12th International Conference on Domain Decomposition Methods
Editors: Tony Chan, Takashi Kako, Hideo Kawarada, Olivier Pironneau,

1. Analysis of a Multigrid Algorithm for the Mortar Finite Element Method

Dietrich Braess[1]

Introduction

The *mortar method* has attracted much interest as a special *domain decomposition* method. It has been analysed in a series of papers (see e. g. [BM97, BMP94, BDW00, Woh99a]) in particular for second order elliptic boundary value problems

$$\begin{aligned} -\operatorname{div} a(x) \operatorname{grad} u(x) &= f(x) && \text{in } \Omega, \\ a(x)\frac{\partial u}{\partial n} &= g(x) && \text{on } \Gamma_N \subset \partial\Omega, \\ u &= 0 && \text{on } \Gamma_D := \partial\Omega \setminus \Gamma_N. \end{aligned} \tag{1}$$

Here $a(x)$ is a (sufficiently smooth) uniformly positive definite matrix in the bounded domain $\Omega \subset \mathbb{R}^d$, Γ_D is a subset of the boundary Γ of Ω, and $\Gamma_N := \Gamma \setminus \Gamma_D$.

Let Ω be decomposed into non-overlapping subdomains $\Omega_k, k = 1, \dots, K$,

$$\bar{\Omega} \;=\; \bigcup_{k=1}^{K} \bar{\Omega}_k, \quad \Omega_k \cap \Omega_l = \emptyset \;\text{ for } k \neq l. \tag{2}$$

Let $H^s(\Omega)$ denote the usual Sobolev spaces endowed with the Sobolev norms $\|\cdot\|_{s,\Omega}$, and $H^1_{0,D}(\Omega)$ be the closure in H^1 of all C^∞-functions vanishing on Γ_D. The natural space associated to the domain decomposition (2) is the product space

$$X_\delta := \{v \in L_2(\Omega) : v|_{\Omega_k} \in H^1(\Omega_k),\ k = 1, \dots, K,\ v|_{\Gamma_D} = 0\}, \tag{3}$$

endowed with the (broken) norm

$$\|v\|_{1,\delta} := \left(\sum_{k=1}^{K} \|v\|^2_{1,\Omega_k} \right)^{1/2}. \tag{4}$$

The space $H^1_{0,D}(\Omega)$ is determined as a subspace of X_δ by appropriate linear constraints. Corresponding discretizations lead to *saddle point problems.* In this paper we present a multigrid method for the *efficient* solution of such *indefinite* systems of equations. According to standard multigrid convergence theory the main tasks are to establish appropriate approximation properties in terms of *direct estimates* as well as to design suitable *smoothing procedures* which give rise to corresponding *inverse estimates.*

The discretization error of the mortar finite elements can be analyzed either by the theory of nonconforming elements and the lemma of Berger, Scott, and Strang

[1]Faculty of Mathematics, Ruhr-University, 44780 Bochum, Germany,
braess@num.ruhr-uni-bochum.de

(Strang's second lemma), or by the theory of saddle point problems. Up to now most investigations have used the first approach. It has the advantage that the analysis can be performed with standard Sobolev spaces.

On the other hand, the framework of mixed methods is more appropriate when the computions are performed for the saddle point formulation and fast solvers are to be developed. It seems to be necessary to use mesh-dependent norms if Brezzi's theory is applied. Ellipticity of the variational form, boundedness of the functional in the definition of the constraints, and the inf-sup condition have to be guaranted. We will follow this scheme.

We note that there is also an alternative which can be found in [BB99] and [Woh99a]. The finite element spaces for the direct variables and the Lagrange multipliers need not be balanced so strictly if the error estimates are derived in a two-stage process. First, the direct variables are treated as nonconforming elements. Having an error estimate for them, only the inf-sup condition and no ellipticity assumption is required when the error of the Lagrange multipliers are treated; cf. Remark 1.

There is a correspondence between all the approaches. Roughly speaking, the terms in the formula of the lemma of Berger, Scott, and Strang are obtained by arguments which are refound in the analysis of the mixed method and vice versa; there are, however, some tiny but very sophisticated differences. *Although we admit that the finite element functions are not continuous at the cross points, the subset of the functions without jumps at cross points is responsible for the stability of the mortar elements.*

A suitable smoothing procedure for the multigrid algorithm that is consistent with the approximation properties above is obtained by a method known from the Stokes problem. The paper concludes with a numerical example.

