12th International Conference on Domain Decomposition Methods
Editors: Tony Chan, Takashi Kako, Hideo Kawarada, Olivier Pironneau,

30. A heterogeneous domain decomposition for initial– boundary value problems with conservation laws and electromagnetic fields

C.A. Coclici, W.L. Wendland[1], J. Heiermann, M. Auweter–Kurtz[2]

Introduction

In this paper a nonoverlapping domain decomposition method for the numerical treatment of compressible viscous plasma flows inside a self–field magnetoplasmadynamic (MPD) accelerator is developed. The high–enthalpy magneto–plasma flow is modelled by a system of conservation laws extended by partial differential equations describing the electromagnetic field. Due to the tremendous computational time needed for the numerical solution of the complex equations, the flow–field domain is decomposed into two model zones, characterized by different physical properties of the flow. The complete model of the extended Navier–Stokes equations in the near field of the accelerator is coupled with a simplified model of the extended Euler equations in the far field. The coupling is realized by appropriate transmission conditions at the artificial coupling boundary.

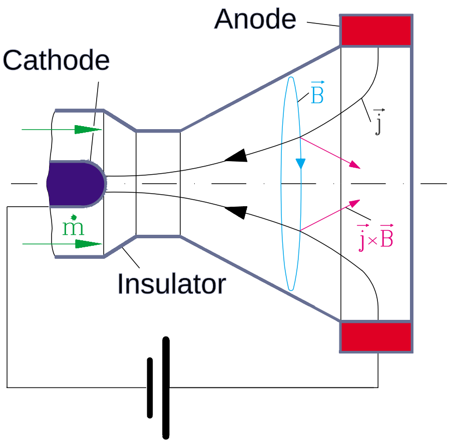

Figure 1: MPD thruster

The principle of a self–field thruster is shown in Figure 1. A cold gas (argon) enters the accelerator and is heated up by an electric discharge to a hot plasma. The plasma expands thermally and accelerates into a test tank in the laboratory. In addition, the plasma is accelerated by electromagnetic Lorentz forces. The flow is described by the conservation equations for mass, momentum and energy for the heavy particles (argon atoms Ar^0 and ions Ar^{1+}, Ar^{2+}), by the conservation equation for the electron and the ionization energy, and by the Maxwell equations of classical electrodynamics.

Furthermore, reaction equilibrium, thermal non–equilibrium (two–fluid model), and laminar flow are assumed at this time.
The complete system of governing equations is employed within an essentially smaller near–field region Ω_1, containing the thruster, and is coupled by appropriate

[1]Mathematical Institute A, University of Stuttgart, {coclici, wendland}@mathematik.uni-stuttgart.de
[2]Institute of Space Systems, University of Stuttgart {heierman, auweter}@irs.uni-stuttgart.de

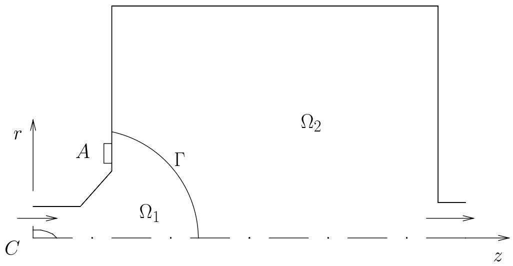

Figure 2: Decomposition of the computational domain

transmission conditions across the artificial boundary Γ with a simplified model in the complementary far field Ω_2, corresponding to the test tank. Generally, the far–field simplifications should be chosen in such a way, that on one hand the flow in the far–field domain is still modelled accurately enough, and on the other hand, the numerical treatment can be performed efficiently.

