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28. The Mortar Element Method for 3D Maxwell's equations: analysis and application to magnetodynamics

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Introduction

In this paper, we describe the main ideas of the mortar element method combined with $H(\mathbf{curl})$–conforming finite elements for the numerical approximation of Maxwell's equations. This method turns out to be a new non–conforming, non–overlapping domain decomposition technique where non–matching grids are allowed at the interface between adjacent sub–domains. We report the results on the method's convergence and error estimate together with the description of the main implementation details and some numerical results.

Position of the problem

We are interested in a system that can be modeled by the set of Maxwell's equations when the displacement currents are neglected:

$$\begin{aligned} &(a) \qquad \mathbf{curl}\,\mathbf{E} = -\partial_t \mathbf{B} \qquad && \text{in } \Omega\times]0,T[\\ &(b) \qquad \mathbf{curl}\,\mathbf{H} = \mathbf{J} && \text{in } \Omega\times]0,T[\\ &(c) \qquad \mathbf{J} = \sigma\mathbf{E} && \text{in } \Omega\times]0,T[\\ &(d) \qquad \mathbf{B} = \mu\,\mathbf{H} && \text{in } \Omega\times]0,T[\end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^3$ is bounded, $\mathbf{E}$, $\mathbf{H}$ are the electric and magnetic fields, $\mathbf{B}$ the magnetic induction and $\mathbf{J}$ the current density. In system (1), ∂_t stands for the first derivative in time.

The physical parameters of the problem are: the magnetic permeability μ and the electric conductivity σ. Without loss of generality we assume that μ is a positive constant and σ is simply bounded. We set $\mathcal{C} = supp\{\sigma\}$ the conducting part and we assume that $\mathcal{C}$ is simply connected.

In three–dimensional magnetodynamic applications, system (1) is usually reformulated in terms of a primary variable which is either the magnetic field $\mathbf{H}$ or the magnetic vector potential $\mathbf{A}$. In both cases, by re–writing system (1) in terms of the chosen primary variable, we obtain the following parabolic equation which is the object of our study:

$$\partial_t(\alpha\ \mathbf{u}) + \mathbf{curl}\,(\beta\ \mathbf{curl}\,\mathbf{u}) = \mathbf{f} \qquad \text{in } \ \Omega\times]0,T[\,, \tag{2}$$

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with α and β two functions related to the physical parameters σ and μ and $\mathbf{f}$ related to the external sources. Equation (2) can be non–strictly parabolic: this may occur when working with the magnetic vector potential. In this case, a jauge condition (e.g., $\mathrm{div}(\mathbf{u}) = 0$ where $\alpha = 0$) has to be added to equation (2) to ensure the uniqueness of the solution. Moreover, we assume:

$$\mathbf{u} \wedge \mathbf{n} = \mathbf{0} \ \text{ on } \ \partial\Omega\times]0,T[\qquad \text{and} \qquad \mathbf{u}(\mathbf{x},0) = 0 \ \text{ a.e. in } \ \mathbf{x} \in \overline{\mathcal{C}}. \tag{3}$$

We introduce the following Hilbert spaces (endowed with the corresponding graph norms)

$$\begin{array}{rcl} H(\mathbf{curl},\Omega) & = & \left\{ \mathbf{u} \in L^2(\Omega)^3 \mid \mathbf{curl}\,\mathbf{u} \in L^2(\Omega)^3 \right\}, \\ H_0(\mathbf{curl},\Omega) & = & \left\{ \mathbf{u} \in H(\mathbf{curl},\Omega) \mid (\mathbf{u} \wedge \mathbf{n})_{|\partial\Omega} = \mathbf{0} \right\}. \end{array} \tag{4}$$

The variational formulation of the problem (2) reads:

$$\begin{array}{l} \text{Find } \mathbf{u} \in H_0(\mathbf{curl},\Omega) \text{ such that } \forall \mathbf{v} \in H_0(\mathbf{curl},\Omega) : \\ \quad \int_\Omega \partial_t\alpha \ \mathbf{u}\cdot\mathbf{v} \ d\mathbf{x} \ + \int_\Omega \beta \ \mathbf{curl}\,\mathbf{u}\cdot\mathbf{curl}\,\mathbf{v} \ d\mathbf{x} \ = \int_\Omega \mathbf{f}\cdot\mathbf{v} \ d\mathbf{x} \, . \end{array} \tag{5}$$

It can be proved that this problem admits a unique solution when suitably interpreted in time and a jauge condition is imposed where $\alpha = 0$. Note that when $\alpha = 0$ everywhere, (2) is the magnetostatic problem and (5) its variational formulation.

The main concern of our work is to propose an efficient domain decomposition method for this type of equations, discretized by using edge element approximation in three dimensions which allows for non-matching grids. The outline of the paper is the following: in the second section the mortar element method is proposed, the analysis is sketched and some details of the implementation are given. The third section is devoted to the applications: we present some preliminary numerical results in the magnetostatic case and the governing equations for the magnetodynamic problem in moving geometries. Numerical simulations in the latter case are in progress.

