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11. Domain Decomposition and Splitting Methods for Mortar Mixed Approximations to Parabolic Problems

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Introduction

Mixed Finite Element (MFE) methods have become popular for the numerical simulation of single phase flow in porous media due to their good approximation of the flux variable and their local and global mass conservation properties. In many situations such as flow around wells or through conductive faults, the complexity of the geometry, the heterogeneities of the media, or the singularities of the data may require the use of flexible meshes including hybrid meshes or local refinements to capture the spatial behavior of the solution. In that case, non-overlapping domain decomposition techniques with Mortar elements at the interfaces of the decomposition have proven to be efficient since they enable to define the grids independently in the subdomains regions (see [GW88], [Yot96]), [ACWY96]).

On the other hand, the transient behavior of the solution may also warrant the use of different time steps in the different subdomains.

The idea of the domain decomposition method introduced in this paper is to combine Mortar Mixed Finite Element methods for the space discretization with operator splitting techniques for the time discretization in order to obtain (1) a fully parallel algorithm and (2) the possibility to use flexible meshes and local time steppings in the subdomains.

We consider a domain $\Omega \subset \mathbb{R}^d$ of boundary Γ and the parabolic equation

$$\begin{cases} \partial_t p + \nabla \cdot u = f, \ u = -K\nabla p \text{ in } \Omega, \\ p = g \text{ on } \Gamma, \ p|_{t=0} = p_0, \end{cases} \tag{1}$$

where K is a symmetric matrix, positive definite uniformly in $\overline{\Omega}$.

Most domain decomposition algorithms for such parabolic problems involve, at each time step, the solution of an elliptic problem, using classical domain decomposition iterative algorithms for elliptic equations. The present domain decomposition approach takes advantage of the parabolic structure of the problem to obtain, through operator splitting, a non-iterative method in the sense that the subdomains problems are solved only once at each time step. Other related non-iterative domain decomposition and splitting methods for parabolic problems can be found in [MPW98], [CL96], and [Dry91], and the references therein. The main originality of our method is to allow

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by construction non-matching grids at the interfaces of the domain decomposition.

Notation: for two positive functions $A(v)$ and $B(v)$, the notation $A \lesssim B$ means that there exists a constant C, independent of the various parameters, such that for all v one has $A(v) \leq CB(v)$.

Mixed Finite Element Domain Decomposition Method

Let us consider a domain decomposition of Ω into N non-overlapping subdomains $\Omega_i, i = 1, \ldots, N$ such that $\Omega_i \cap \Omega_j = \emptyset$ for all $i \neq j$, and $\overline{\Omega} = \bigcup_{i=1}^N \overline{\Omega}_i$. We set $\Gamma_i := \partial\Omega_i/\Gamma$. For $i, j \in I := \{i, j \text{ s.t. } i \neq j \text{ and } \mathrm{mes}_{d-1}\partial\Omega_i \cap \partial\Omega_j \neq 0\}$, we denote by $\Gamma_{i,j} := \partial\Omega_i \cap \partial\Omega_j$ the interface between two subdomains, and by $\gamma := \bigcup_{i,j\in I} \Gamma_{i,j}$, the skeleton of the domain decomposition.

On each subdomain Ω_i, we introduce the function spaces $M_i := L^2(\Omega_i)$ and $V_i = H(\Omega_i; \mathrm{div}) := \{v \in L^2(\Omega_i)^d \text{ s.t. } \nabla \cdot v \in M_i\}$, endowed with their usual norms denoted by $\|q_i\|_{0,i}$ and $\|v_i\|_{V_i} := \left(\|v_i\|_{0,i}^2 + \|\nabla \cdot v_i\|_{0,i}^2\right)^{1/2}$. On the domain Ω, we define the product spaces $M := \bigoplus_{i=1}^N M_i$ and $V := \bigoplus_{i=1}^N V_i$ endowed with their Hilbertian product norms $\|q\|_0$ and $\|v\|_V$.

