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31. A Defect Correction Method for the Retrieval of Acoustics Waves

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Introduction

For a given mathematical problem and a given approximate solution, the residue or defect may be defined as a quantity to measure how well the problem has been solved. Such information may then be used in a simplified version of the original mathematical problem to provide an appropriate correction quantity. The correction can then be applied to correct the approximate solution in order to obtain a better approximate solution to the original mathematical problem. Such idea has been around for a long time and in fact has been used in a number of different ways.

A famous example of defect correction is the computation of a solution to the nonlinear equation $f(x) = 0$. Suppose $\bar{x}$ is an approximate solution, then $-f(\bar{x})$ is the defect. One possible version of the original problem is to define $\bar{f}(x) \equiv f'(\bar{x})(x - \bar{x}) + f(\bar{x}) = 0$. In fact, if one replaces $x - \bar{x}$ as v, then v is the correction which is obtained by solving $f'(\bar{x})v = -f(\bar{x})$ and an updated approximation can be obtained by evaluating $x := \bar{x} + v$. Most defect correction are used in conjunction with discretisation methods and two-level multigrid methods [BS84]. This paper is not intended to give an overview of defect correction methods but to use the basic concept of a defect correction in conjunction with fluctuations in flow field variables for sound and noise retrieval.

Recall that sound waves - manifested as pressure fluctuations - are typically several orders of magnitude smaller than the pressure variations in the flow field that account for flow acceleration. Furthermore, they propagate at the speed of sound in the medium, not as a transported fluid quantity. A decomposition of variables was first introduced in [DLP97] and has been further examined in [Dja98] to include three types of components. These components include (1) the mean flow, (2) flow perturbations or aerodynamic sources of sound, and (3) the acoustic perturbation. We have demonstrated the accurate computation of (1) and (2) in [DLP98]. Mathematically, the flow variable U may be written as $\bar{u} + u$ where $\bar{u}$ denotes the mean flow and part of aerodynamic sources of sound and u denotes the remaining part of the aerodynamic sources of sound and the acoustic perturbation.

While flow perturbation or aerodynamic sources of sound may be easier to recover, it is not true for the acoustic perturbation because of its comparatively small magnitude. In fact, the solutions of the Reynolds averaged Navier-Stokes equations reveal only a truncated part of the full physical quantities. This paper follows the basic principle of the defect correction as discussed above and applies the concept to the recovery of the propagating acoustic perturbation. The method relies on the use of a lower order partial differential equation defined on the same computational domain

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where a residue exists such that the acoustic perturbation may be retrieved through a properly defined coarse mesh.

This paper is organised as follows. First, derivation of a lower order partial differential equation resulting from the Navier-Stokes equations is given. Second, accurate representation of residue on a coarse mesh is discussed. The coarse mesh is designed in such a way as to allow various frequencies of noise to be studied. Suitable interpolation operators are studied for the two different meshes. Third, 1-D and 2-D examples are used to illustrate the concept. Finally, future work is discussed.

The Defect Correction Method

The aim here is to solve the non-linear equation

$$\mathcal{L}U := \mathcal{L}(\bar{u} + u) = 0 \tag{1}$$

where $\mathcal{L}$ is a non-linear operator depending on $U := \bar{u} + u$. Using the concept of decomposition of variable, U is now written as $\bar{u} + u$, where $\bar{u}$ is the mean flow and u is the acoustic perturbation. Note that $u \ll \bar{u}$. In the case of sound generated by the motion of fluid, it is natural to imagine $\mathcal{L}$ as the Navier-Stokes operator. For a 2-D problem,

$$\bar{u} = \begin{bmatrix} \bar{\rho} \\ \bar{v}_1 \\ \bar{v}_2 \end{bmatrix} \qquad u = \begin{bmatrix} \rho \\ v_1 \\ v_2 \end{bmatrix}$$

where ρ is the density of fluid and v_1 and v_2 are the velocity components along the two spatial axes. Using the summation notation of subscripts, the 2-D Navier-Stokes problem $\mathcal{L}u = 0$ is written as

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_j}{\partial x_j} = 0$$

and

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} + \frac{1}{\rho}\frac{\partial P}{\partial x_i} - \frac{f_i}{\rho} = 0$$

where P is the pressure and f_i is the external force along i-th axis.