Definition and analysis of the mortar element method

Since the definition and analysis of a domain decomposition procedure for (2) is strictly related to the choice of the spatial discretization, in this section, without loss of generality, we consider the following model problem:

$$\begin{array}{l} \text{Find } \mathbf{u} \in \ H_0(\mathbf{curl},\Omega) \text{ such that } \forall \ \mathbf{v} \in \ H_0(\mathbf{curl},\Omega) \\ \displaystyle\int_\Omega \mathbf{curl}\,\mathbf{u}\cdot\mathbf{curl}\,\mathbf{v} \ d\mathbf{x} \ + \int_\Omega \mathbf{u}\cdot\mathbf{v} \ d\mathbf{x} \ = \int_\Omega \mathbf{f}\cdot\mathbf{v} \ d\mathbf{x} \, . \end{array} \tag{6}$$

This problem admits obviously a unique solution in $H_0(\mathbf{curl},\Omega)$ and it is worth noting that it is strictly related to (2): when the parameters of the problem are set equal to 1 and an implicit time stepping procedure is applied, (6) is the problem that we have to solve at each time step. The case of vanishing α will be the object of further remarks.

Approximation spaces

Partition of the domain and local spaces

Assume here that the domain Ω is a convex bounded (Lipschitz) polyhedral[4] subset of $\mathbb{R}^3$. Let $\Omega_k \subseteq \Omega$, for $1 \leq k \leq K$, be a non–overlapping, polyhedral partition of Ω, that is:

$$\overline{\Omega} = \cup_{k=1}^{K} \overline{\Omega}_k \quad \text{with} \quad \Omega_k \cap \Omega_l = \emptyset \text{ if } k \neq l. \tag{7}$$

For every k $(1 \leq k \leq K)$ we denote by $\mathbf{n}_k$ the outer normal to Ω_k and we call $\left\{\Gamma^{k,i}\right\}_{1 \leq i \leq F(k)}$ the $F(k)$ faces of the polyhedron Ω_k. We define the **skeleton** Σ as $\Sigma = \cup_{k=1}^{K} \cup_{l=1}^{K} \partial\Omega_k \setminus \partial\Omega$. Let $\tau^{k,i}$ be the counterclockwise tangential vector to $\partial\Gamma^{k,i}$; we define also the outer normal to $\partial\Gamma^{k,i}$ as $\mathbf{n}^{k,i} = \tau^{k,i} \wedge \mathbf{n}_k$.

Among several possibilities we choose a splitting of the skeleton Σ as the disjoint union of some closed faces $\{\overline{\Gamma}^{k,i}\}_{k,i}$ that we call *slave faces*. A unique set of indices corresponds to this choice and we denote it by:

$$\mathcal{I}_M = \left\{ m = (k,i) \text{ such that } \Gamma^{k,i} \text{ is a slave face } \right\}.$$

To shorten the notations we denote the slave faces by Γ^m $(1 \leq m \leq M)$ and, as prescribed, we have: $\overline{\Sigma} \equiv \cup_{m=1}^{M} \overline{\Gamma}^m$ and $\Gamma^m \cap \Gamma^n = \emptyset$ if $m \neq n$. Moreover we set: $\overline{\gamma}^{l,k} = \partial\Omega_l \cap \partial\Omega_k$. We define the following "broken" space:

$$X := \left\{ \mathbf{u} \in L^2(\Omega)^3 \mid \mathbf{u}_{|\Omega_k} \in H(\mathbf{curl}, \Omega_k) \ , \ (\mathbf{u} \wedge \mathbf{n})_{|\,\partial\Omega \cap \partial\Omega_k} = \mathbf{0} \ 1 \leq k \leq K \right\}. \tag{8}$$

As standard, X is a Hilbert space endowed with the following broken norm: $||\mathbf{u}||^2_{\star,\mathbf{curl}} := \sum_{k=1}^{K} ||\mathbf{u}_{|\Omega_k}||^2_{\mathbf{curl},\Omega_k}$.

For each index k $(1 \leq k \leq K)$, we introduce a regular quasi–uniform triangulation $\mathcal{T}^k_{h(k)}$ on the sub–domain Ω_k and we denote by h the maximum of the mesh sizes. These partitions can be composed either of tetrahedra or parallelepipeds; they are completely independent and thus, in general, non–matching at the interfaces $\{\gamma^{k,l}\}_{k,l}$.

Let $\hat{K}$ be the reference tetrahedron or cube. For every $K_i \in \mathcal{T}^k_{h(k)}$, we denote by $\mathcal{F}_i : \hat{K} \to K_i$ a bijective mapping from $\hat{K}$ to K_i. These mappings can be chosen as *linear* both in the case of tetrahedra and parallelepipeds: $\mathcal{F}_i(\hat{\mathbf{x}}) = \mathcal{B}_i \hat{\mathbf{x}} + \mathbf{c}_i$ where $\mathcal{B}_i \in \mathbb{R}^{3\times 3}$ is an invertible matrix and $\mathbf{c}_i \in \mathbb{R}^3$ is a constant field. Over each sub–domain Ω_k we define the finite dimensional space which is at the base of the domain decomposition method:

$$Y^k_h := \{\mathbf{v}^k_h \in H(\mathbf{curl}, \Omega_k) \mid \mathcal{B}_i^T(\mathbf{v}^k_{h\,|K_i} \circ \mathcal{F}_i) \in \mathcal{P}_{p(k)} \ \forall K_i \in \mathcal{T}^k_{h(k)}\}, \tag{9}$$