The Continuous Problem

For convenience, we assume that the domain $\Omega \subset \mathbb{R}^d$ and the subdomains Ω_k in (2) are polygonal. If Ω_k and Ω_l share a common interface, we set $\bar{\Gamma}_{kl} := \bar{\Omega}_k \cap \bar{\Omega}_l$. The interior faces form the *skeleton*

$$\mathcal{S} := \bigcup_{k,l} \Gamma_{kl}. \tag{5}$$

Γ_{kl}, Γ_N, and Γ_D will always be assumed to be the union of polygonal subsets of the boundaries of the Ω_k. Often such a decomposition is called *geometrically conforming.*

In order to characterize $H^1_{0,D}(\Omega)$ as a subspace of X_δ, recall that for any (sufficiently regular) manifold Γ the Sobolev spaces $H^s(\Gamma)$ can be defined by their intrinsic norms (see [LM72, Section 7.3]), or alternatively, when Γ is part of a boundary, as a *trace space.* In fact, whenever $s - 1/2$ is *not* an integer,

$$\|v\|_{s-1/2,\Gamma} := \inf_{w \in H^s(\Omega), w|_\Gamma = v} \|w\|_{s,\Omega}$$

is an equivalent norm for $H^{s-1/2}(\partial\Omega)$. Moreover, if Γ' is a smooth subset of Γ, $H^s_{00}(\Gamma')$ consists of those elements $v \in H^s(\Gamma')$ whose extension $\tilde{v}$ of v by zero to all of Γ belongs to $H^s(\Gamma)$, cf. [LM72, p. 66], in particular,

$$\begin{aligned} H_{00}^{1/2}(\Gamma_{kl}) &= \{v \in H^{1/2}(\Gamma_{kl}) : \tilde{v} \in H^{1/2}(\partial\Omega_k),\ \tilde{v}|_{\Gamma_{kl}} = v;\ \tilde{v} = 0 \text{ on } \partial\Omega_k \backslash \Gamma_{kl}\}, \\ \|v\|_{H_{00}^{1/2}(\Gamma_{kl})} &:= \|\tilde{v}\|_{1/2,\partial\Omega_k}. \end{aligned} \tag{6}$$

We note that $H_{00}^{1/2}(\Gamma_{kl})$ is an *interpolation space* between $L_2(\Gamma_{kl})$ and $H_0^1(\Gamma_{kl})$

$$H_{00}^{1/2}(\Gamma_{kl}) = [H_0^1(\Gamma_{kl}), L_2(\Gamma_{kl})]_{1/2},$$

while

$$H^{1/2}(\Gamma_{kl}) = [H^1(\Gamma_{kl}), L_2(\Gamma_{kl})]_{1/2}.$$

This can be realized, e. g., by the K-method [LM72, pp. 64–66, pp. 98–99].

It is appropriate to characterize $H_{0,D}^1(\Omega)$ as a subspace of

$$X_{00} := \{v \in X_\delta : [v] \mid_{\Gamma_{kl}} \in H_{00}^{1/2}(\Gamma_{kl})\ \forall\, \Gamma_{kl} \subset \mathcal{S}\}, \tag{7}$$

endowed with the norm

$$\|v\|_X^2 := \sum_k \|v\|_{1,\Omega_k}^2 + \sum_{\Gamma_{kl} \subset \mathcal{S}} \|[v]\|^2_{H_{00}^{1/2}(\Gamma_{kl})}. \tag{8}$$

The trace terms in (8) arise from the fact that X_{00} is a proper subspace of X_δ, and they motivate our later treatment of the finite element discretization. Specifically we have

$$H_{0,D}^1(\Omega) = \{v \in X_{00} : (\mu, [v])_{0,\Gamma_{kl}} = 0 \quad \forall \mu \in H_{00}^{-1/2}(\Gamma_{kl}),\ \Gamma_{kl} \subset \mathcal{S}\}. \tag{9}$$

Here and in the sequel we write $H_{00}^{-1/2}$ and $H^{-1/2}$ for the dual of $H_{00}^{1/2}$ and $H^{1/2}$, respectively.