Suppose (1) may be split and re-written as

$$\mathcal{L}(\bar{u} + u) \equiv \mathcal{L}\bar{u} + E\{\bar{u}\}u + K[\bar{u}, u] \tag{2}$$

where $E\{\bar{u}\}$ is an operator depending on the knowledge of $\bar{u}$ and $K[\bar{u}, u]$ is a functional depending on the knowledge of both $\bar{u}$ and u. Following the concept of defect correction, $\bar{u}$ may be considered as an approximate solution to (1). Hence one can evaluate the residue of (1) as

$$R \equiv \mathcal{L}(\bar{u} + u) - \mathcal{L}\bar{u} = -\mathcal{L}\bar{u}$$

which may then be substituted into (2) to give

$$E\{\bar{u}\}u + K[\bar{u}, u] = R \tag{3}$$

In many cases, $K[\bar{u}, u]$ is small and can then be neglected. In those cases, the problem in (3) is a linear problem and may be solved more easily to obtain the acoustics fluctuation u. A non-linear iterative solver is required in order to obtain u for cases when $K[\bar{u}, u]$ is not negligible. Finally, to obtain the approximate solution $\bar{u}$, one only needs to solve $\mathcal{L}\bar{u} = 0$.

Expanding $\mathcal{L}(\bar{u} + u) = 0$ for $\mathcal{L}$ being the Navier-Stokes operator and re-arranging we obtain

$$\frac{\partial \rho}{\partial t} + \bar{v}_j \frac{\partial \rho}{\partial x_j} + \bar{\rho}\frac{\partial v_j}{\partial x_j} + [v_j \frac{\partial(\bar{\rho} + \rho)}{\partial x_j} + \rho\frac{\partial(\bar{v}_j + v_j)}{\partial x_j}] = -[\frac{\partial \bar{\rho}}{\partial t} + \bar{v}_j \frac{\partial \bar{\rho}}{\partial x_j} + \bar{\rho}\frac{\partial \bar{v}_j}{\partial x_j}]$$

and

$$\frac{\partial v_i}{\partial t} + \bar{v}_j \frac{\partial v_i}{\partial x_j} + \frac{1}{\bar{\rho}}\frac{\partial P}{\partial x_i} - \frac{f_i}{\bar{\rho}}$$

$$+[\frac{\rho}{\bar{\rho}}\frac{\partial(\bar{v}_i + v_i)}{\partial t} - (v_j + \frac{\rho}{\bar{\rho}}(\bar{v}_j + v_j))\frac{\partial(\bar{v}_i + v_i)}{\partial x_j}] = -[\frac{\partial \bar{v}_i}{\partial t} + \bar{v}_j \frac{\partial \bar{v}_i}{\partial x_j} + \frac{1}{\bar{\rho}}\frac{\partial \bar{P}}{\partial x_i} - \frac{\bar{f}_i}{\bar{\rho}}] \quad (4)$$

It can be seen that (4) may be written in the form of (3) where

$$E\{\bar{u}\}u = \begin{bmatrix} \frac{\partial \rho}{\partial t} + \bar{v}_j \frac{\partial \rho}{\partial x_j} + \bar{\rho}\frac{\partial v_j}{\partial x_j} \\ \frac{\partial v_i}{\partial t} + \bar{v}_j \frac{\partial v_i}{\partial x_j} + \frac{1}{\bar{\rho}}\frac{\partial P}{\partial x_i} - \frac{f_i}{\bar{\rho}} \end{bmatrix} \quad (5)$$

$$K[\bar{u}, u] = \begin{bmatrix} v_j \frac{\partial(\bar{\rho}+\rho)}{\partial x_j} + \rho\frac{\partial(\bar{v}_j+v_j)}{\partial x_j} \\ \frac{\rho}{\bar{\rho}}\frac{\partial(\bar{v}_i+v_i)}{\partial t} - (v_j + \frac{\rho}{\bar{\rho}}(\bar{v}_j + v_j))\frac{\partial(\bar{v}_i+v_i)}{\partial x_j} \end{bmatrix} \quad (6)$$

$$R = \begin{bmatrix} -[\frac{\partial \bar{\rho}}{\partial t} + \bar{v}_j \frac{\partial \bar{\rho}}{\partial x_j} + \bar{\rho}\frac{\partial \bar{v}_j}{\partial x_j}] \\ -[\frac{\partial \bar{v}_i}{\partial t} + \bar{v}_j \frac{\partial \bar{v}_i}{\partial x_j} + \frac{1}{\bar{\rho}}\frac{\partial \bar{P}}{\partial x_i} - \frac{\bar{f}_i}{\bar{\rho}}] \end{bmatrix} \equiv -\mathcal{L}\bar{u} \quad (7)$$

From the knowledge of physics of fluids, the acoustic perturbations ρ and v_j are of very small magnitude (this is not true for their derivatives), therefore, K may be considered negligible due to the reason that any feedback from the propagating waves to the flow can be completely ignored, except in some cases of acoustic resonance, which is impossible to occur in the examples of this paper. Hence the equation $E\{\bar{u}\}u = R$, with E given by (5), which is known as the linearised Euler equation, can be solved in an easier way. The remaining question is to obtain the approximate solution $\bar{u}$ to the original problem (2). It is well known that CFD analysis packages provide excellent methods for the solution of $\mathcal{L}\bar{u} = 0$. Therefore one requires to use a Reynolds averaged Navier-Stokes package supplemented with turbulence models such as [CPC95] to provide a solution of $\bar{u}$. Physically, one requires $\bar{u}$ to be as accurate as possible to capture all the physics such as turbulence and vortices.