In the non-overlapping domain decomposition framework, the smoothness assumptions on the solution will be as usual measured in the broken norms $\|\cdot\|_{\mathcal{H}^r(\Omega)}$ related to the product spaces $\mathcal{H}^r(\Omega) := \bigoplus_{i=1}^N H^r(\Omega_i),\ r \geq 0$. On the skeleton γ, we define the norm $\|\mu\|_{\frac{1}{2},\gamma} := \sup_{v\in V} \frac{\sum_{i=1}^N \int_{\Gamma_i} (v\cdot n_i)\mu d\gamma}{\|v\|_V}$, and we shall denote by $H^{\frac{1}{2}}(\gamma)$, the subspace of $L^2(\gamma)$ of functions μ such that $\|\mu\|_{\frac{1}{2},\gamma} < \infty$.

We consider, on the domain decomposition $(\Omega_i)_{i=1,\ldots,N}$, a Mortar Mixed Finite Element (MMFE) discretization of (1), introduced in [GW88] for matching grids, and extended in [Yot96], [ACWY96] to the case of non-matching grids at the interfaces between the subdomains Ω_i. In that case, a so called Mortar space $\Lambda_h \subset L^2(\gamma)$ is introduced on the skeleton γ. Then, equation (1) is discretized on each subdomain by a Mixed Finite Element Method, and the matching at the interfaces is written in the weak sense through the continuity of the orthogonal projection on Λ_h of the normal fluxes defined on each sides of $\Gamma_{i,j}$.

Let $\mathcal{T}_{i,h}$ be a quasi-uniform mesh of Ω_i. We consider, on these grids, MFE approximation spaces $V_{i,h} \subset V_i$, $M_{i,h} \subset M_i$ of order $k+1$, that can be either the RT_k or BDF_k or BDFM_k MFE discretizations (see [RT91] or [BF91]). In addition we shall assume in the sequel that $\nabla \cdot V_{i,h} = M_{i,h}$.

On the domain Ω, we define the product spaces $M_h := \bigoplus_{i=1}^N M_{i,h} \subset M$ and $V_h := \bigoplus_{i=1}^N V_{i,h} \subset V$. The dual space of V_h (resp. M_h) is denoted by V_h' (resp. M_h') endowed with the dual norm $\|\cdot\|_{V_h'}$ (resp. $\|\cdot\|_{M_h'}$). We shall denote by $\langle\cdot,\cdot\rangle$ the duality pairing.

We reproduce the choice of the Mortar space Λ_h as described in [Yot96]. Let $\mathcal{T}_{i,j,h}$,

$i, j \in I$ be a quasi-uniform mesh of $\Gamma_{i,j}$ and $\Lambda_{i,j.h}$ a finite element space on $\mathcal{T}_{i,j,h}$, either continuous or discontinuous, and of order $k+2$. The Mortar space on the skeleton γ is the product space $\Lambda_h := \bigoplus_{i,j\in I} \Lambda_{i,j,h} \subset L^2(\gamma)$.

In order to write the MMFE variational formulation of (1), we define the operators $S_h, A_h : V_h \to V_h'$, $B_h^t : \Lambda ambda_h \to V_h'$, $\mathrm{div}_h : V_h \to M_h'$, $T_h^t : H^{1/2}(\Gamma) \to V_h'$ such that for all $v_h = (v_{i,h})_{i=1,\ldots,N}, w_h = (w_{i,h})_{i=1,\ldots,N} \in V_h$, $q_h = (q_{i,h})_{i=1,\ldots,N} \in M_h$, $\mu_h \in \Lambda ambda_h$, $\varphi \in H^{1/2}(\Gamma)$:

$$\begin{array}{l} \langle S_h v_h, w_h\rangle := \sum_{i=1}^N \int_{\Omega_i} K^{-1} v_{i,h} \cdot w_{i,h} dx, \\ \langle A_h v_h, w_h\rangle := \sum_{i=1}^N \int_{\Omega_i} (\nabla \cdot v_{i,h})(\nabla \cdot w_{i,h}) dx, \\ \langle \mathrm{div}_h v_h, q_h\rangle := \sum_{i=1}^N \int_{\Omega_i} (\nabla \cdot v_{i,h}) q_{i,h} dx, \\ \langle B_h^t \mu_h, v_h\rangle := \sum_{i=1}^N \int_{\Gamma_i} \mu_h (v_{i,h} \cdot n_i) d\gamma, \ \langle T_h^t \varphi, v_h\rangle := \int_\Gamma \varphi (v_h \cdot n) d\sigma. \end{array} \tag{2}$$