where $\mathcal{P}_{p(k)}$ denotes a family of Nédélec type finite elements for Maxwell's equations of degree $p(k)$ (see [N80, N86] for a complete definition). Furthermore we set:

$$X_h \quad := \quad \left\{ \mathbf{v}_h \in X \mid \mathbf{v}^k_h := \mathbf{v}_{h\,|\,\Omega_k} \in Y^k_h \ \text{ and } \ (\mathbf{v}^k_h \wedge \mathbf{n})_{\partial\Omega \setminus \partial\Omega_k} = \mathbf{0} \right\}, \tag{10}$$

and in the following we denote the elements $\mathbf{v}_h \in X_h$ both as functions and as K-uplets $\mathbf{v}_h = (\mathbf{v}^1_h, \mathbf{v}^2_h, \cdots, \mathbf{v}^K_h)$ where $\mathbf{v}^k_h \in Y^k_h$ $(1 \leq k \leq K)$. We use both notations

[4]The whole theory applies even when Ω is a regular bounded subset of $\mathbb{R}^3$. Of course, in this case the subdomain $\{\Omega_k\}_k$, defined afterwards, can not be polyhedra but curved polyhedra.

since it is never misleading. Since we deal with tangential traces of these vector fields, we introduce some further definitions. For any index $m = (k,i) \in \mathcal{I}_M$, we set $T_h^{k,i} = \big\{(\mathbf{v}_h^k \wedge \mathbf{n}_k)_{|\Gamma^{k,i}} \mid \mathbf{v}_h^k \in Y_h^k\big\}$ and its subset $T_{h,0}^{k,i} = \big\{\lambda_h \in T_h^{k,i} \mid (\lambda_h \cdot \mathbf{n}^{k,i})_{\,|\partial\Gamma^{k,i}} = 0\big\}$.

Let Γ^m be a *slave face* with $m = (k,i)$ the corresponding indices and $\mathbf{v}_h \in X_h$: for almost every $\mathbf{x} \in \Gamma^m$ there exists an l $(1 \le l \le K\,,\, l \ne k)$, such that $\mathbf{x} \in \Gamma^m \cap \gamma^{k,l}$. At this point $\mathbf{x}$, we have two fields, namely $\mathbf{v}_h^k$ and $\mathbf{v}_h^l$. In general, since the macro–decomposition is non–conforming, the value of l depends on the point $\mathbf{x}$ and we denote by I_m the set of indices l $(1 \le l \le K\,,\, l \ne k)$ such that $\Gamma^m \cap \gamma^{k,l} \ne \emptyset$. We then set $\mathbf{v}_h^{-k}(\mathbf{x}) = \mathbf{v}_h^l(\mathbf{x}), \quad \forall \mathbf{x} \in \Gamma^m \cap \gamma^{k,l}, \quad l \in I_m$. The function $\mathbf{v}_h^{-k}(\mathbf{x})$ is defined at almost every $\mathbf{x} \in \Gamma^m$. Due to the non–conformity of the macro–decomposition, $\mathbf{v}_h^{-k}(\mathbf{x})$ is not in general the tangential trace at Γ^m of a field $\mathbf{v}$ in $H(\mathbf{curl}, \Omega_k)$.

Constraint problem and matching condition

Let $\mathbf{v} \in H_0(\mathbf{curl}, \Omega)$, we have that $(\mathbf{v}^{-k} \wedge \mathbf{n})_{|\Gamma^m} = (\mathbf{v}^k \wedge \mathbf{n})_{|\Gamma^m}$ in $\big(H_{00}^{1/2}(\Gamma^m)\big)'$. The purpose of this section is to express how to impose this condition on the discrete broken space in a weak sense. To this aim, we define, for any $m \in I_m$, $M_h^m \subseteq T_h^m$, $\quad \dim\{M_h^m\} = \dim\{T_{h,0}^m\}$. We set:

$$M_h := \big\{\psi_h \in L^2(\Sigma)^2 \mid \forall m \in I_m \ ,\ \psi_{h\,|\Gamma^m} \in M_h^m\big\}. \tag{11}$$

As before we also adopt the vector notation $\psi_h = (\psi_h^1, \psi_h^2, \dots, \psi_h^M)$ when it is convenient. Then, we propose the following non–conforming approximation space for $H_0(\mathbf{curl}, \Omega)$:

$$X_h^c = \Big\{\mathbf{v} \in X_h \mid \forall m \ ,\ \int_{\Gamma^m} (\mathbf{v}_h^k \wedge \mathbf{n}_k - \mathbf{v}_h^{-k} \wedge \mathbf{n}_k) \cdot \psi_h \ d\Gamma = 0 \ \forall \psi_h \in M_h^m\Big\}. \tag{12}$$

The discrete problem reads: find $\mathbf{u}_h \in X_h^c$ such that $\forall\, \mathbf{v}_h \in\ X_h^c$:

$$\sum_{k=1}^{K} \Big[\int_{\Omega_k} \mathbf{curl}\, \mathbf{u}_h^k \cdot \mathbf{curl}\, \mathbf{v}_h^k \ d\mathbf{x} \ + \int_{\Omega_k} \mathbf{u}_h^k \cdot \mathbf{v}_h^k \ d\mathbf{x}\ \Big] = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}_h \ d\mathbf{x} \qquad \forall\, \mathbf{v}_h \in\ X_h^c. \tag{13}$$

The numerical properties of the space X_h^c depend strongly on the choice of the Lagrange multiplier space M_h. In the following, we discuss our choice for this space and, we proceed to the analysis of the method. We refer to [Hop99] for a different approach using the first family of edge elements.