The axisymmetric plasma flow is described in cylindrical coordinates by the vector–valued function

$$\mathbf{W} = \mathbf{W}(r,z;\,t) := \big[\,\mathbf{w}, p_H, T_H;\; \mathbf{w}_e;\; \mathbf{w}_{EB}\,\big]^\top(r,z;\,t), \quad (r,z)\in\Omega,\; t\in[0,T].$$

Here, $\mathbf{w}=(\rho,\rho v_r,\rho v_z,E_H)^\top$ collects the conservative variables with the density ρ, the velocity vector $\mathbf{v}=(v_r,v_z)^\top$, and the energy of the heavy particles E_H. The pressure and the temperature of the heavy particles are denoted by p_H and T_H, respectively. The function $\mathbf{w}_e=(\mathrm{e}_{ei},p_e,T_e)^\top$ describes the electron component of the plasma, with e_{ei} containing the electron and the ionization energy, and with p_e and T_e representing the pressure and the temperature of the electron component, respectively. Finally, $\mathbf{w}_{EB} = (\mathbf{E},\mathbf{B},\mathbf{j})^\top$ contains the electromagnetic field $(\mathbf{E},\mathbf{B})$ and the electric current density $\mathbf{j}$.

Modelling of the near field

The heavy–particle flow is modelled by the compressible Navier–Stokes equations which are extended due to the influence of an arc discharge. They take in cylindrical coordinates the form

$$\frac{\partial \mathbf{w}_1}{\partial t} + \mathrm{div}_{(r,z)}\mathbf{F}(\mathbf{W}_1) = \mathrm{div}_{(r,z)}\mathbf{R}(\mathbf{w}_1,\nabla_{(r,z)}\mathbf{w}_1) + \mathbf{G}(\mathbf{W}_1) \;\text{ in }\; \Omega_1\times[0,T]. \qquad (1)$$

The function $\mathbf{F}$ contains the convective part of the Navier–Stokes equations (here, with the pressure field $p = p_H + p_e$), and, in addition, an electromagnetic pressure term derived from the source terms. We represent $\mathbf{F}$ as

$$\mathbf{F} = (\mathbf{F}_r,\mathbf{F}_z)(\mathbf{W}) = (\mathbf{f}_r,\mathbf{f}_z)(\mathbf{w},\mathbf{w}_e) + (\mathbf{g}_r,\mathbf{g}_z)(\mathbf{w}_{EB}),$$

where, with the purely azimuthal magnetic field $\mathbf{B} = (0,B,0)^\top$ and with the magnetic permeability of vacuum $\mu_0 > 0$,

$$\begin{aligned}
\mathbf{f}_r(\mathbf{w},\mathbf{w}_e) &:= \big(\rho v_r,\; \rho v_r^2 + (p_H+p_e),\; \rho v_r v_z,\; [E_H + (p_H+p_e)]\; v_r\big)^\top,\\
\mathbf{f}_z(\mathbf{w},\mathbf{w}_e) &:= \big(\rho v_z,\; \rho v_z v_r,\; \rho v_z^2 + (p_H+p_e),\; [E_H + (p_H+p_e)]\; v_z\big)^\top,\\
\mathbf{g}_r(\mathbf{w}_{EB}) &:= \big(0,\, B^2,\, 0,\, B^2 v_r\big)^\top\!/(2\mu_0),\;\; \mathbf{g}_z(\mathbf{w}_{EB}) := \big(0,\, 0,\, B^2,\, B^2 v_z\big)^\top\!/(2\mu_0).
\end{aligned}$$

The viscous terms are collected in the function $\mathbf{R} = (\mathbf{R}_r, \mathbf{R}_z)(\mathbf{w}, \nabla_{(r,z)}\mathbf{w})$ where

$$\begin{aligned}\mathbf{R}_r(\mathbf{w}, \nabla_{(r,z)}\mathbf{w}) &:= \big(0,\ \tau_{rr},\ \tau_{rz},\ \tau_{rr}v_r + \tau_{rz}v_z + \lambda_H\ \partial T_H/\partial r\big)^\top,\\ \mathbf{R}_z(\mathbf{w}, \nabla_{(r,z)}\mathbf{w}) &:= \big(0,\ \tau_{zr},\ \tau_{zz},\ \tau_{zr}v_r + \tau_{zz}v_z + \lambda_H\ \partial T_H/\partial z\big)^\top,\end{aligned}$$

