = \big\{ g_\delta \in L^2(\Omega);\ \forall K \in \mathcal{T}_h^n,\ g_{\delta\,|K} \in \mathcal{P}_{\ell^*}(K) \big\}. \tag{11}$$

where ℓ^* is a nonnegative integer. We consider two types of indicators.

- Error indicators linked to the finite elements

For each K in $\mathcal{T}_h^n$, we denote by $\mathcal{E}_K$ the set of edges of K which are not contained in $\partial\Omega$. In what follows, h_K stands for the diameter of K and h_e for the length of any e in $\mathcal{E}_K$ (or in any $\mathcal{E}^m$).

The residual error indicator η_K associated with any triangle in $\mathcal{T}_h^n$ is now defined in a completely standard way, see [Ver96, (1.18)]:

$$\eta_K = h_K\, \|f_\delta + \Delta u_\delta\|_{L^2(K)} + \frac{1}{2} \sum_{e \in \mathcal{E}_K} h_e^{\frac{1}{2}}\, \|\, [\partial_n u_\delta]\, \|_{L^2(e)}, \tag{12}$$

where ∂_n denotes the normal derivative on e and $[\cdot]$ the jump through e. Note that the term "residual" here means that, when suppressing all the δ in the previous line, the quantity in the right-hand side is zero.

- Error indicators linked to the edges of the skeleton

Like in [BV96, (3.3)], for $1 \le m \le M^n$, we associate with each e in $\mathcal{E}^m$ the indicator η_e defined as

$$\eta_e = h_e^{-\frac{1}{2}}\, \|\, [u_\delta]\, \|_{L^2(e)}. \tag{13}$$

There also, this quantity vanishes when suppressing the δ.

Remark 4 *It is readily checked that, for all m, $1 \le m \le M^n$, and for all e in $\mathcal{E}^m$, the quantity η_e is equivalent to the norm $\|\, [u_\delta]\, \|_{H^{\frac{1}{2}}(e)}$. However it is much easier to compute.*

We sum up in the two following propositions the estimates concerning these indicators, and we refer to [BM00, §3] for their detailed proofs. The first one relies on the formula, obtained by local integration by parts,

$$\begin{aligned}\|u-u_\delta\|_{H^1_\delta(\Omega)} \le \quad & c\Big(\sum_{K\in\mathcal{T}_h^n}\Big(\|f_\delta+\Delta u_\delta\|_{L^2(K)}\,\frac{\|v-v_\delta\|_{L^2(K)}}{\|v\|_{H^1_\delta(\Omega)}}\\ &+\|f-f_\delta\|_{L^2(K)}\,\frac{\|v-v_\delta\|_{L^2(K)}}{\|v\|_{H^1_\delta(\Omega)}}\\ &-\frac{1}{2}\sum_{e\in\mathcal{E}_K}\|[\partial_n u_\delta]\|_{L^2(e)}\,\frac{\|v-v_\delta\|_{L^2(e)}}{\|v\|_{H^1_\delta(\Omega)}}\Big)\\ &+|\sum_{m=1}^{M^n}\sum_{e\in\mathcal{E}^m}\int_e \partial_n(u-u_\delta)\,[u_\delta]\,d\tau|^{\frac{1}{2}}\Big),\end{aligned} \tag{14}$$

where v is equal to $u-u_\delta$ and v_δ is any "conforming" approximation of v, *i.e.* which belongs to $\mathbb{X}_\delta\cap H^1_0(\Omega)$. The first three terms are evaluated by constructing an extension of Clément's operator to the present situation, while estimating the last one relies on conditions (6). The arguments for proving the second proposition are standard for residual indicators, see [Ver96, Chap.3].