We now turn the problem (1) into a weak form based on the above characterization of $H_{0,D}^1(\Omega)$. Let

$$a(u,v) := \sum_k \int_{\Omega_k} (a(x)\nabla u(x)) \cdot \nabla v(x) dx, \tag{10}$$

$$b(v,\mu) := \sum_{\Gamma_{kl} \subset \mathcal{S}} (\mu, [v])_{0,\Gamma_{kl}}. \tag{11}$$

Setting

$$M := \prod_{\Gamma_{kl} \subset \mathcal{S}} H_{00}^{-1/2}(\Gamma_{kl}),$$

we consider the variational problem: *find* $(u, \lambda) \in X_{00} \times M$ *such that*

$$\begin{aligned} a(u,v) + b(v,\lambda) &= (f,v)_{0,\Omega} + (g,v)_{0,\Gamma_N}, & v \in X_{00}, \\ b(u,\mu) &= 0, & \mu \in M. \end{aligned} \tag{12}$$

From the definition of the trace spaces it follows that the operator $B : X_{00} \to \prod_{\Gamma_{kl}} H_{00}^{1/2}(\Gamma_{kl})$, $v \mapsto Bv$ defined by $(Bv, \mu)_{0,\mathcal{S}} = \sum_{\Gamma_{kl} \subset \mathcal{S}} (\mu, [v])_{0,\Gamma_{kl}}$ for any $\mu \in M$, is bounded.

Moreover, the saddle point problem (12) satisfies the *inf-sup condition.* A straight forward proof can be found in [BDW00]. The crucial point is that the jump on Γ_{kl} belongs to $H_{00}^{1/2}(\Gamma_{kl})$, and it can be extended without interference to other parts of the skeleton.

Furthermore, we know from (9) that $H^1_{0,D}(\Omega) = V := \ker B$. Since $\|v\|_X = \|v\|_{1,\Omega}$ for $v \in H^1_{0,D}$, the bilinear form $a(\cdot,\cdot)$ is V-elliptic, i.e., elliptic on the kernel of B.

The discrete problem

In the discussion of the finite element discretization of (12), we will restrict ourselves to the bivariate case, $d = 2$. For each subdomain Ω_k we choose a family of (conforming) triangulations $\mathcal{T}_{k,h}$ independently of the neighboring subdomains; i.e., the nodes in $\mathcal{T}_{k,h}$ that belong to Γ_{kl} need not match the nodes of $\mathcal{T}_{l,h}$. The corresponding spaces of piecewise linear finite elements on $\mathcal{T}_{k,h}$ are denoted by $S_h(\mathcal{T}_{k,h})$. Following [BB99, BM97, BMP94] we set

$$X_h := X_\delta \cap \prod_{k=1}^{K} S_h(\mathcal{T}_{k,h}), \tag{13}$$

i.e., the functions in X_h are not required to be continuous at the cross-points of the polygonal subdomains Ω_k and $X_h \not\subset X_{00}$. We associate with each interface Γ_{kl} the *nonmortar side* which, by the usual convention, is Ω_k while Ω_l is the *mortar side.* Let $M_{kl,h}$ be the space of all continuous piecewise linear functions on Γ_{kl} on that partition induced by the triangulation $\mathcal{T}_{k,h}$ on the nonmortar side, under the additional constraint that the elements in $M_{kl,h}$ are *constant* on the two intervals containing the end points of Γ_{kl}. Thus the dimension of $M_{kl,h}$ agrees with the dimension of $\tilde{T}_{kl,h} := S_h(\mathcal{T}_{k,h}) \cap H^1_0(\Gamma_{kl}) \subseteq H^{1/2}_{00}(\Gamma_{kl})$. The space of discrete multipliers is defined as

$$M_h := \prod_{\Gamma_{kl}\subset\mathcal{S}} M_{kl,h}. \tag{14}$$

The kernel of the restriction operator is

$$V_h := \{v_h \in X_h : b(v_h,\mu_h) = 0 \ \text{ for } \ \mu_h \in M_h\}. \tag{15}$$