with the heat conductivity $\lambda_H > 0$ of the heavy–particle flow, and with

$$\tau_{rr} = \mu\left[2\frac{\partial v_r}{\partial r} - \frac{2}{3}\operatorname{div}\mathbf{v}\right],\ \tau_{rz} = \tau_{zr} = \mu\left[\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right],\ \tau_{zz} = \mu\left[2\frac{\partial v_z}{\partial z} - \frac{2}{3}\operatorname{div}\mathbf{v}\right]$$

defining the components of the viscous part of the stress tensor; $\mu > 0$ represents the viscosity coefficient. The function $\mathbf{G}$ contains the electromagnetic force and heat terms as well as quantities describing the heat transfer due to the collisions between the plasma components:

$$\begin{aligned}\mathbf{G}(\mathbf{W}) := \Bigg(0,\ &\frac{1}{r}\left[p_H + p_e - \frac{2}{3}\mu\Big(2\frac{v_r}{r} - \frac{\partial v_r}{\partial r} - \frac{\partial v_z}{\partial z}\Big) - \frac{B^2}{2\mu_0}\right],\ 0,\\ &\Big(p_e + \frac{B^2}{2\mu_0}\Big)\operatorname{div}\mathbf{v} - \frac{v_r}{r}\frac{B^2}{\mu_0} + \sum_{\nu=0}^{2} n_\nu n_e \alpha_{e\nu}(T_e - T_H)\Bigg)^\top.\end{aligned}$$

Here, n_ν ($\nu = 0, 1, 2$) and n_e are the densities of the heavy particles and of the electrons, respectively, and $\alpha_{e\nu}$ are heat transfer coefficients. Note that, by including the Lorentz terms $\mathbf{j}\times\mathbf{B}$ as $B^2/(2\mu_0)$ in the fluxes, our formulation observes as much conservation as possible. Consequently, conservative numerical methods (as e.g. the finite volume method) are good candidates to be used for the numerical treatment of the problem. The conservation of the electron and ionization energy is given in $\Omega_1 \times [0, T]$ by

$$\begin{aligned}\frac{\partial \mathrm{e}_{ei}}{\partial t} + \operatorname{div}(\mathrm{e}_{ei}\mathbf{v}) - \operatorname{div}(\lambda_{ei}\nabla T_e) &= -p_e \operatorname{div}\mathbf{v} + \frac{5}{2}\frac{k}{e}\mathbf{j}\cdot\nabla T_e - \frac{1}{n_e e}\mathbf{j}\cdot\nabla p_e\\ &\quad + \sum_{\nu=0}^{2} n_\nu n_e \alpha_{e\nu}(T_H - T_e) + \frac{|\mathbf{j}|^2}{\sigma}.\end{aligned} \tag{2}$$

Here, λ_{ei} denotes the heat conductivity for the electron component of the flow, k is the Boltzmann constant, and σ is the electric conductivity. The Maxwell equations and Ohm's law for plasmas read

$$\operatorname{rot}\mathbf{B} = \mu_0\mathbf{j},\quad \operatorname{rot}\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t},\quad \operatorname{div}\mathbf{B} = 0; \qquad \mathbf{E} = \frac{\mathbf{j}}{\sigma} - \mathbf{v}\times\mathbf{B} + \beta\,\mathbf{j}\times\mathbf{B} - \beta\,\nabla p_e$$

(β – Hall parameter), leading to the discharge equation

$$\frac{\partial^2 B}{\partial r^2} + \frac{\partial^2 B}{\partial z^2} + \left[\frac{1}{r} - \mu_0\sigma v_r\right]\frac{\partial B}{\partial r} - \mu_0\sigma v_z\frac{\partial B}{\partial z} - \left[\frac{1}{r^2} + \mu_0\sigma\Big(\frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z}\Big)\right]B = F_B, \tag{3}$$

where $F_B = F_B(\sigma, p_e, \mu_0)$ denotes a source term. Additional equilibrium reactions are incorporated into our model but, for brevity, they are not given here explicitly (for more details, see e.g. [Sle99]).

