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32. Numerical solution of vascular flows by heterogeneous domain decomposition methods

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Introduction

In this note we investigate a problem arising from fluid dynamics for hemodynamics, using heterogeneous domain decomposition techniques. In particular we will couple Navier-Stokes equations with Oseen or Stokes equations, as advocated in papers [FGQ99] and [FGQ00].

Our interest is twofold. On one hand we would like to assess the quality of the coupled heterogeneous models; in particular we want to compare two options where the Oseen flux or the Stokes flux is matched continuously at the interface. On the other hand, we wish to carry out iterative substructuring method to solve the coupled problem. This iterative procedure has been introduced and analyzed in [FGQ00] for a general problem.

More generally in multi-field domain decomposition problems, different physical, mathematical or numerical models are adopted in different parts of the computational domain. One motivation is to develop parallel algorithms, the other is to provide an efficient way to reduce the complexity of the problem in certain regions, by using there a simpler mathematical model.

Given a bounded domain $\Omega \subset \mathbb{R}^2$, with a Lipschitz boundary $\partial\Omega$, $T > 0$, a vector field $\mathbf{f}$, a constant viscosity $\nu > 0$, we are interested in approximating the velocity field $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ and the pressure field $p = p(\mathbf{x}, t)$ for the incompressible Navier-Stokes equations:

$$\partial_t \mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega \times (0, T) \tag{1}$$

by a multi-field approach. The idea is to consider two disjoint subregions Ω_1 and Ω_2 of Ω such that $\overline{\Omega}_1 \cup \overline{\Omega}_2 = \overline{\Omega}$, and to couple the Navier-Stokes equations (1) restricted to the subregion Ω_1 with the following linear Oseen equations

$$\partial_t \mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{u}_\infty\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega_2 \times (0, T), \tag{2}$$

where $\mathbf{u}_\infty$ is a prescribed solenoidal vector field. Sometimes the Oseen equations can be replaced by the Stokes equations, which are a special case of (2) with $\mathbf{u}_\infty = \mathbf{0}$.

The Navier-Stokes subregion Ω_1 can be a suitable internal domain of Ω and the Oseen subregion Ω_2 an exterior subdomain. Otherwise Ω_1 can be the part of Ω where the flow is quite perturbed by the presence of an obstacle. On the common boundary

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of the two subdomains, $\Gamma := \partial\Omega_1 \cap \partial\Omega_2$, correct transmission conditions have to be imposed.

The mathematically admissible transmission conditions at subdomain interfaces have been determined and analyzed in [FGQ99]. A Dirichlet/Neumann iterative procedure among subdomains has been proposed to solve the coupled Navier-Stokes/Oseen (or Navier-Stokes/Stokes) problem and its analysis was carried out in [FGQ00].

In the first and second Sections we recall the general problem, while in the last Section we carry out the numerical results and the assessment of the proposed method.

Multi-domain formulations and transmission conditions

We consider a vector field $\mathbf{w} : \Omega :\to \mathbb{R}^2$ such that $\mathbf{w}_i = \mathbf{w}_{|\Omega_i}$, for $i = 1, 2$ and

$$\mathbf{w}_1 = \mathbf{u}_{|\Omega_1} \quad \text{and} \quad \mathbf{w}_2 \text{ is equal either to } \mathbf{u}_{\infty|\Omega_2} \text{ or to } \mathbf{0}. \tag{3}$$

The multi-domain formulation corresponding to (1) (restricted to Ω_1) - (2), is: find $\mathbf{u}_i : \Omega_i \to \mathbb{R}^2$ and $p_i : \Omega_i \to \mathbb{R}$, for $i = 1, 2$ satisfying

$$\begin{aligned}
\partial_t \mathbf{u}_i - \nu\Delta\mathbf{u}_i + (\mathbf{w}_i \cdot \nabla)\mathbf{u}_i + \nabla p_i = \mathbf{f}, &\quad \text{in } \Omega_i \times (0, T) \quad i = 1, 2 &(4)\\
\nabla \cdot \mathbf{u}_i = 0 &\quad \text{in } \Omega_i \times (0, T) \quad i = 1, 2 &(5)\\
\mathbf{u}_1 = \mathbf{u}_2 &\quad \text{on } \Gamma \times (0, T) &(6)\\
-p_1\mathbf{n} + \nu(\mathbf{n} \cdot \nabla)\mathbf{u}_1 = -p_2\mathbf{n} + \nu(\mathbf{n} \cdot \nabla)\mathbf{u}_2 &\quad \text{on } \Gamma \times (0, T) &(7)
\end{aligned}$$

and suitable boundary conditions on $\partial\Omega \times (0, T)$, where $\mathbf{u}_i = \mathbf{u}_{|\Omega_i}$, $p_i = p_{|\Omega_i}$, for $i = 1, 2$, and $\mathbf{n}$ denotes the normal unit vector on Γ directed from Ω_1 to Ω_2.