As already anounced in the introduction we will employ mesh-dependent norms as in [AT95, Woh99b]. Setting

$$\|w\|_{1/2,h,\Gamma_{kl}} := h^{-1/2}\|w\|_{0,\Gamma_{kl}},$$

let

$$\begin{aligned} \|v_h\|^2_{1,h} &:= \|v_h\|^2_{1,\delta} + \sum_{\Gamma_{kl}\subset\mathcal{S}} \|[v_h]\|^2_{1/2,h,\Gamma_{kl}} \\ &= \|v_h\|^2_{1,\delta} + \sum_{\Gamma_{kl}\subset\mathcal{S}} h^{-1}\|[v_h]\|^2_{0,\Gamma_{kl}}, \end{aligned} \tag{16}$$

$$\|\mu\|^2_{-1/2,h} := \sum_{\Gamma_{kl}\subset\mathcal{S}} \|\mu\|^2_{-1/2,h,\Gamma_{kl}} = \sum_{\Gamma_{kl}\subset\mathcal{S}} h\|\mu\|^2_{0,\Gamma_{kl}}. \tag{17}$$

Obviously, (16) corresponds to (8). Whenever a distinction of local mesh sizes matters, the global h in (16)–(17) has to be replaced by the mesh size h_k of the non-mortar side in the summands for Γ_{kl}. In this framework,

$$\begin{aligned} a(u_h, v_h) + b(v_h, \lambda_h) &= (f, v_h)_{0,\Omega} + (g, v_h)_{0,\Gamma_N}, & v_h \in X_h, \\ b(u_h, \mu_h) &= 0, & \mu_h \in M_h, \end{aligned} \tag{18}$$

is a stable discretization of (12).

When verifying this, one crucial point of the analysis is the proof of the inf-sup condition. This is well-known for the saddle point formulation, but the reader may wonder that we find the arguments for the inf-sup condition (often very concealed) also in the analysis by the theory of nonconforming elements. It is done for the following reason. Given $u \in H^2(\Omega_k)$, by the classical theory there is a finite element function $v_h \in X_\delta$ such that $\|u - v_h\|_{1,\delta}$ can be easily estimated. The lemma of Berger, Scott, and Strang, however, requires a good approximation by an element that satisfies the mortaring condition. Now Fortin's theory (see [BF91] or [Bra97, p. 130]) yields this property whenever the inf-sup condition holds.

There is one more point that is found in all treatments of mortar elements which we know. Although the analysis in the papers aim at different norms (the usual Sobolev norms or mesh-dependent norms), they start with an inf-sup condition for the L_2 inner product on the skeleton. We will exemplify a simple proof. Here the inf-sup condition is stated in terms of a projection operator.

To this end we consider the trace space on an interface Γ_{kl} and let

$$\xi_0 < \xi_1 < \ldots < \xi_{p-1} < \xi_p$$

be a partition of the interval $[\xi_0, \xi_p]$ which represents Γ_{kl}. Motivated by the setting (14) of $\tilde{T}_{kl,h}$ and $M_{kl,h}$ we consider two subspaces of the space of continuous piecewise linear functions on $[\xi_0, \xi_p]$. Let $\tilde{T}_{kl,h}$ be the subspace of those functions that vanish at the endpoints ξ_0 and ξ_p, and let $M_{kl,h}$ be the subspace of those functions that are constant on the first and on the last interval. So $\tilde{T}_{kl,h}$ and $M_{kl,h}$ have the same dimension $p-1$.

Lemma 1 *The projectors $Q_h : L_2[\xi_0, \xi_p] \to \tilde{T}_{kl,h}$ defined by*

$$(Q_h f, v)_0 = (f, v)_0 \quad \textit{for } v \in M_{kl,h}, \tag{19}$$

are uniformly bounded in L_2, specifically

$$\|Q_h f\|_0 \le \frac{4}{3}\|f\|_0 \quad \textit{for } f \in L_2[\xi_0, \xi_p]. \tag{20}$$