A Two-Level Numerical Scheme

In order to simulate accurately the approximate solution, $\bar{u}$, to the original problem, $\mathcal{L}U = 0$, the QUICK differencing scheme [Leo79] is used which produces sufficiently accurate results of $\bar{u}$ for the purpose of evaluating the residue as defined in (7). A sufficiently fine mesh has to be used in order to preserve vorticity motion. However, situations are different for the numerical solutions of linearised Euler equations [DLP97][Dja98], where the mesh has to be much coarser in order to obey Courant limit and to account for the fact that the acoustic wavelength is larger than the vortex diameter. An account on various high order finite difference schemes and its mesh requirements for the numerical solution of linearised Euler equations can be found in [Dja98]. After evaluating the residue on the fine mesh, it is then required to transfer these residuals onto the coarser mesh. Physically, the residue is in fact the sound source that will disappear without the proper retrieval technique as discussed in this paper.

Let h denote the mesh to be used in the Reynolds averaged Navier-Stokes solver. Instead of evaluating $\bar{u}$, one would solve the discretised approximation $\mathcal{L}_h\bar{u}_h = 0$ to obtain $\bar{u}_h$. The residue on the fine mesh h can be computed as $\mathcal{L}u_h$ by means of a higher order approximation [Dja98]. Let H denote the mesh for the linearised Euler equations solver. Again instead of evaluating u, one would solve the discretised approximation $E_H\{\bar{u}_H\}u_H = R_H$ to obtain u_H. Here R_H is the projection of R onto the mesh H. Let $I_{\{h,H\}}$ be a projection operator to project the residue computed on the fine mesh h to the coarser mesh H. The projected residue can then be used in the numerical solutions of linearised Euler equations. Let $I_{\{H,h\}}$ be an interpolation operator to interpolate the acoustic signals from the coarser mesh back to the finer mesh. Therefore the two-level numerical scheme is

For non-resonance problems:
Solve $\mathcal{L}_h\bar{u}_h = 0$
$R_H := -I_{\{h,H\}}\mathcal{L}\bar{u}_h$
$\bar{u}_H := I_{\{h,H\}}\bar{u}_h$
Solve $E_H\{\bar{u}_H\}u_H = R_H$
$u_h := I_{\{H,h\}}u_H$
$\bar{u} := \bar{u}_h + u_h$

Note that R_H cannot be computed as $\mathcal{L}I_{\{h,H\}}\bar{u}_h$ because $\mathcal{L}$ is a non-linear operator.

In the actual implementation, a pressure-density relation which also defines the speed of sound c in air is used:

$$\frac{\partial P}{\partial \rho} = c^2 \approx 1.4\frac{\bar{P}}{\bar{\rho}} \tag{8}$$

and the first component of the linearised Euler equations in (5) becomes

$$\frac{\partial P}{\partial t} + \bar{v}_j\frac{\partial P}{\partial x_j} + \bar{\rho}c^2\frac{\partial v_j}{\partial x_j} = -c^2[\frac{\partial \bar{\rho}}{\partial t} + \bar{v}_j\frac{\partial \bar{\rho}}{\partial x_j} + \bar{\rho}\frac{\partial \bar{v}_j}{\partial x_j}] \tag{9}$$

The purpose of this substitution is to make sure that the new fluctuations P and v_i do not contain a hydrodynamic component, and hence it allows them to be resolved on regular Cartesian meshes [Dja98] which is essential for the accurate representation of the acoustic waves or the fluctuation quantity u. On the other hand, an unstructured mesh is usually used to obtain $\bar{u}_h$. The two different meshes overlapped one another on the computational domain. The computational domain for the linearised Euler equations is not necessarily exactly the same as the one for the CFD solutions. However, the computational domain for the linearised Euler equations must be large enough to contain the longest wavelength of a particular problem under consideration. The numerical examples as shown in next section do not involve any solid objects, therefore $I_{\{h,H\}}$ and $I_{\{H,h\}}$ are simply arithmetic averaging processes.