Thanks to the locality in the definition (11) of M_h, we focus our attention on a single slave face Γ^m. Γ^m is decomposed by $\mathcal{T}^k_{h(k)|\Gamma^m}$ either into triangles or parallelograms. For every parallelogram (resp. triangle) of $\mathcal{T}^k_{h_{(k)}|\Gamma^m}$, there exists a linear mapping F_i satisfying $T_i = F_i(\hat{T})$ where $\hat{T}$ is the reference square $]-1,1[^2$ (resp. the reference triangle $\hat{T} := \{(x,y) \in \mathbb{R}^2 \mid 0 < x < 1\ ,\ 0 < y < 1 - x\}$). The construction of M_h^m consists in imposing additional constraints at T_h^m on the parallelograms (resp. triangles) which meet the boundary $\partial\Gamma^m$. We denote by BT^m the set of all these elements T_i and assume that the mapping F_i associates to (one of) the boundary edge(s)

($\overline{T}_i \cap \partial\Gamma^m$) an edge of $\hat{T}$ that is parallel to a Cartesian axis (this is exhaustive up to a rotation). Our choice of the Lagrange multiplier space turns out to be:
Case of parallelograms:

$$M_h^m := \left\{ \lambda_h^m \in T_h^m \mid B_i^{-1}(\lambda_h^m \circ F_i) \in \mathbb{Q}_{p,p}(\hat{S}) \times \mathbb{Q}_{p,p-1}(\hat{S}) \ , \ T_i \in BT^m \right\} \tag{14}$$

where $\mathbb{Q}_{p,p'}$ denotes the space of polynomials which are of degree p in the first variable and of degree p' in the second one. Of course, if a corner of Γ^m belongs to the parallelogram $\overline{T}_i$, then the Lagrange multiplier λ_h^m is chosen so that $B_i^{-1}(\lambda_h^m \circ F_i) \in \mathbb{Q}_{p-1,p}(\hat{S}) \times \mathbb{Q}_{p,p-1}(\hat{S})$.
Case of triangles:

$$M_h^m := \left\{ \lambda_h^m \in T_h^m \mid B_i^{-1}(\lambda_h^m \circ F_i) \in P_p(\hat{T}) \times P_{p-1}(\hat{T}) \ , \ T_i \in BT^m \right\}. \tag{15}$$

As before, if a corner of Γ^m belongs to the triangle $\overline{T}_i$, then the Lagrange multiplier λ_h^m is chosen so that $B_i^{-1}(\lambda_h^m \circ F_i) \in P_{p-1}(\hat{T}) \times P_{p-1}(\hat{T})$.

The spaces T_h^m and M_h^m are $H(\mathrm{div})$–conforming and the degrees of freedom are related to the normal components of the fields along the edges. We refer to [BBM00] for a complete characterization.

The following proposition holds in both cases of triangles and parallelograms:

Proposition 1 *Let $\Pi_h^m : L^2(\Gamma^m)^2 \to T_{h,0}^m$ be defined by*

$$\int_{\Gamma^m} (\mathbf{u} - \Pi_h^m \mathbf{u}) \cdot \varphi_h \, d\Gamma \ = 0 \qquad \forall \varphi_h \in M_h^m. \tag{16}$$

There exists a constant C independent of h such that the following stability estimate holds:

$$\forall \mathbf{u} \in L^2(\Gamma^m)^2 \quad , \quad ||\Pi_h^m \mathbf{u}||_{0,\Gamma^m} \leq C \, ||\mathbf{u}||_{0,\Gamma^m}. \tag{17}$$

Remark 1 *If one deals with the first family of Nédélec type finite elements (see [N80]), then at the interface Γ^m the space T_h^m is of Raviart-Thomas type and the Lagrange multiplier space can be similarly defined (see [Hop99]).*

Convergence result

In this section we simply state the convergence results concerning problem (13) whose proofs can be found in [BBM00]. We have

Theorem 1 *Let $\mathbf{u} \in H_0(\mathbf{curl}, \Omega)$ be the solution of problem (6) and $\mathbf{u}_h$ the solution of problem (13) with M_h defined by (14) or (15). We assume that $\mathbf{u}_k \in H^{p+1}(\Omega_k)$ with $\mathbf{curl}\, \mathbf{u}_k \in H^{p+1}(\Omega_k)$ $(1 \leq k \leq K)$ and we suppose that there exists a uniform constant γ such that $\max_k\{h_k\} \leq \gamma \min_k\{h_k\}$. We set $h := \max_k\{h_k\}$. The following estimate holds:*

$$||\mathbf{u} - \mathbf{u}_h||_{\star,\Omega} \leq C_1 \, h^p \left(\sum_{k=1}^{K} ||\mathbf{u}||_{p+1,\Omega_k}^2 \right)^{\frac{1}{2}} + C_2 \, h^p \sqrt{|\ln h|} \left(\sum_{k=1}^{K} ||\mathbf{curl}\, \mathbf{u}||_{p+1,\Omega_k}^2 \right)^{\frac{1}{2}} \tag{18}$$

where C_1, C_2 are uniform constants depending only on the macro-decomposition.