Then, the MMFE spatial discretization of (1) looks for $(p_h, u_h, p_{\gamma,h}) \in M_h \times V_h \times \Lambda_h$ such that

$$\left\{ \begin{array}{l} \partial_t p_h + \mathrm{div}_h u_h = i_{M_h}^t f, \\ S_h u_h = \mathrm{div}_h^t p_h - B_h^t p_{\gamma,h} - T_h^t g, \\ B_h u_h = 0, \\ p_h|_{t=0} = p_{0,h}. \end{array} \right. \tag{3}$$

The stationnary MMFE approximation (3) is analysed in [Yot96] and [ACWY96]. In order to obtain a well posed problem, one has to assume that the Mortar space Λ_h verifies a compatibility condition with the normal traces on γ of V_h. In particular this condition ensures that the operator B_h^t is into as well as the property

$$\{q_h, \text{ t. q. } \langle \mathrm{div}_h v_h, q_h\rangle = 0, \text{ for all } v_h \in W_h := \mathrm{Ker} B_h\} = \{0\},$$

which all together guarantees existence and uniqueness of the solution. We refer to [Yot96] for the proof, under this assumption, of optimal error estimates for the solutions $u_h, p_h, p_{\gamma,h}$ of the stationnary problem.

An equivalent flux formulation

As a preliminary step towards the time discretization by an operator splitting technique, it is useful to introduce an equivalent flux formulation of (3) obtained by elimination of the discrete pressure unknown in (3). This formulation will also be crucial to analyse the stability and the error estimates of our method.

Proposition 1 *Let us define $\lambda_h := \partial_t p_{\gamma,h}$ and $g_0 := g|_{t=0}$. Then problem (3) is equivalent to the following flux formulation:*

$$\left\{ \begin{array}{l} S_h \partial_t u_h + A_h u_h + B_h^t \lambda_h + T_h^t \partial_t g = \mathrm{div}_h^t f, \\ B_h u_h = 0, \\ u_h|_{t=0} = u_h^0, \end{array} \right. \tag{4}$$

given the initialization

$$\left\{ \begin{array}{l} S_h u_h^0 = \mathrm{div}_h^t p_{0,h} - B_h^t p_{\gamma,h}^0 - T_h^t g_0, \\ B_h u_h^0 = 0, \end{array} \right. \tag{5}$$

and the pressure equation

$$\left\{ \begin{array}{l} \partial_t p_h + \mathrm{div}_h u_h = i^t_{M_h} f, \\ \partial_t p_{\gamma,h} = \lambda_h, \\ p_h|_{t=0} = p_{0,h},\ p_{\gamma,h}|_{t=0} = p^0_{\gamma,h}. \end{array} \right. \tag{6}$$

proof: the proof relies on elementary algebra using the assumption on the MFE spaces that $\nabla \cdot V_h = M_h$, and assuming enough regularity on the trace g.

Time discretization by operator splitting

The flux formulation (4) is a mixed problem formally equivalent to the Stokes equation. The idea of the time discretization by operator splitting is then to apply to the flux formulation (4) a projection scheme introduced by Chorin [Cho68] and analysed in [Ran92] in the framework of the Navier-Stokes equations.

In the framework of the MMFE method, the projection scheme splits the system (4) into two successive steps: (i) advance in time with a fixed λ_h given by the previous time step, (ii) orthogonal projection (with respect to the scalar product $\langle S_h \cdot, \cdot \rangle$) of the flux onto W_h, and updating of λ_h. The initialization of the flux is still given by equation (5). This scheme requires to be given an approximation $\lambda^0_h \in \Lambda_h$ of $\lambda|_{t=0}$. At first order accuracy in time, we shall see that it is sufficient to set $\lambda^0_h = 0$. However, in order to expect second order accuracy, a first order accurate approximation of λ^0_h has to be obtained by one time step calculation of the fully coupled system.