Numerical Examples

To test the feasibility of this approach the simple 1-D example is considered of an initial-value wave propagation problem with exact solution

$$\begin{aligned} P &= f(x-ct)+f(x+ct) \\ \bar{\rho}cv_1 &= f(x-ct)-f(x+ct) \\ f(x) &= \begin{cases} \frac{A}{2}(1+\cos 2\pi\frac{x}{\lambda}), |x| < \frac{\lambda}{2} \\ 0, |x| \geq \frac{\lambda}{2} \end{cases} \end{aligned} \tag{10}$$

where A is the amplitude and λ is the wavelength of the pressure pulses that start from the origin ($x = 0$) at $t = 0$.

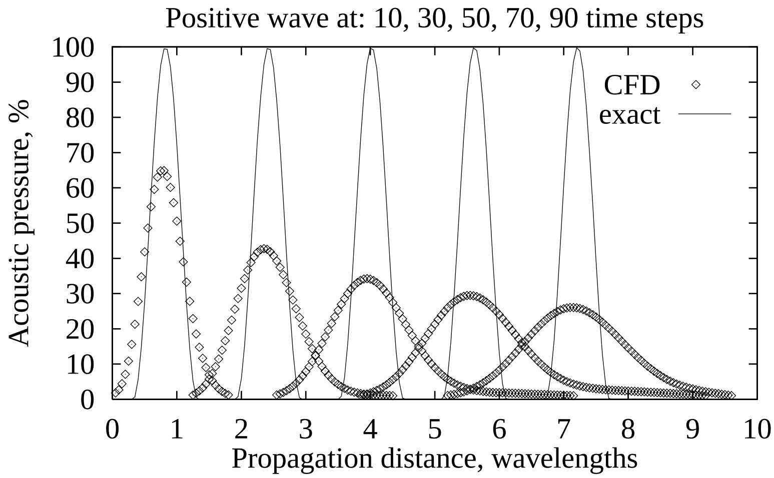

Figure 1: Analytic and preliminary CFD solutions of test problem

For the fine grid problem, $\mathcal{L}_h\bar{u}_h = 0$, the initial conditions $P = 2f(x)$ and $\bar{v}_1 = 0$ were prescribed for the CFD solution which uses a structured finite volume code [CHA95] with QUICK differencing scheme for the momentum equations. The time dependent result pictured in Figure 1 agrees with the analytic solution only in phase, but not in amplitude. Refining of the mesh does not improve the result at all. Since the problem is symmetrical with respect to the origin, only the right part ($x > 0$) is shown (and solved for). Uniform mesh was used to avoid averaging of the residual sources with this test. The time step of the CFD simulation can be several times larger than the time step of the explicit Euler solver (which has to obey the Courant limit). In this example the CFD makes 12.5 time steps per cycle with 20 points per wavelength. In fact, time steps smaller than this produce greater numerical errors over the same propagation distance. This is most probably due to the false diffusion of the CFD schemes which accumulates with every time step.

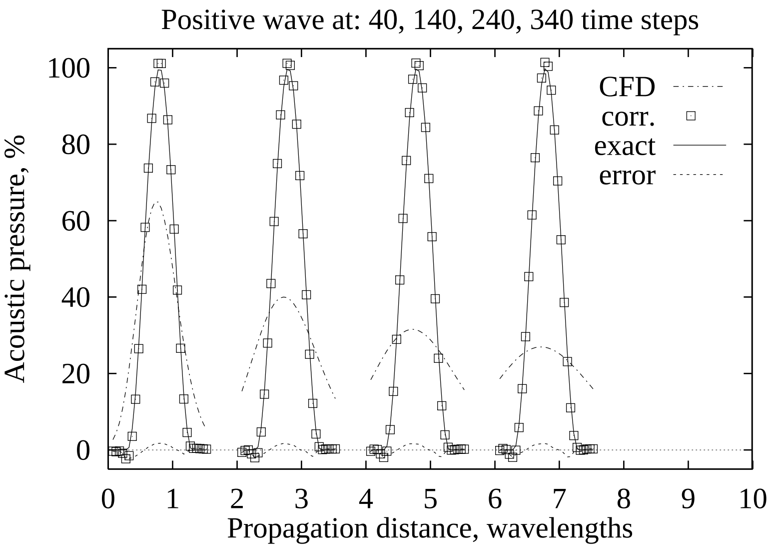

Figure 2: Defect corrected solution of test problem

The coarse grid solver, $E_H\{\bar{u}_H\}u_H = R_H$, or the acoustic module starts with zero initial conditions, and gradually accumulates the differences between the real pressure and velocity fields and their CFD representations. This process is driven by the source terms of (4) which are discretised in a time-accurate way. The solution in Figure 2 is obtained with second order approximation of the CFD quantities along the temporal axis, and its maximum error is about 2%. If linear approximations are used (which require only two stored CFD steps) the overall error becomes a little higher than 6%.