In order to get a more profound understanding of the complex physical processes involved, theoretical and numerical investigations have been performed [Sle99, WKAK98]. Continuing this work, the mathematical formulations of the conservation equations have been extended in [HAKE+99], where an advanced numerical finite–volume code has been written in order to capture the plasma flow physics accurately.

However, due to the high complexity of the model and the tremendous computational costs, the system has been discretized only in the vicinity Ω_1 of the MPD thruster, identified here as the near field. Γ is there considered as outflow (freestream) boundary and characteristic boundary conditions, using data obtained from measurements, are used. In that model one faces the problem that a certain amount of ambient (cold) far–field gas *recirculates* into the hot plasma jet in the near field. Consequently, parts of the outflow boundary Γ get "inflow" properties and require additional information about the flow quantities. This makes the numerical treatment of the plasma flow in the far field Ω_2 necessary. We consider in Ω_2 a simplified model which takes into account the physical properties of the flow, and couple this model to the complete system in Ω_1. Our coupling procedure extends previous work on heterogeneous domain decomposition in aerodynamics (see, e.g. [QS95, Coc98, CW01]) to the case of compressible magneto–plasma flows.

Simplified modelling of the far field

In a first approximation we assume that far away from the MPD accelerator the shear stresses $\tau_{rr}, .., \tau_{zz}$ and the heat conduction terms $\lambda_H \partial_{r(z)} T_H$, defining the quantity $\mathbf{R}$, are strongly dominated by the convective part. Hence, we assume the heavy–particles flow to be inviscid in Ω_2. At the moment, we also assume that the magnetic field $\mathbf{B}$ vanishes identically in Ω_2. The system of conservation laws takes the simplified form

$$\frac{\partial \mathbf{w}_2}{\partial t} + \operatorname{div}_{(r,z)}(\mathbf{f}_r, \mathbf{f}_z)(\mathbf{w}_2, \mathbf{w}_{e,2}) = \mathbf{H}(\mathbf{w}_2, \mathbf{w}_{e,2}) \quad \text{in} \quad \Omega_2 \times [0, T], \tag{4}$$

with the simplified source term

$$\mathbf{H}(\mathbf{w}_2, \mathbf{w}_{e,2}) := \left(0,\ \frac{p_H + p_e}{r},\ 0,\ p_e \operatorname{div} \mathbf{v} + \sum_{\nu=0}^{2} n_\nu n_e \alpha_{e\nu}(T_e - T_H)\right)^{\top}.$$

Furthermore, as a consequence of $\mathbf{j} = \operatorname{rot} \mathbf{B}/\mu_0 \equiv 0$ in Ω_2, the equation of conservation of electron and the ionization energy (2) becomes in $\Omega_2 \times [0, T]$

$$\frac{\partial \mathrm{e}_{ei}}{\partial t} + \operatorname{div}(\mathrm{e}_{ei}\mathbf{v}) - \operatorname{div}(\lambda_{ei} \nabla T_e) = -p_e \operatorname{div} \mathbf{v} + \sum_{\nu=0}^{2} n_\nu n_e \alpha_{e\nu}(T_H - T_e). \tag{5}$$