$$\begin{array}{ll} (i) & S_h \frac{\tilde{u}^{n+1}_h - u^n_h}{\Delta t} + A_h \tilde{u}^{n+1}_h + B^t_h \lambda^n_h + T^t_h \frac{g^{n+1} - g^n}{\Delta t} = \mathrm{div}^t_h f^{n+1}, \\ (ii) & \left\{ \begin{array}{l} S_h \frac{u^{n+1}_h - \tilde{u}^{n+1}_h}{\Delta t} + B^t_h (\lambda^{n+1}_h - \lambda^n_h) = 0, \\ B_h u^{n+1}_h = 0, \end{array} \right. \end{array} \tag{7}$$

The pressures p^n_h and $p^n_{\gamma,h}$ are obtained by discrete integration in time.

$$\left\{ \begin{array}{l} \frac{p^{n+1}_h - p^n_h}{\Delta t} + \mathrm{div}_h \tilde{u}^{n+1}_h = i^t_{M_h} f^{n+1},\ p^0_h = p_{0,h}, \\ \frac{p^{n+1}_{\gamma,h} - p^n_{\gamma,h}}{\Delta t} = \lambda^{n+1}_h,\ p^0_{\gamma,h} \text{ given by (5)}. \end{array} \right. \tag{8}$$

As for the semi-discrete formulation, the space-time discretization (7)-(8) admits an equivalent mixed pressure-flux formulation which, from elementary algebra, writes:

$$\begin{array}{ll} (i) & \left\{ \begin{array}{l} \frac{p^{n+1}_h - p^n_h}{\Delta t} + \mathrm{div}_h \tilde{u}^{n+1}_h = i^t_{M_h} f^{n+1}, \\ S_h \tilde{u}^{n+1}_h = \mathrm{div}^t_h p^{n+1}_h - B^t_h (2p^n_{\gamma,h} - p^{n-1}_{\gamma,h}) - T^t_h g^{n+1}, \end{array} \right. \\ (ii) & \left\{ \begin{array}{l} S_h u^{n+1}_h = \mathrm{div}^t_h p^{n+1}_h - B^t_h p^{n+1}_{\gamma,h} - T^t_h g^{n+1}, \\ B_h u^{n+1}_h = 0, \end{array} \right. \end{array} \tag{9}$$

with $p^0_h := p_{0,h}$ and $p^{-1}_{\gamma,h} := p^0_{\gamma,h} - \Delta t \lambda^0_h$. We note that (9) corresponds, at step (i), to a second order linear extrapolation in time of the interface pressure $p^{n+1}_{\gamma,h} \simeq 2p^n_{\gamma,h} - p^{n-1}_{\gamma,h}$.

The main advantage of the projection scheme is that the prediction step (i) can be solved in a fully parallel way on each subdomain independently, while the projection step (ii) reduces to inverse the interface problem related to the operator $B_h S^{-1}_h B^t_h$.

Let us restrict ourselves to the assumption that only RT_0 mixed finite elements are used in the neighborhood of the skeleton γ. Then, a mass condensation of the matrix representing the operator S_h in the nodal basis can be locally performed, preserving the order of approximation of the discretization. It results that the interface operator matrix in the nodal basis of Λ_h is diagonal and can be readily inverted in $\mathcal{O}(N_{\Lambda_h})$ operations where N_{Λ_h} is the dimension of Λ_h.

More generally, the interface problem can be efficiently solved by a conjugate gradient iterative algorithm preconditioned by the approximate interface matrix obtained by mass condensation of S_h in the neighborhood of γ.

Stability analysis

Let $Z_h := B_h S_h^{-1} B_h^t$ denote the interface operator related to the projection step (ii). For any $\mu \in L^2(\gamma)$, we set $\|\mu\|_{Z_h} := \langle Z_h \mu, \mu \rangle^{\frac{1}{2}}$, which defines a semi-norm on $L^2(\gamma)$ and a norm on Λ_h. On the other hand, we define $\|B_h^t \mu\|_{V_h'} := \sup_{v_h \in V_h} \frac{\sum_{i=1}^N \int_{\Gamma_i} (v_h \cdot n_i) \mu d\gamma}{\|v_h\|_V}$, semi-norm on $L^2(\gamma)$ (and norm on Λ_h).

The stability analysis of the incremental scheme is done in its equivalent flux formulation (7)-(8) in order to avoid to deal with the three steps equations (9). It is then formally similar to the analysis performed for Navier Stokes equations (see [She92], [GQ98]) with necessary adaptations to the framework of domain decomposition and MMFE.