Proposition 1 *There exists a constant c independent of δ such the following error estimate holds between the solutions u of problem* (1) *and u_δ of problem* (7)*:*

$$\|u-u_\delta\|_{H^1_\delta(\Omega)} \le c\Big(\sum_{K\in\mathcal{T}_h^n}\big(\eta_K^2+h_K^2\,\|f-f_\delta\|^2_{L^2(K)}\big)+\sum_{m=1}^{M^n}\sum_{e\in\mathcal{E}^m}\eta_e^2\Big)^{\frac{1}{2}}. \tag{15}$$

Proposition 2 *There exists a constant c' independent of δ such that the following estimate holds for all K in $\mathcal{T}_h^n$:*

$$\eta_K \le c'\,\big(\|u-u_\delta\|_{H^1_\delta(\Xi_K)}+(\sum_{K'\subset\Xi_K}h_{K'}^2\,\|f-f_\delta\|^2_{L^2(K')})^{\frac{1}{2}}\big), \tag{16}$$

where Ξ_K is the union of at most four triangles such that at least an edge of K is contained in an edge of such triangles. There exists a constant c'' independent of δ such that the following estimate holds for all m, $1\le m\le M^n$, and for all e in $\mathcal{E}^m$:

$$\eta_e \le c''\,\|u-u_\delta\|_{H^1_\delta(\Xi_e)}, \tag{17}$$

where Ξ_e is the union of the triangle of $\mathcal{T}_h^{n,k(m)}$ that intersects γ^m and a triangle K contained in an $\overline{\Omega}^{n,k_i(m)}$ such that e is an edge of K.

Remark 5 *The constants c, c' and c'' in the previous propositions only depend on the regularity parameter σ of the initial family of triangulations $(\mathcal{T}_h^0)_{h^0}$ and on the constant λ introduced in* (4)*. Their dependency with respect to λ is explicitly written in* [BH00].

Proposition 2 states a local version of the upper bounds for the error indicators. However a global version can be proven by similar arguments. When compared with (15), this global estimate proves that the error $\|u - u_\delta\|_{H^1_\delta(\Omega)}$ is equivalent to the Hilbertian sum of the indicators, up to some terms concerning the approximation of the data f. These terms are most often negligible, so that the combined two types of indicators lead to an optimal evaluation of the error.

The adaptivity algorithm and numerical experiments

Thanks to the previous choice of the discrete problem, the algorithm for mesh adaptivity is now straightforward. Assume that we are given a triangulation $\mathcal{T}_h^n$ and the corresponding skeleton $\mathcal{S}^n$. We solve the associated problem (7) and compute all the error indicators η_K and η_e, next the meanvalue $\overline{\eta}^n$ of the η_K, $K \in \mathcal{T}_h^n$, and the mean value $\overline{\eta}_*^n$ of the η_e, $e \in \mathcal{E}^m$, $1 \leq m \leq M^n$. The next triangulation $\mathcal{T}_h^{n+1,k}$ and skeleton $\mathcal{S}^{n+1}$ are then built in two steps.

- Step 1. For all K in $\mathcal{T}_h^n$, there exists an integer k such that

$$2^k\,\overline{\eta}^n \leq \eta_K \leq 2^{k+1}\,\overline{\eta}^n. \tag{18}$$

If k is positive, we cut the triangle K into 2^{2k} equal subtriangles by iteratively joining the middle of the edges. This allows for defining an intermediary skeleton $\mathcal{S}_*^n$.

- Step 2. We only consider the e in $\mathcal{E}^m$, $1 \leq m \leq M^n$, such that

$$\eta^e \geq 2\,\overline{\eta}_*^n. \tag{19}$$

If this edge remains in $\mathcal{S}_*^n$ after Step 1, we cut the triangles on both sides of e, such that e becomes "conforming", *i.e.* it is no longer contained in the next skeleton $\mathcal{S}^{n+1}$.

We stop the algorithm either after a finite number of iterations or when the following condition is satisfied for a given tolerance ε:

$$\sum_{K\in\mathcal{T}_h^n} \eta_K^2 + \sum_{m=1}^{M^n} \sum_{e\in\mathcal{E}^m} \eta_e^2 \leq \varepsilon^2. \tag{20}$$

We now present some numerical results in the case where Ω is the L-shaped domain $]-1,1[^2 \backslash [0,1]^2$ and the data f is equal to 1. We work with piecewise quadratic functions ($\ell = 2$) and an initial mesh $\mathcal{T}_h^0$ made of 64 triangles. The dimension of the corresponding space $\mathbb{X}_\delta$ is 105.