Transmission conditions

These conditions should be chosen in such a way, that on one hand, the fundamental physical laws are respected, and on the other hand, the resulting coupled problem is well–posed and consistent with the full original problem. The continuity of the characteristic variables could be chosen as transmission condition, but according to the theory of hyperbolic equations, this can be required only across that part of the interface, where the corresponding characteristics enter the hyperbolic region, see e.g. [Hir88]. We also refer to [QS95, Coc98, CW01], where the continuity of the Riemann invariants across the inflow part and compatibility conditions across the outflow part of the boundary are used. In accordance with the conservation laws, the continuity of the normal flux yields a transmission condition on the complete interface Γ: the total flux associated with the full model in Ω_1 (containing the inviscid as well as the viscous contributions) is set equal to the normal inviscid flux, that results from the simplified equations in Ω_2:

$$-\big[\mathbf{R}_r(\mathbf{w}_1,\nabla\mathbf{w}_1)n_r+\mathbf{R}_z(\mathbf{w}_1,\nabla\mathbf{w}_1)n_z\big]+\big[\mathbf{f}_r(\mathbf{w}_1,\mathbf{w}_{e,1})+\mathbf{g}_r(\mathbf{w}_{EB,1})\big]\,n_r$$

$$+\big[\mathbf{f}_z(\mathbf{w}_1,\mathbf{w}_{e,1})+\mathbf{g}_z(\mathbf{w}_{EB,1})\big]\,n_z=\mathbf{f}_r(\mathbf{w}_2,\mathbf{w}_{e,2})\,n_r+\mathbf{f}_z(\mathbf{w}_2,\mathbf{w}_{e,2})\,n_z \qquad (6)$$

across Γ. The flux condition has successfully been used for pure flow problems (see, for example, [QS95, CW01]). However, it implies that the solutions of the coupled problem may exhibit jumps at the interface, depending on the magnitude of the viscosity and heat transfer terms neglected in the far field (for more details, see [Coc98]). Since the solution of the original problem should satisfy the *natural* transmission conditions at the artificial interface (i.e. continuity of the solution and of the total normal flux), the approximate extended Navier–Stokes / extended Euler solution can only be a first approximation and needs to be corrected by special terms accounting for the loss of continuity and maintaining the continuity of the normal flux. A boundary layer correction for a simplified transmission problem is presented in [CW00] in the framework of singular perturbation theory. This analysis is extended for the problem under consideration in [CMW00].
In order to assure the electron heat transfer across the interface, we impose the continuity of the co–normal derivative of the electron temperature T_e:

$$\Big[\lambda_{ei,1}\frac{\partial T_{e,1}}{\partial\mathbf{n}}\Big](r,z)=\Big[\lambda_{ei,2}\frac{\partial T_{e,2}}{\partial\mathbf{n}}\Big](r,z)\quad\text{for all}\quad(r,z)\in\Gamma. \qquad (7)$$

Finally, we impose $B\equiv 0$ on Γ.

Numerical aspects and results

In the numerical code, the extended conservation laws (1) and (4), describing the heavy–particle motion, as well as the electron and the ionization energy equations (2) and (5) are solved on an unstructured, dual mesh by using a second–order finite volume upwind scheme based on explicit Euler time–stepping. The discharge equation (3) is currently solved by triangular finite elements with linear ansatz functions using an SOR scheme. For a detailed description we refer to [HAKE$^+$99]. A finite volume formulation is in preparation for this conservation equation.

The full computational domain including the near field of the MPD accelerator and the far field corresponding to the tank, are shown in Figure 3. The area of the far field is about 80 times larger than that of the near field, emphasizing the necessity of simplifying the mathematical model in the far field.