Note that the first term comes from the best approximation error and the second one from the consistency error.

Remark 2 *The same error estimate holds when the coefficients are not set equal to one but they jump through the different sub–domains: the constant in front of the right hand side will depend on the size of their jumps.*

Remark 3 - *Imposing a jauge condition* - *When the parameter α is vanishing on a part of the domain, equation (6) must be replaced by:*

$$\int_\Omega \alpha \mathbf{u}\cdot\mathbf{v}\ d\mathbf{x}\ +\int_\Omega \beta\ \mathbf{curl}\,\mathbf{u}\cdot\mathbf{curl}\,\mathbf{v}\ d\mathbf{x}\ =\int_\Omega \mathbf{f}\cdot\mathbf{v}\ d\mathbf{x}\,;\quad div\ (\mathbf{u})_{|\{\alpha=0\}}=0. \tag{19}$$

The definition of the proposed method in this case would involve a non–conforming mortar element approximation of the mixed problem related to (19) which is still not understood. Nevertheless, it is worth noticing that when the partition (7) is chosen in a way that $\alpha = 0$ in one sub–domain only, say $\Omega_{\bar{k}}$, the sub–domains are decomposed in polyhedra and none of the faces of $\Omega_{\bar{k}}$ is slave; then problem (19) can be suitably approximated. The discrete problem is: find $\mathbf{u}_h \in X_h^c$ such that $\forall \mathbf{v}_h \in X_h^c$ and $p_h \in \mathcal{S}^{p+1}(\mathcal{T}_h^{\bar{k}}, \Omega_{\bar{k}}) \cap H_0^1(\Omega_{\bar{k}})$:

$$\int_{\Omega\setminus\Omega_{\bar{k}}} \alpha\mathbf{u}_h\cdot\mathbf{v}_h\ d\mathbf{x}\ +\int_\Omega \beta\ \mathbf{curl}\,\mathbf{u}_h\cdot\mathbf{curl}\,\mathbf{v}_h\ d\mathbf{x} \quad=\quad \int_\Omega \mathbf{f}\cdot\mathbf{v}_h\ d\mathbf{x} \tag{20}$$

$$\textit{and}\qquad \int_{\Omega_{\bar{k}}} \mathbf{u}_h\cdot \textit{grad}\ p_h\ d\mathbf{x} \quad=\quad 0 \tag{21}$$

where $\mathcal{S}^{p+1}(\mathcal{T}_h^{\bar{k}}, \Omega_{\bar{k}})$ is the standard scalar space of Lagrange finite elements of degree $p+1$. Making use of the approximation results proved in [ABDG98] and the ones of the previous section, it can be proved that (20)–(21) admits a unique solution and the error estimate (18) holds true when $\mathbf{u}$ is solution of (19) and $\mathbf{u}_h$ of (20)–(21).

On the other hand, when the quantity of interest is the magnetic vector potential, a unique solution can be selected by using a suitable iterative solver and expressing $\mathbf{J} = \mathbf{curl}\ \mathbf{T}$ for a vector $\mathbf{T}$. Note that only the **curl** *of the magnetic vector potential is needed: so, the magnetic induction is uniquely determined in any case.*

Reduction of the computational cost

The use of the second family of Nédélec type finite elements is often out of range in realistic three–dimensional computations and the use of the first family is often preferred. In the standard approximation context, with respect to the first one, the second family does not give a substantial improvement in the accuracy while it increases the number of degrees of freedom. In this section we show how these two families of edge elements can be merged in a way to obtain, on one hand, "quasi-optimal" convergence of the mortar element method and, on the other hand, a sensible reduction in the algebraic system dimension. We consider here the case where each sub–domain is discretized by a finite number of tetrahedra, first or second order edge elements are chosen and we will focus the attention on one slave face Γ^m.

First family: six degrees of freedom per tetrahedron – Given a tetrahedron K, let $\mathbf{r}_j$ (j=1,4) be the position vectors of its vertices and $\lambda_j(\mathbf{r})$ be the barycentric coordinate of a point $P \in K$ (with vector position $\mathbf{r}$) with respect to the vertex j. It is clear that $\lambda_j(\mathbf{r})$ is a linear function in the tetrahedron with $\lambda_j(\mathbf{r}_k) = \delta_{jk}$ $(j,k \in \{1,2,3,4\})$. The vector basis function corresponding to an edge e_{ij} going from $\mathbf{r}_i$ to $\mathbf{r}_j$, is given by

$$\mathbf{w}_{ij}(\mathbf{r}) = \lambda_i(\mathbf{r}) \text{ grad } \lambda_j(\mathbf{r}) - \lambda_j(\mathbf{r}) \text{ grad } \lambda_i(\mathbf{r}) \quad , \quad i,j = 1,2,3,4,\, i < j\,; \tag{22}$$

let us denote by $\mathcal{P}_1(K)$ the space generated by the basis functions settled in (22). The interpolating function $\mathbf{u}_h$ on K for the vectorial state variable $\mathbf{u} \in (C^0(\overline{K}))^3$ has the following form

$$\mathbf{u}_h = \sum_{i=1}^{3} \sum_{j=i+1}^{4} \mathbf{w}_{ij}\ \alpha_{ij}(\mathbf{u}) \quad \text{with} \quad \alpha_{ij}(\mathbf{u}) = |e_{ij}|\,(\mathbf{u} \cdot \mathbf{t}_{e_{ij}})(\mathbf{x}_{ij}^M)$$

where $|e_{ij}|$ is the length of e_{ij}, $\mathbf{x}_{ij}^M$ its midpoint and $\mathbf{t}_{e_{ij}}$ its tangent unit vector.