Theorem 1 *Let $t_n := n\Delta t$, and assume $\partial_t g \in L^2(0, t_m; H^{\frac{1}{2}}(\Gamma))$, $\sum_{n=0}^{m-1} \Delta t \|f^{n+1}\|_0^2 \lesssim 1$, then the incremental projection scheme (7)-(8) or (9) is unconditionally stable in the sense that for all Δt*

$$\left\{ \begin{array}{l} \|u_h^m\|_0^2 + \Delta t^2 \|\lambda_h^m\|_{Z_h}^2 + \sum_{n=0}^{m-1} \Delta t \|\nabla \cdot \tilde{u}_h^{n+1}\|_0^2 \\ \qquad \lesssim \|u_h^0\|_0^2 + \Delta t^2 \|\lambda_h^0\|_{Z_h}^2 + \Delta t \sum_{n=0}^{m-1} \|f^{n+1}\|_0^2 + \int_0^{t_m} \|\partial_t g(s)\|_{H^{\frac{1}{2}}(\Gamma)}^2 ds, \\ \|p_h^m\|_0^2 \lesssim \|p_{0,h}\|_0^2 + \sum_{n=0}^{m-1} \Delta t \|\nabla \cdot \tilde{u}_h^{n+1}\|_0^2 + \Delta t \sum_{n=0}^{m-1} \|f^{n+1}\|_0^2, \\ \|B_h^t p_{\gamma,h}^m\|_{V_h'} \lesssim \|u_h^m\|_0 + \|p_h^m\|_0 + \|g^m\|_{H^{\frac{1}{2}}(\Gamma)}, \end{array} \right. \tag{10}$$

with constants independent of h, Δt, N and depending only on t_m and K.

Error estimates

We denote by $(u, p) \in C^0(0, t_m; H(\Omega; \mathrm{div})) \times C^0(0, t_m; M)$ the weak solution of (1) on the interval $[0, t_m]$. We shall assume that the pressure p and its derivative $\partial_t p$ are globally in $H^1(\Omega)$ in order to define the interface pressure $p_\gamma := p|_\gamma$ and its derivative $\lambda := \partial_t p|_\gamma = \partial_t p_\gamma$ in $H^{1/2}(\gamma)$. We set $t_n = n\Delta t$ and $u^n := u(t_n)$, $p^n := p(t_n)$, $\lambda^n := \lambda(t_n)$, $p_\gamma^n := p_\gamma(t_n)$.

The dependence of the semi-norm $\|\cdot\|_{Z_h}$ on the mesh size h, as given by the estimate $\|\mu\|_{Z_h} \lesssim h^{-\frac{1}{2}} \|\mu\|_{L^2(\gamma)}$ $\forall \mu \in L^2(\gamma)$, deteriorates the convergence of the method. We can prove the following theorem.

Theorem 2 *Let $(u, p) \in C^0(0, t_m; H(\Omega; \mathrm{div})) \times C^0(0, t_m; M)$, be the weak solution of (1) such that $p \in C^1(0, t_m; H^1(\Omega))$. Pour $1 \leq r \leq k+1$ et $u \in H^1(0, t_m; \mathcal{H}^r(\Omega)^d)$,*