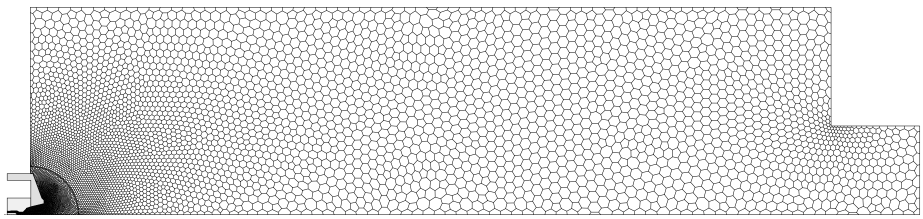

Figure 3: Full computational domain

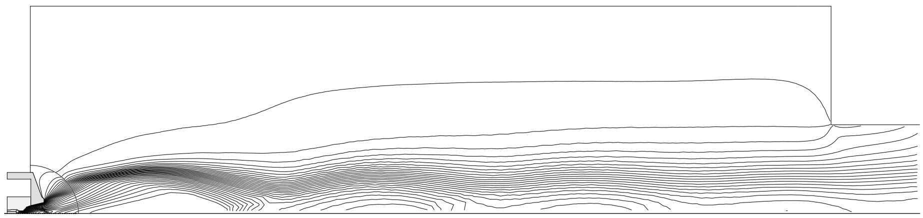

Figure 4: Isolines of v_z

The isolines of the axial velocity component v_z give an overall impression of the plasma flow: The plasma is accelerated in the MPD accelerator, then it is expanded into the tank, and finally it flows out of the tank at the far right.

The coupling domain, where dual cells on both sides of the coupling boundary touch each other, is shown in Figure 5 (left). Both meshes are produced with an advancing front algorithm. The mesh generator enforces the global mesh to be conforming at the artificial coupling boundary.

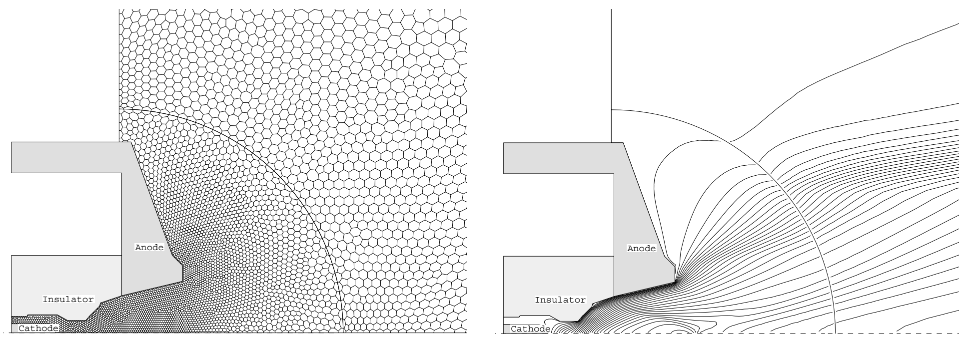

Figure 5: Computational domain for the coupling (left); isolines of v_z (right)

The isolines of the longitudinal velocity v_z for the coupled solution are presented on the right. Obviously, the coupling method works very well for the central, hot plasma jet. Up to one nozzle radius above the centerline, v_z passes smoothly the coupling boundary Γ. However, farther away from the centerline, the isolines become slightly discontinuous and do not cross the interface smoothly.

It turns out that the neglection of the heat conduction terms corresponding to the heavy–particle flow is significant, see Figure 6 (left). While the isolines of the heavy–particle temperature T_H behave smoothly across the part of the interface Γ contained in the central plasma jet, we can see the discontinuities of the solution in the region above very clearly. The explanation for this is that the heavy–particle heat conduction is still physically relevant with respect to the inviscid Euler energy flux, such that the heavy particle heat conduction cannot be neglected in this geometrical decomposition. The local discontinuity of T_H also causes a slight discontinuity and non–smoothness of v_z in the critical region. Therefore, the far field domain will be further decomposed into a small intermediate domain attached to the near field and the complementary far–field region. In the intermediate field, the heavy–particle heat conduction will be considered, while the components of the viscous stress tensor will be neglected. Also a rigorous dimension analysis of the flow quantities is necessary to justify the use of the intermediate model.