Proof: For $u_h := Q_h f \in \tilde{T}_{kl,h}$ let $v_h \in M_{kl,h}$ be defined by $v_h(\xi_i) = u_h(\xi_i)$, $i = 1, \ldots, \xi_{p-1}$. The two functions are determined by these $p-1$ values. Thus u_h and v_h agree on $[\xi_1, \xi_{p-1}]$, and $\int_{\xi_1}^{\xi_{p-1}} u_h v_h dx = \frac{1}{2}\int_{\xi_1}^{\xi_{p-1}} (u_h^2 + v_h^2) dx$. On the other hand, one obtains for the first (and last) interval

$$\int_{\xi_0}^{\xi_1} u_h v_h dx = \frac{1}{2} D, \quad \int_{\xi_0}^{\xi_1} u_h^2 dx = \frac{1}{3} D, \quad \int_{\xi_0}^{\xi_1} v_h^2 dx = D,$$

where $D := (\xi_1 - \xi_0) u_h(\xi_1)^2$. Hence,

$$\int_{\xi_0}^{\xi_1} u_h v_h dx = \frac{3}{8} \int_{\xi_0}^{\xi_1} (u_h^2 + v_h^2) dx.$$

Summing over all intervals and using Young's inequality yields

$$\|f\|_0 \|v_h\|_0 \geq (f, v_h)_0 = (u_h, v_h)_0 \geq \frac{3}{8} \left(\|u_h\|_0^2 + \|v_h\|_0^2 \right) \geq \frac{3}{4} \|u_h\|_0 \|v_h\|_0, \tag{21}$$

which proves (20). ■

Let $\mu_h \in M_h$ and Γ_{kl} be an interface. It follows from the lemma that $(\mu_h, w_{kl})_{0,\Gamma_{kl}}$ is large if $w_{kl} := Q_{h,kl}\, \mu_h$. Specifically we conclude that

$$\inf_{\mu_h \in M_h} \sup_{v_h \in X_h} \frac{b(v_h, \mu_h)}{\|v_h\|_{0,\mathcal{S}}\, \|\mu_h\|_{0,\mathcal{S}}} \geq \frac{3}{4}.$$

The proof of the Brezzi condition for the correct norms usually proceeds in a standard way. Interpolation theory yields an inverse estimate

$$\|w_{kl}\|_{H_{00}^{1/2}} \leq c h^{-1/2} \|w_{kl}\|_{0,\Gamma_{kl}} = c \|w_{kl}\|_{1/2,h,\Gamma_{kl}}. \tag{22}$$

There is an extension v such that

$$[v] = w_{kl} \quad \text{on } \Gamma_{kl}\,,$$

and the $\|\cdot\|_1$-norm of the extension is bounded by the $H_{00}^{1/2}$ norm above. Thus the same construction is good for the proof of the Brezzi condition for the mesh-dependent norms or for the Sobolev norms.

Theorem 1 *Assume that the triangulation in each subdomain Ω_k is quasiuniform. The discretizations (18) based on the spaces X_h, M_h defined by (13) and (14), respectively, satisfy the LBB-condition, i.e., there exists some $\beta > 0$ such that*

$$\inf_{\mu_h \in M_h} \sup_{v_h \in X_h} \frac{b(v_h, \mu_h)}{\|v_h\|_{1,h} \|\mu_h\|_{-1/2,h}} \geq \beta, \tag{23}$$

and

$$\inf_{\mu_h \in M_h} \sup_{v_h \in X_h} \frac{b(v_h, \mu_h)}{\|v_h\|_X\, \|\mu_h\|_{H_{00}^{-1/2}}} \geq \beta \tag{24}$$

holds uniformly in h.

We want to stress one point. Since we admit that the finite element functions in X_h can be discontinuous at the cross points, their jumps on the interface Γ_{kl} are only in $H^{1/2}(\Gamma_{kl})$. Nevertheless, the construction for the proof of the inf-sup condition yielded finite element functions with jumps in the subspaces $H_{00}^{1/2}(\Gamma_{kl})$, and we conclude that the subspace of finite elements with this property is thick enough. Therefore it is natural that the Lagrange multipliers are equipped with the norms $\|\mu_h\|_{H_{00}^{-1/2}(\Gamma_{kl})}$ in those investigations in which norms of the classical Sobolev spaces rather than mesh-dependent norms are preferred.