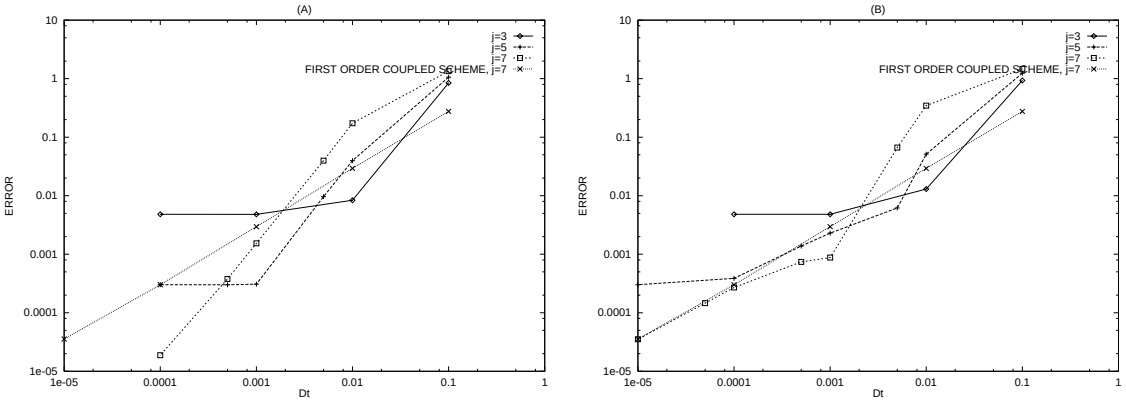

Figure 1: Convergence history of the pressure error in $l^\infty(L^2(\Omega))$ norm: (A) first order incremental and coupled schemes, (B) second order incremental and first order coupled schemes.

$\partial_{t^2} u \in L^2(0,t_m;V')$, $\partial_t \lambda \in L^2(0,t_m;L^2(\gamma))$, $\partial_{t^2} g \in L^2(0,t_m;H^{\frac{1}{2}}(\Gamma))$, $\partial_{t^2} p \in L^2(0,t_m;L^2(\Omega))$, $p \in W^{1,\infty}(0,t_m;\mathcal{H}^{r+1}(\Omega))$, $\sum_{n=0}^{m-1} \Delta t \|\nabla \cdot u^{n+1}\|^2_{\mathcal{H}^r(\Omega)} \lesssim 1$, *we have*

$$\begin{aligned} &\|u^m - u_h^m\|_0 + \|p^m - p_h^m\|_0 + \|B_h^t(p_\gamma^m - p_{\gamma,h}^m)\|_{V_h'} \\ &\quad + \Big(\sum_{n=0}^{m-1} \Delta t \|\nabla \cdot (u^{n+1} - \tilde{u}_h^{n+1})\|_0^2\Big)^{\frac{1}{2}} \lesssim \Delta t(1 + h^{-\frac{1}{2}}) + h^r, \end{aligned} \tag{11}$$

with a constant depending only on t_m, K. To obtain these estimates, it suffices to choose for $p_{0,h}$ the orthogonal projection of p_0 onto M_h and $\lambda_h^0 = 0$.

Numerical example

Let us consider in dimension $d = 1$, the interval $\Omega =]-1,1[$ splitted into two subdomains $\Omega_1 =]-1,0[$ and $\Omega_2 =]0,1[$, and equation (1) for $g = 0$ and $K = 1$ with exact solution $p(x,t) = \cos\frac{\pi x}{2}(\cos 6t + 2)$. This problem is discretized on a uniform mesh of size $h = 2^{-j}, j \in \mathbb{N}$ using RT_0 MFE with mass condensation. Figure 1 reports the convergence history of the error $p_h^n - p^n$ in $l^\infty(L^2(\Omega))$ norm for 3 different time discretizations: (a) the incremental scheme (9), (b) the incremental scheme with second order Crank-Nicholson time discretization in the subdomains at step (i), (d) the first order Euler backward fully coupled discretization.

From the numerical results displayed Figures 1, we see that the error behaves like $\min(\frac{\Delta t}{h^{1/2}}, \frac{\Delta t^2}{h})$ for the incremental projection scheme (a), like $\min(\frac{\Delta t}{h^{1/2}}, \frac{\Delta t^2}{h}) + \Delta t$ for the incremental projection scheme (b). The same results can be observed for the flux u and the interface pressure p_γ.

These results suggest that the error is the sum of the error produced by the coupled scheme and the splitting error (between the coupled scheme and the projection scheme) of order $\min(\frac{\Delta t}{h^{1/2}}, \frac{\Delta t^2}{h})$.

Conclusion

The method introduced in this paper combines Mortar Mixed Finite Element domain decomposition spatial discretization with projection schemes, in order to obtain a fully parallel algorithm for parabolic equations. In addition this method enables the use of hybrid meshes and local time steppings.

Although the scheme is shown to be unconditionally stable, the convergence is obtained only if the condition $\Delta t \lesssim h^{\frac{1}{2}}$ is verified. This is the price to pay to decouple the interface problem from the computation of the subdomain solutions.

This strategy has proven to be efficient to solve single phase Darcy flows around 2D wells and faults with high physical heterogeneities and complex geometries, and we refer to [Gai00] where such numerical tests are reported.

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