Second family: twelve degrees of freedom per tetrahedron – A complete linear interpolation of a three-dimensional vector in a tetrahedron needs twelve degrees of freedom. The corresponding edge element can be obtained by taking two unknowns over each edge of the tetrahedron. Keeping the same notations as the ones used to introduce the first family of edge elements, one of the possibilities is to define the vector basis functions corresponding to an edge e_{ij} going from $\mathbf{r}_i$ to $\mathbf{r}_j$, as follows

$$\mathbf{w}_{ij}(\mathbf{r}) = \lambda_i(\mathbf{r}) \text{ grad } \lambda_j(\mathbf{r}) \quad , \quad i,j = 1,2,3,4,\, i \neq j\,; \tag{23}$$

let us denote by $\mathcal{P}_2(K)$ the space generated by the basis functions defined in (23). The interpolating function $\mathbf{u}_h$ on K for the vectorial state variable $\mathbf{u} \in (C^0(\overline{K}))^3$ has the following form

$$\mathbf{u}_h = \sum_{i=1}^{4} \sum_{j \neq i, j=1}^{4} \mathbf{w}_{ij}\ \beta_{ij}(\mathbf{u}) \quad \text{with} \quad \begin{aligned} \beta_{ij}(\mathbf{u}) &= |e_{ij}|\,(\mathbf{u} \cdot \mathbf{t}_{e_{ij}})(\mathbf{x}_i) \\ \beta_{ji}(\mathbf{u}) &= |e_{ij}|\,(\mathbf{u} \cdot \mathbf{t}_{e_{ij}})(\mathbf{x}_j) \end{aligned}$$

where $\mathbf{x}_i$ and $\mathbf{x}_j$ are the end points of the edge e_{ij}.

Merging the two families – In paper [BBM00], the authors have shown that the mortar method combined with edge elements in three dimensions leads to an approximation which is slightly sub-optimal with the second family and give indications that with the first family non-optimal results could be feared. On the other hand, by using the second family of edge elements in one domain, the number of unknowns for a given mesh is multipled by two. To overcome the difficulties, the idea is based on the following two facts:

- taking the difference of $\mathbf{w}_{ij}$ and $\mathbf{w}_{ji}$ defined in (23) we get the old element $\mathbf{w}_{ij}$ defined in (22); moreover, one element $\mathbf{v} \in \mathcal{P}_1$ can be thought as an element $\mathbf{v} \in \mathcal{P}_2$ with the corresponding degrees of freedom $(\beta_{ij}, \beta_{ji}) = (\alpha_{ij}, -\alpha_{ij})$;
- the Lagrange multipliers of the mortar method are defined locally on Γ^m.

The compromise to have a good approximation without too many unknowns is to limit the use of the second family to all edges that belong to the interface Γ^m. The first family is then adopted to approximate the problem solution along all edges that do not belong to the interface (i.e. over each tetrahedron that does not meet the interface). The space of edge elements $\mathcal{P}$ involved in the definition (9) is the following:

$$\mathcal{P}(K) = \{\mathbf{u} \,|\, \mathbf{u}_{|e} \in \mathcal{P}_1 \,,\, \forall e \notin \partial K \cap \Gamma^m \text{ and } \mathbf{u}_{|e} \in \mathcal{P}_2 \,,\, \forall e \in \partial K \cap \Gamma^m \,\} \,. \tag{24}$$

From the implementation point of view, the merging can be done by introducing a rectangular matrix R_K that depends on the current tetrahedron K as follows:

$$\begin{aligned}
&R_K \in \mathcal{M}(6,12) \quad && \partial K \cap \Gamma^m = \emptyset \text{ or reduced to one point} \\
&R_K \in \mathcal{M}(7,12) \quad && \partial K \cap \Gamma^m \text{ consists of one edge of } K \,, \\
&R_K \in \mathcal{M}(9,12) \quad && \partial K \cap \Gamma^m \text{ consists of one face of } K \,, \\
&R_K \in \mathcal{M}(11,12) \quad && \partial K \cap \Gamma^m \text{ consists of two faces of } K \,.
\end{aligned}$$

$\mathcal{M}(n,m)$ denotes the set of matrices with n rows and m columns. Moreover, the local

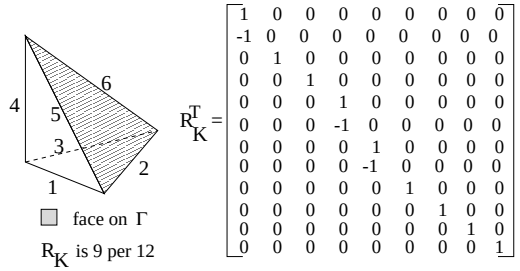

Figure 1: The elements of matrix R_K^T for a given tetrahedron K; to each edge among those with numbers 1, 3, 4, is associated one circulation and to those with numbers 2, 5, 6, are associated two.

stiffness matrix associated to each tetrahedron is built using the second family for only those elements K that meet the interface, i.e. $S_K \in \mathcal{M}(12,12)$ if $\partial K \cap \Gamma^m \neq \emptyset$ nor to one point. In this case, the assembling process does not involve the full matrix S_K but the smaller one given by $R_K S_K R_K^T$ (we have got rid of the additional unknowns for all edges of K that do not lie on Γ^m).