After the inf-sup condition has been established, only approximation properties are required for the proof of the error estimate. Assume that the problem is H^2-regular, i.e., $u \in H^2$. Let $w_h \in X_h$ be finite element function with $\|u - w_h\|_{1,\delta} \le ch\|u\|_2$ that need not satisfy the mortar condition. Similarly, we have $\|\frac{\partial u}{\partial n} - \mu_h\|_{0,\Gamma_{kl}} \le ch^{1/2}\|u\|_2$, for some $\mu_h \in M_h$ and all Γ_{kl}, (and this term appear also in the usual bounds of the consistency error of the second Strang lemma). The regularity assumption and a density argument assert that the first equation in (12) holds for all $v \in X_{00} + X_h$. Hence,

$$\begin{aligned} a(u_h - w_h, v_h) + b(v_h, \lambda_h - \mu_h) &= \langle l, v\rangle \quad \forall v_h \in X_h, \\ b(u_h - w_h, \mu) &= 0, \qquad \forall \mu \in M_h, \end{aligned} \tag{25}$$

where $\langle l, v\rangle := a(u - w_h, v) + b(v, \lambda - \mu_h)$. By construction $|\langle l, v\rangle| \le ch\,\|u\|_2\,\|v\|_{1,h}$. From this bound and the stability of (25) we obtain the error estimate

$$\|u - u_h\|_{1,h} + \|\lambda - \lambda_h\|_{-1/2,h} \le ch\|u\|_2 \tag{26}$$

and by a duality argument

$$\|u - u_h\|_0 + h\|\lambda - \lambda_h\|_{-1/2,h} \le ch^2\|u\|_2. \tag{27}$$

For details the reader is refered to [BDW00].

Remark 1 *We have provided the well-known arguments in the derivation of the error estimates since we want to be more specific about the remark at the end of the introduction.*

In establishing (25) we have used ellipticity of $a(\cdot,\cdot)$, boundedness of $b(\cdot,\cdot)$, and the inf-sup condition. On the other hand, if a bound for $\|u - u_h\|_{1,\delta}$ has been determined elsewhere, following [BB99, Woh99a] the first equation in (25) may be rewritten

$$b(v_h, \lambda_h - \mu_h) = b(v_h, \lambda - \mu_h) - a(u_h - u, v_h) \quad \forall v_h \in X_h. \tag{28}$$

From (24) we know that we obtain $\|\lambda_h - \mu_h\|_{H_{00}^{1/2}} \le \beta^{-1} b(v_h, \lambda_h - \mu_h)/\|v_h\|_X$ with an appropriate test function v_h. Specifically, the right test function has its jumps on the interfaces in $H_{00}^{1/2}$, and it is not an obstacle that $b(\cdot,\cdot)$ is not bounded on $H^{1/2} \times H_{00}^{-1/2}$. After applying the triangle inequality the error of the Lagrange multipliers is established in the $H_{00}^{-1/2}$ norm. This technique circumvents the fact that the bilinear form b is not bounded on $H^{1/2} \times H_{00}^{-1/2}$. – The unboundedness is an obstacle for the direct application of Brezzi's theory.

Finally, we note that recently other finite elements for the Lagrange multipliers have been suggested. Computations are easier if they are obtained from a dual basis [Woh99c].

Multigrid Convergence Analysis

The saddle point problem (18) gives rise to a linear system of the form

$$\begin{pmatrix} A & B^T \\ B & \end{pmatrix} \begin{pmatrix} u_h \\ \lambda_h \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}, \tag{29}$$

where the dimension of the vectors coincides with the dimension of the finite element spaces X_h and M_h, respectively. For convenience, the same symbol is taken for the finite element functions and their vector representations, and the index h is suppressed whenever no confusion is possible.

The finite element basis functions are assumed to be normalized such that the Euclidean norm of the vectors $\|\cdot\|_{\ell_2}$ is equivalent to the L_2-norm of the functions, i.e.

$$\|v_h\|_{\ell_2} \approx \|v_h\|_{0,\Omega} \quad \text{for } v_h \in X_h. \tag{30}$$

When the equations (29) are solved by a multigrid algorithm, the design of the smoothing procedure is the crucial point. Motivated by [BS97] our smoothing procedure will be based on the following concept. Let C be a preconditioner for A which, in particular, is normalized so that

$$v^T A v \leq v^T C v, \quad v \in X_h, \tag{31}$$

and for which the linear system

$$\begin{pmatrix} C & B^T \\ B & \end{pmatrix} \begin{pmatrix} v \\ \mu \end{pmatrix} = \begin{pmatrix} d \\ e \end{pmatrix} \tag{32}$$

is easily solvable. In actual computations the vectors v, μ are obtained by implementing $S\mu = BC^{-1}d - e, v = C^{-1}(d - B^T\mu)$, where $S := BC^{-1}B^T$ is the *Schur complement* of C in (32).