There are no external sources of mass and no external forces are acting on the fluid in this example. Also, the viscous stresses can be completely ignored with these 1-D sound waves: $\overline{f_i} = 0$ (see equation 4).

In Figure 2 the defect corrected solution (which is the sum of the CFD solution and the linearised Euler solution) is shown at regular intervals in order to trace the wave propagating from left to right. The time step with the acoustic module is 4 times smaller than the CFD step, and this is equivalent to 50 time steps per cycle. Since the acoustic procedure is fully explicit, these correction steps are computationally very inexpensive (the acoustic module needs less than 10 s to compute the correction of this example including the input and output of disk files). It can be seen that the result of this one-dimensional test is very encouraging.

As a more realistic example, a 2-D production of sound waves due to the generation of a vortex series within a flow medium as depicted in Figure 3 is examined. There are no solid bodies in the flow domain, and no acoustic source cells have been predefined. The vortices are initiated by the time-dependent source patch in the middle of the left boundary of the CFD domain. There is a background flow at a rate of 160 m/s from left to right which is not shown in Figure 3. An additional source of mass is associated with the sinusoidal in time (with an amplitude of 12 m/s) source of momentum in the vertical direction, active during time steps 1–30. Both of these cooperate in the production of acoustic waves which originate at the source patch.

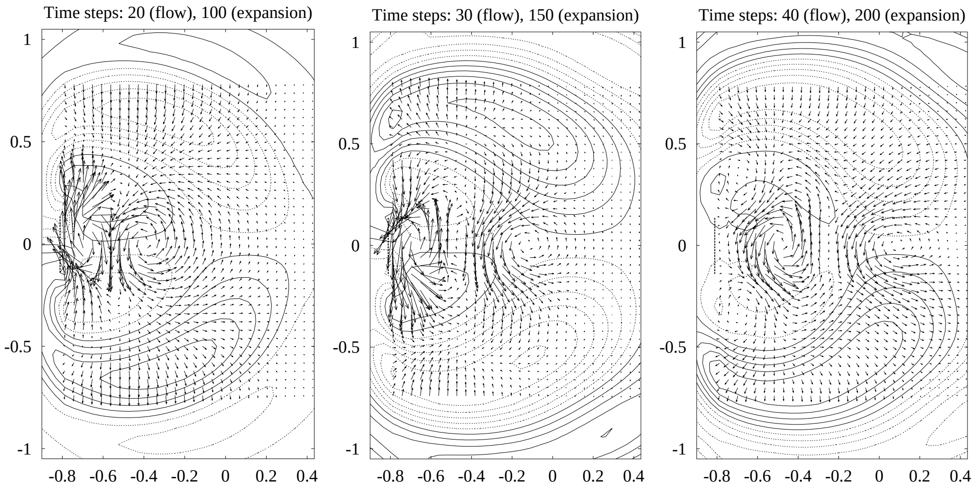

Figure 3: Vortex generation and acoustic correction pressure contours (positive - solid lines, negative - dashed lines, spacing: 20 Pa). Velocity vector scale: 3 m/s to 0.2 m. Vertical dashed line marks vortex generation source patch.

The same fully-implicit in time CFD code [CHA95] with QUICK differencing scheme for the momentum equations was used to simulate the generation and the convection of vortices. The mesh density is indicated in Figure 3 by the density of the arrows representing velocity vectors. As expected, no acoustic waves can be identified in the resulting CFD pressure field. After the correction steps (re-discretisation, mapping of residuals, and linearised Euler solution), the missing part of the pressure field is obtained, and it is shown in Figure 3 by contours. In this case of regular meshes with no solid objects, the mapping procedure is simple: two CFD cells in the vertical direction constitute one acoustic cell, and simple arithmetic averaging is used for the mapping.

Figure 3 shows clearly the acoustic waves that have been produced at the vortex generation patch propagating upwards, downwards, and out of the domain. There is no analytical validation for this example, but the results obtained are physically correct.

Conclusions

A framework of defect correction has been established for computational aeroacoustics. Based on this defect correction framework, it is possible to study aerodynamic sound generation in a systematic way. A 1-D example with validation and a 2-D example with physically correct results are shown. The authors are currently investigating a 2-D example with proper validation and are extending the concept to 3-D problems.

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