Dealing with the first family

The use of the first family inside each sub-domain together with the second one at the interface glued together with the mortar element method as defined in the second section does not pollute the general accuracy of the problem. In order to analyse this we refer to the standard tool for the analysis of non-conforming approximation: the Berger-Scott-Strang Lemma. This Lemma allows to state:

$$|\mathbf{u} - \tilde{\mathbf{u}}_h|_{*,\Omega} \leq \inf_{\mathbf{v}_h \in \tilde{X}_h^c} |\mathbf{u} - \mathbf{u}_h|_{*,\Omega} + \sup_{\mathbf{v}_h \in \tilde{X}_h^c} \frac{\sum_{k=1}^K \langle \mathbf{v}_h^k \wedge \mathbf{n}_k, \mathbf{curl}\ \mathbf{u} \rangle_{-\frac{1}{2},\frac{1}{2},\partial\Omega_k}}{|\mathbf{v}_h|_{*,\Omega}} \tag{25}$$

where $\tilde{X}_h^c$ denotes the subspace of X_h^c composed of all functions that are of the first family inside the subdomains as described in the previous subsection, and $\tilde{\mathbf{u}}_h$ denotes the solution of problem (13) where X_h^c is replaced by $\tilde{X}_h^c$. The first contribution is known as the best fit of $\mathbf{u}$ by elements of $\tilde{X}_h^c$ and the second contribution is the consistancy error. This second contribution is exactly the same as in the analysis of problem (13) while the former is analyzed following the same steps of the proof of Theorem 2.3: starting from the local approximation od $\mathbf{u}_{|\Omega_k}$ by elements of the first family (e.g. the interpolation $\mathcal{I}_h^k \mathbf{u}_{|\Omega_k}$), we correct the trace value on the slave side of the interfaces by substructing from $\mathcal{I}_h^k \mathbf{u}_{|\Omega_k}$ the function obtained by prolongating by 0 the difference $\Pi_h^m(\mathcal{I}_h^k \mathbf{u}_{|\Omega_k} - \mathcal{I}_h^{-k} \mathbf{u}_{|\Omega_{-k}})$ between the current value on the slave face and the value derived from the application of the mortar condition.

Since the local interpolation operator has the same asymptotic approximation properties in the first and second family, the previous correction is optimal and we can state that the same error bound holds for $\tilde{\mathbf{u}}_h$ as what is stated in Theorem 2.3.

Applications

The flexibily and performance of a numerical method for the simulation of electromagnetic field distributions relies, in several cases, on the possibility of working with non-matching grids at the interface between adjacent sub–domains. One example is given by the treatment of moving structures. Our choice is to work in Lagrangian ones, dealing with non-conforming discretizations at the level of the sliding interface between the stator and the rotor. The second choice is less expensive from the computational point of view if we use a method that avoids re-meshing or interpolation procedures. Another example is the optimization of the structure shape for an electromagnetic device. We can re-mesh either the whole domain or only a region containing the shape to be optimized: in the second case it may be useful to work with non-matching grids to simplify the local re-meshing task and successive solution of the problem. A third example consists in the possibility of coupling variational methods of different others or with unknowns associated to different geometric entities.

Some preliminary results in magnetostatics

Currently, the work in progress consists in applying the described method to compute the distribution of induced currents in moving structures: this is an information of great importance for performances prediction and devices design. Nevertheless, the magnetostatic problem is of great interest due to the fact that we have to face all the difficulties of the method's implementation even if the geometry does not move. The movement treatment would add the additional cost of discretizing the coupling condition at each new position of the free part.

As an example of application, we present some results obtained by solving the magnetostatic problem in terms of the magnetic vector potential $\mathbf{A}$, i.e. the equation $\mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\mathbf{A}) = \mathbf{J}$, with homogeneous boundary conditions. We consider a hexahedrical domain divided into two sub–domains which are discretized by non–structured tetrahedrical coarse meshes. The computational domain in presented in Figure 2 while the magnetic induction $\mathbf{B} = \mathbf{curl}\,\mathbf{A}$ computed on matching and non–matching grids

is displayed in Figure 3. In both cases, the information (i.e. the tangential component of the unknown) is well transmitted from one domain to the other.

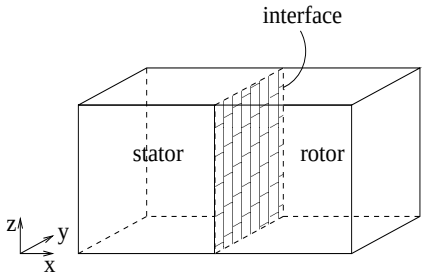

Figure 2: The domain Ω: a flat interface separates the two sub–domains.