Then the iteration that will serve as a smoother in our multigrid scheme has the form

$$\begin{pmatrix} u_h^{j+1} \\ \lambda_h^{j+1} \end{pmatrix} := \begin{pmatrix} u_h^j \\ \lambda_h^j \end{pmatrix} - \begin{pmatrix} C & B^T \\ B & \end{pmatrix}^{-1} \left\{ \begin{pmatrix} A & B^T \\ B & \end{pmatrix} \begin{pmatrix} u_h^j \\ \lambda_h^j \end{pmatrix} - \begin{pmatrix} f \\ 0 \end{pmatrix} \right\} \tag{33}$$

$$= \begin{pmatrix} u_h^j \\ 0 \end{pmatrix} - \begin{pmatrix} C & B^T \\ B & \end{pmatrix}^{-1} \begin{pmatrix} Au_h^j - f \\ Bu_h^j \end{pmatrix}, \tag{34}$$

where superscripts will denote iteration indices. It is important to note that u_h^{j+1} always satisfies the constraint, i.e.,

$$Bu_h^{j+1} = 0, \tag{35}$$

see [BS97]. Specifically the implementations are based on (33) in order to have auxiliary problems with small (correction) vectors, while the representation (34) shows that the next iterate is independent of the old Lagrange multiplier λ_h^j.

Now we assume that the reader is familiar with the general concept of multigrid algorithms [Hac85] and knows some simple application. This is sufficient since the finite element spaces $X_h \subset X_\delta$ and $M_h \subset M$ are nested and the coarse grid correction of the multigrid scheme can be performed in the standard manner, see e.g. [BS97] or [Hac85, p. 235].

As usually, the analysis of the multigrid method will be based on two different norms. The fine topology will be defined by the norm

$$|||v_h, \mu_h|||_2 := \|Av_h + B^T\mu_h\|_{\ell_2}, \tag{36}$$

i.e., by a discrete analogon of the H^2-norm, and the coarse one by the L_2-norm

$$\|v_h\|_{0,\Omega}.$$

The latter expression is independent of the Lagrange multiplier since the iteration (34) is independent of the multiplier in the previous step. We recall that $\lambda_{\max}(A) = O(h^{-2})$.

Smoothing property: Assume that $\lambda_{\max}(A) \le \alpha \le c\lambda_{\max}(A)$. If m smoothing steps of the relaxation (34) with $C := \alpha I$ are performed, then

$$|||u_h^m - u_h, \lambda_h^m - \lambda_h|||_2 \le \frac{ch^{-2}}{m}\|u_h^0 - u_h\|_{0,\Omega}. \tag{37}$$

Approximation property: For the coarse grid correction u_{2h} one has

$$\|u_h - u_{2h}\|_{0,\Omega} \le ch^2|||u_h, \lambda_h|||_2. \tag{38}$$

The proof of the two properties are now quite standard. The verification of the smoothing property is performed by purely algebraic manipulations [BDW00, BS97]. The approximation property looks very much like the L_2-error estimate (27). Indeed, it is derived from the latter by a duality argument; cf. [BDW00] or [Bra97, Lemma V.2.8].

Recently, a version was implemented as a *cascadic multigrid* algorithm; see [BDL99]. In that context it is shown that the Lagrange multipliers must be treated in a different way than the u-variables if the iteration (33) is built into a conjugate gradient method.