Figure 2 presents the initial mesh $\mathcal{T}_h^0$ and the first five refined meshes $\mathcal{T}_h^n$, $n = 1$, 2, 3, 4, 5, according to the previous algorithm.

Figure 3 presents the isovalue curves of the error indicators η_K, $K \in \mathcal{T}_h^n$, for each of the previous $\mathcal{T}_h^n$.

Table 1 presents for each triangulation $\mathcal{T}_h^n$ the number of triangles N_T^n, the total number of mortars M^n, the dimension $\dim \mathbb{X}_\delta$ of the corresponding space $\mathbb{X}_\delta$, the maximal value K^n of the k and finally the Hilbertian sum η^n_{norm} of all indicators η_K and η_e. It can be observed that, for a fixed initial mesh, this sum decreases at each iteration n.

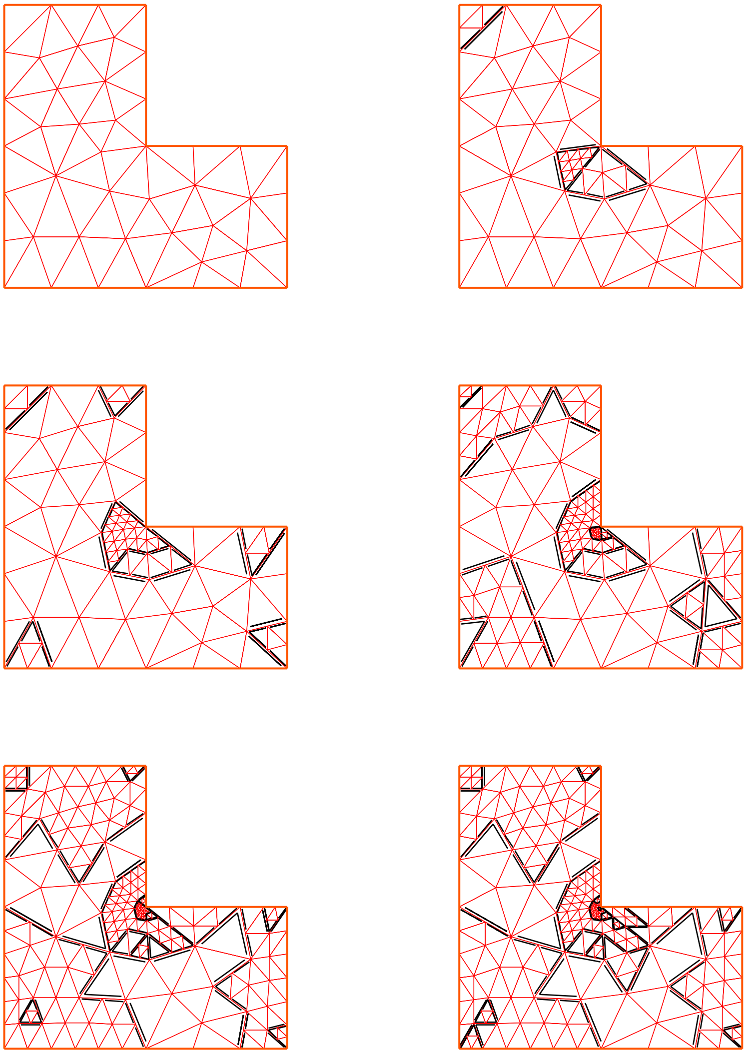

Figure 2: The sequence of adapted meshes

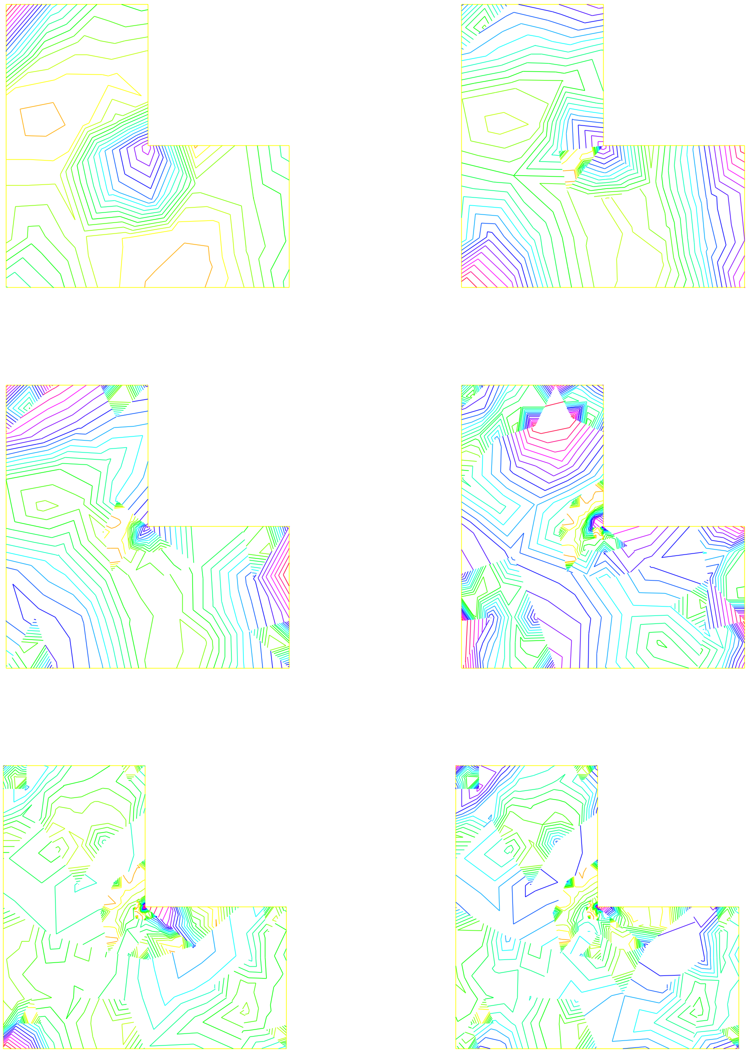

Figure 3: Isovalue curves of the indicators

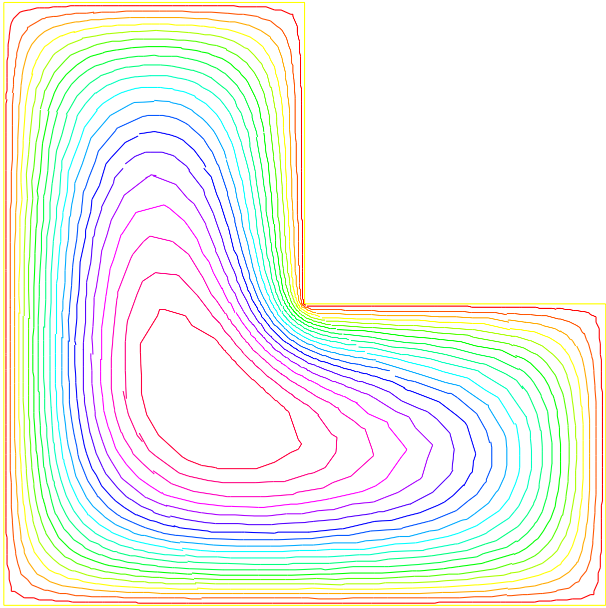

Figure 4: Isovalue curves of the solution

n	0	1	2	3	4	5
N_T^n	64	88	121	214	301	325
M^n	0	8	18	32	47	59
$\dim \mathbb{X}_\delta$	105	171	256	466	661	730
K^n	0	2	2	4	4	5
$\eta_{\rm norm}^n$	0.3438	0.2187	0.1587	0.0927	0.0677	0.0562

Table 1

The isovalue curves of the discrete solution obtained on the mesh $\mathcal{T}_h^5$ are presented in Figure 4 .

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