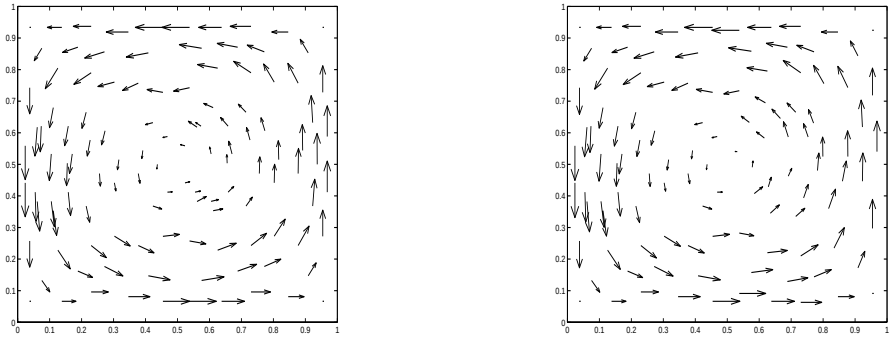

Figure 3: Field $\mathbf{B}$ on the plane $y = .25$ computed on matching (left) and non-matching (right) grids. The stored magnetic energy is ≈ 1.9 MJ in both cases.

Formulation and discretization of the magnetodynamic problem

We are given with a domain $\Omega \subset \mathbb{R}^3$, decomposed in a rotating part (rotor) Ω_1 and a static one (stator) $\Omega_2 = \Omega \setminus \bar{\Omega}_1$. Ω_1 is a cylinder that turns around its axis. Let $\theta \in C^1(0,T)$ be the law of rotation, i.e., $\theta(t)$ denotes the rotation angle at time t and $r_t : \Omega_1 \to \Omega_1$ the rotation operator which turns the domain Ω_1 with an angle $\theta(t)$ and r_{-t} its inverse. Here we suppose for simplicity that $\alpha > 0$ everywhere.

In both domains Ω_1 and Ω_2, we have to solve the equation (2) while the transmission conditions at Γ take into account the movement. They are:

$$r_t \mathbf{u}_1(r_{-t}\mathbf{x}, t) \wedge \mathbf{n}_\Gamma = \mathbf{u}_2(\mathbf{x}, t) \wedge \mathbf{n}_\Gamma \ , \tag{26}$$

$$r_t \beta(r_{-t}\mathbf{x}, t) \mathbf{curl}\, \mathbf{u}_1(r_{-t}\mathbf{x}, t) \wedge \mathbf{n}_\Gamma = \beta(\mathbf{x}, t) \mathbf{curl}\, \mathbf{u}_2(\mathbf{x}, t) \wedge \mathbf{n}_\Gamma \ . \tag{27}$$

Set $\mathcal{H} = H(\mathbf{curl}, \Omega_1) \times H_{0,\partial\Omega}(\mathbf{curl}, \Omega_2)$, we then are led to introduce

$$\mathcal{H}^t = \{\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2) \in \mathcal{H} \mid r_t \mathbf{u}_1(r_{-t}\mathbf{x}, t) \wedge \mathbf{n}_\Gamma = \mathbf{u}_2(\mathbf{x}, t) \wedge \mathbf{n}_\Gamma \ \forall \mathbf{x} \in \Gamma\}. \tag{28}$$

The problem obtained by considering equation (2) in both domain together with homogeneous boundary condition at $\partial\Omega$ and the transmission conditions (26-27) admits a unique solution $\mathbf{u} \in L^\infty(0,T,\mathcal{H}) \cap H^1(0,T,L^2(\Omega))$ when suitably interpreted in a variational sense both in time and space. Note that here the essential transmission condition (26) is strongly imposed in the definition of the functional space, while the natural one (27) is weakly imposed through the variational formulation (this is a consequence of the integration by parts). We are now in the position of making a discretization of this problem and the key point will be the discrete counterpart of the time-dependent constraint characterizing the definition of the space $\mathcal{H}^t$.

The mortar element method proposed in the second section provides an "optimal" spatial discretization of the stated problem. The computational domain is split up into two sub–domains Ω_1 and Ω_2 and the skeleton consists of 3 interfaces (see the Figure 4). Over each sub–domain, we consider the finite element discretization derived in the

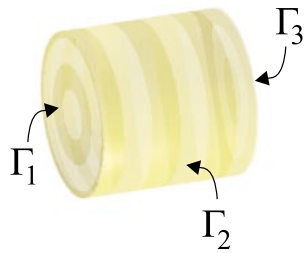

Figure 4: Interfaces for the definition of the mortar element method.

first part of the second section. We call $\mathcal{H}_h^t$ the resulting broken edge element space. The Lagrange multiplier spaces are chosen according to the second section , namely we have M_h^i, $i = 1,2,3$. At each interface, the matching condition turns out to be time-dependent, namely, for any $i = 1,2,3$ and $\mathbf{u}_h = (\mathbf{u}_{1,h}, \mathbf{u}_{2,h}) \in \mathcal{H}_h^t$ we have:

$$\int_{\Gamma_i} \left(r_t \mathbf{u}_{1,h}(r_{-t}\mathbf{x}, t) - \mathbf{u}_{2,h}(\mathbf{x}, t) \right) \wedge \mathbf{n}_\Gamma \cdot \psi_h^i d\Gamma = 0 \quad \forall \psi \in M_h^i.$$

The problem is then discretized in time by means of an implicit Euler method. The analysis of such a formulation is available in the 2D case together with some numerical results (see [BMR99], [Rap00]), and it is in progress for the 3D problem.

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