Numerical Example

We report on one of the examples in [BDW00] with big jumps of coefficients and several cross points. The equation (1.1) is considered with scalar diffusion coefficients that are constant on each subdomain. In Figure 1 large bricks are separated by thin channels. Fixing the diffusion constant for the bricks to $a_0 = 1$, we test the cases where the channels have higher or lower permeability ($a_1 = 10^6$ or $a_1 = 10^{-6}$, resp.). We perform the cg-method with V(1,1)-cycle and two inner iterations. The convergence rates are stable if the mortar side is on the side with the smaller diffusion constant and large step size, resp.; otherwise the method may fail. The results in Figure 1 for the case $a_1 = 10^6$ show clearly that the diffusion is faster in the small channels.

level	elements	$a_0 = 1,\, a_1 = 1$	$a_0 = 1,\, a_1 = 10^6$	$a_0 = 1,\, a_1 = 10^{-6}$
3	6784	0.21	0.12	0.08
4	27136	0.21	0.14	0.07
5	108544	0.21	0.14	0.08
inner iterations		1–2	1–2	1–2

Table 1: Convergence for the example with several cross-points for MG with V(1,1)-cycle

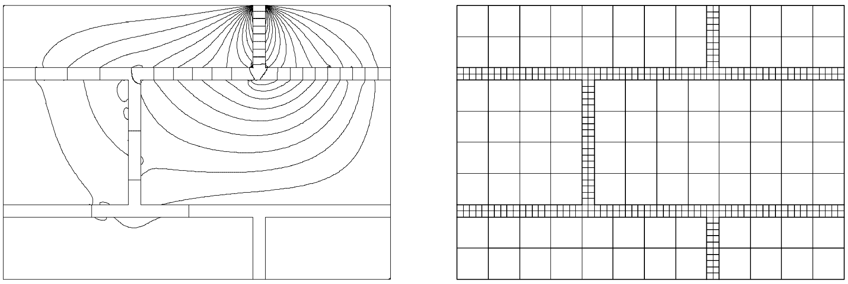

Figure 1: Example with several cross-points

References

[AT95]A. Agouzal and J.-P. Thomas. Une méthode d'élement finis hybridesen décomposition de domaines. *RAIRO M²AN*, 29:749–764, 1995.

[BB99]Faker Ben Belgacem. The mortar finite element method with Lagrange multipliers. *Numer. Math.*, 84(2):173–197, 1999.

[BDL99]D. Braess, P. Deuflhard, and K. Lipnikov. A cascadic multigrid algorithm for mortar elements. Technical report, Konrad-Zuse-Zentrum, Berlin, 1999.

[BDW00]D. Braess, W. Dahmen, and C. Wieners. A multigrid algorithm for the mortar finite element method. *SIAM J. Numer. Anal.*, 37:48–69, 2000.

[BF91]F. Brezzi and M. Fortin. *Mixed and Hybrid Finite Element Methods*. Springer-Verlag, New York – Berlin – Heidelberg, 1991.

[BM97]Faker Ben Belgacem and Yvon Maday. The mortar element method for three dimensional finite elements. *RAIRO Mathematical Modelling and Numerical Analysis*, 31(2):289–302, 1997.

[BMP94]Christine Bernardi, Yvon Maday, and Anthony T. Patera. A new non conforming approach to domain decomposition: The mortar element method. In Haim Brezis and Jacques-Louis Lions, editors, *Collège de France Seminar*. Pitman, 1994. This paper appeared as a technical report about five years earlier.

[Bra97]D. Braess. *Finite Elements: Theory, Fast Solvers, and Applications in Solid Mechanics*. Cambridge University Press, Cambridge, 1997.

[BS97]D. Braess and R. Sarazin. An efficient smoother for the Stokes problem. *Applied Numerical Mathematics*, 23:3–19, 1997.

[Hac85]W. Hackbusch. *Multi-Grid Methods and Applications*. Springer-Verlag, Berlin

– Heidelberg – New York, 1985.

[LM72]J.L. Lions and E. Magenes. *Non-Homogeneous Boundary Value Problems.* Springer-Verlag, New York – Berlin – Heidelberg, 1972.

[Woh99a]B. Wohlmuth. Discretization methods and iterative solvers based on domain decomposition. Technical report, Habilitation, Department of Mathematics, Augsburg, 1999.

[Woh99b]B. Wohlmuth. Hierarchical a posteriori error estimators for mortar finite element methods with lagrange multipliers. *SIAM J. Numer. Anal.*, 36:1636–1658, 1999.

[Woh99c]B. Wohlmuth. A mortar finite element method using dual spaces for the lagrange multipliers. Technical report, Department of Mathematics, Augsburg, 1999.