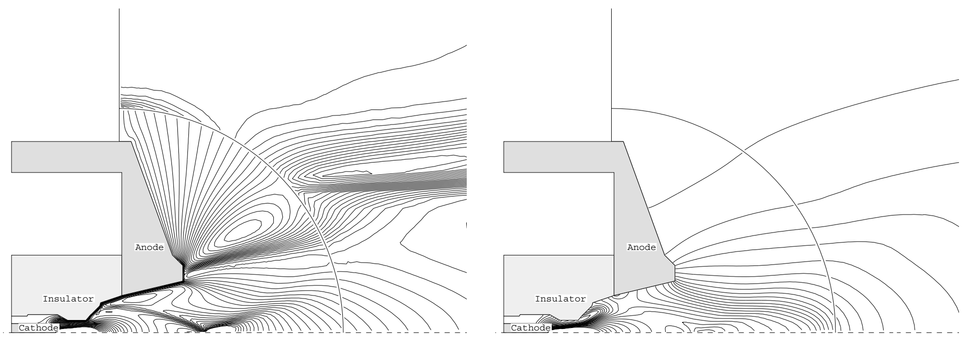

Figure 6: Isolines of the heavy–particles temperature T_H (left) and of the electron temperature T_e (right)

The approximate coupled solution also shows that the electron temperature T_e passes the artificial interface smoothly, as can be seen in Figure 6 (right). Thus, the *natural* transmission condition (7), used for the coupling of the equations (2) and (5) is justified also numerically.

Finally, we outline that by using the heterogeneous domain decomposition, the very complex compressible magneto–plasma flow has been computed for the first time within the whole MPD accelerator plus tank configuration, and that the influence of the far field through the recirculating amount of gas has been simulated numerically. Our investigation shows that the heterogeneous domain decomposition method is an excellent tool which can be efficiently used in the numerical treatment of nonlinear boundary value problems of high complexity.

Acknowledgement

This work has been supported by the German Research Foundation DFG (Deutsche Forschungsgemeinschaft) through the projects "Domain Decomposition Methods and Far–Field Boundary Conditions for Initial Boundary Value Problems with Conservation Laws and Electromagnetic Fields" (We 659/35–1) and "Conservation Equations and Numerical Solutions for Technically Relevant Magneto–Plasmas" (Au 85/9–1,2). The authors thank H.J. Kaeppeler for many stimulating discussions and helpful contributions.

References

[CMW00]C.A. Coclici, G. Morosanu, and W.L. Wendland. Asymptotic correction of the numerical solution to a heterogeneous domain decomposition for compressible plasma flows. In preparation, 2000.

[Coc98]C.A. Coclici. *Domain Decomposition Methods and Far–Field Boundary Conditions for Compressible Flows Around Airfoils*. PhD thesis, Mathematical Institute A, University of Stuttgart, Pfaffenwaldring 57, D-70569 Stuttgart, Germany, 1998.

[CW00]C.A. Coclici and W.L. Wendland. The coupling of hyperbolic and elliptic boundary value problems with variable coefficients. *Math. Meth. Appl. Sci.*, 23:401–440, 2000.

[CW01]C.A. Coclici and W.L. Wendland. Analysis of a heterogeneous domain decomposition for compressible viscous flow. *Math. Models Meth. Appl. Sci.*, 4(11), 2001. accepted for publication.

[HAKE+99]J. Heiermann, M. Auweter-Kurtz, A. Eberle, H.J. Kaeppeler, U. Iben, and P.C. Sleziona. Recent improvements of numerical methods for the simulation of mpd thruster flow on adaptive meshes. In *Proceedings of the 26th International Electric Propulsion Conference, Kitakyushu*, 1999.

[Hir88]C. Hirsch. *Numerical Computation of Internal and External Flows*. Wiley, 1988.

[QS95]A. Quarteroni and L. Stolcis. Homogeneous and heterogeneous domain decomposition for compressible fluid flows at high reynolds numbers. *Numerical Methods for Fluid Dynamics*, 5:113–128, 1995.

[Sle99]P.C. Sleziona. *High Enthalpy Flows for Spaceflight Applications (in German)*. Shaker Verlag, Aachen, 1999.

[WKAK98]H.P. Wagner, H.J. Kaeppeler, and M. Auweter-Kurtz. Instabilities in mpd thruster flows: 1. space charge instabilities in unbounded and inhomogeneous plasmas. *Journal of Physics D: Appl. Phys.*, 31:519–528, 1998.