The choice $\mathbf{w}_1 = \mathbf{u}_1$ and $\mathbf{w}_2 = \mathbf{0}$ corresponds to a *Navier-Stokes/Stokes* coupling, while the choice $\mathbf{w}_1 = \mathbf{u}_1$ and $\mathbf{w}_2 = \mathbf{u}_{\infty|\Omega_2}$ corresponds to a *Navier-Stokes/Oseen* coupling.

The transmission conditions (6) and (7) ensure the continuity of the velocity field and the continuity of the normal stress across the interface, respectively.

For the Navier-Stokes/Oseen coupling, the transmission condition (7) can be replaced on Γ by the following one [FS98]:

$$-p_1\mathbf{n} + \nu(\mathbf{n} \cdot \nabla)\mathbf{u}_1 - \frac{1}{2}(\mathbf{w}_1 \cdot \mathbf{n})\mathbf{u}_1 = -p_2\mathbf{n} + \nu(\mathbf{n} \cdot \nabla)\mathbf{u}_2 - \frac{1}{2}(\mathbf{w}_2 \cdot \mathbf{n})\mathbf{u}_2, \tag{8}$$

and it is associated to the skew-symmetric form of the convective term in (4).

Besides, from now on, given a sufficiently regular vector field $\mathbf{w}$, we set:

$$\begin{aligned}
&T_S(\mathbf{u}, p)\mathbf{n} = -p\mathbf{n} + \nu(\mathbf{n} \cdot \nabla)\mathbf{u} &&\textit{Stokes normal stress},\\
&T_O(\mathbf{w}; \mathbf{u}, p)\mathbf{n} = -p\mathbf{n} + \nu(\mathbf{n} \cdot \nabla)\mathbf{u} - \tfrac{1}{2}(\mathbf{w} \cdot \mathbf{n})\mathbf{u} &&\textit{Oseen normal stress}.
\end{aligned} \tag{9}$$

The mathematical justification for the use of either (7) or (8) is provided in [FGQ99].

The time-dependent system (4)-(7) can be discretised in time, e.g., by a finite-difference scheme, so that a steady problem has to be solved at each time step. The discretisation of time derivative gives rise to a mass term with constant coefficient α dependent from the time scheme.

The boundary conditions we will consider for the coupled problem (4)-(7) will be of Dirichlet type on $\partial\Omega_D$ (e.g., no-slip boundary conditions $\mathbf{u} = \mathbf{0}$ on fixed walls, or inflow conditions $\mathbf{u} = \mathbf{g}$, for a suitable given vector field $\mathbf{g}$) and of Neumann type on $\partial\Omega_N$ (such as $T_S(\mathbf{u}, p)\mathbf{n} = \mathbf{0}$).

Dirichlet/Neumann iterations

In order to solve the multi-domain problem (4)-(7) an iterative procedure was introduced in [FGQ99], based on the solution of a sequence of boundary value problems on each subdomain, plus relaxation conditions at the interface Γ, (see [QV99], Ch. 3). In the current case, the idea consists of solving problems like (4)-(5) for every $i = 1, 2$, for which the transmission conditions (6) and (7) (or (8)) provide Dirichlet and Neumann boundary conditions on the interface Γ, respectively.

Precisely, a Dirichlet/Neumann iteration scheme for problem (4)-(5) with transmission conditions (6), (8), can be set up as follows: given $\boldsymbol{\lambda}^0$ defined on Γ, for each $k \geq 1$ find $(\mathbf{u}_1^k, p_1^k)$ such that:

$$\begin{array}{lll} \alpha\mathbf{u}_1^k - \nu\Delta\mathbf{u}_1^k + (\mathbf{w}_1^k \cdot \nabla)\mathbf{u}_1^k + \nabla p_1^k = \mathbf{f}, & \nabla \cdot \mathbf{u}_1^k = 0 & \text{in } \Omega_1 \\ \mathbf{u}_1^k = \boldsymbol{\lambda}^{k-1} & & \text{on } \Gamma \end{array} \tag{10}$$

then find $(\mathbf{u}_2^k, p_2^k)$ such that:

$$\begin{array}{lll} \alpha\mathbf{u}_2^k - \nu\Delta\mathbf{u}_2^k + (\mathbf{w}_2^k \cdot \nabla)\mathbf{u}_2^k + \nabla p_2^k = \mathbf{f}, & \nabla \cdot \mathbf{u}_2^k = 0 & \text{in } \Omega_2 \\ T_O(\mathbf{w}_2; \mathbf{u}_2^k, p_2^k)\mathbf{n} = T_O(\mathbf{w}_1^k; \mathbf{u}_1^k, p_1^k)\mathbf{n} & & \text{on } \Gamma \end{array} \tag{11}$$

where, for $k \geq 1$, the interface values are updated as follows:

$$\boldsymbol{\lambda}^k = \theta\mathbf{u}_{2|\Gamma}^k + (1-\theta)\boldsymbol{\lambda}^{k-1} \quad \text{on } \Gamma,$$

and θ is a positive relaxation parameter that will be determined in order to ensure, and possibly, to accelerate the convergence of the iterative scheme. We note that the restrictions $\mathbf{u}_{2|\Gamma}^k$ will be understood in the sense of the traces and in the linear case, (i.e. when $\mathbf{w}$ is given independently of $\mathbf{u}$), $\mathbf{w}_1^k = \mathbf{w}_1$ for all $k \geq 1$.

In the case in which the Stokes interface condition (7) is considered, the last equation in (11) is replaced by

$$T_S(\mathbf{u}_2^k, p_2^k)\mathbf{n} = T_S(\mathbf{u}_1^k, p_1^k)\mathbf{n} \quad \text{on } \Gamma. \tag{12}$$

We point out that "parallel" versions of the previous iterative schemes are obtained replacing $\mathbf{u}_1^k$ by $\mathbf{u}_1^{k-1}$ and p_1^k by p_1^{k-1} (and $\mathbf{w}_1^k$ by $\mathbf{w}_1^{k-1}$) in the last set of equations (11) (in (12)).

The convergence of this iterative scheme for a suitable range $(0, \theta^*)$ of relaxation parameters has been proven in [FGQ00], using Schauder fixed point theorem for the Steklov-Poincaré operator in the space of traces on Γ.

Numerical results

We consider the two dimensional model of pulsatile Newtonian flow in the human carotid bifurcation. This model problem is considered in biomechanical literature as a simplification of the more complex 3-D problem. The computational domain is shown in Fig. 3. The basic shape of the model agrees with the model of Bharadvaj et al. ([BMG82]) and the geometry parameters are based upon the data described by Ku et al. ([KGZG85]). Using the common carotid diameter $D = 0.62$cm as characteristic length and a reference blood viscosity $\nu = 0.035$, the maximum Reynolds number within a period of the motion is $Re_{\max} \simeq 800$. The assumed pulse frequency is 72 strokes per minute, so that the motion is periodic with period $T = 5/6$.

At the inflow boundary (the left vertical side) a fully developed time-dependent velocity profile $\mathbf{g}(x_2, t)$, such that $g_2(x_2, t) = h(x_2) \cdot \phi(t)$, is prescribed (where $\phi(t)$ is the function described in Fig. 1 (top) and $h(x_2)$ is a parabolic profile); at the rigid walls the no-slip condition $\mathbf{u} = \mathbf{0}$ is applied, while at the outflow boundary a no-friction condition is imposed (i.e. $T_S(\mathbf{u}, p)\mathbf{n} = \mathbf{0}$).

The two-domains formulation (10)-(11) is here extended to four subdomains (see Fig. 2): one Navier-Stokes domain and three Oseen domains with $\mathbf{u}_\infty(t) = \mathbf{u}_{Stokes}(t)$, that is the Stokes solution subjected to the fully developed time-dependent velocity profile $\mathbf{g}(x_2, t)$. The Euler Semi-Implicit (ESI) finite difference scheme is used to discretise the time derivative, with $\Delta t = 10^{-2}$. At each time step of the ESI scheme, we make use of the Dirichlet/Neumann algorithm. The relaxation parameter θ was chosen dinamically so as to minimize the interface error at each D/N step. In order to test the convergence of the D/N algorithm we check that

$$\max_{i=1,2} \left[\|\mathbf{u}_i^k - \mathbf{u}_i^{k-1}\|_{H^1(\Omega_i)} / \|\mathbf{u}_i^k\|_{H^1(\Omega_i)} \right] \leq 5 \cdot 10^{-6},$$

where k is the iteration counter. The numerical approximation is carried out by considering stabilised Spectral Element Methods, with 25 elements and polynomial degree $N = 5$.

In Fig. 1 the two components of the velocity are shown for the full Navier-Stokes approximation and the NS/OS coupling with either Oseen flux or Stokes flux across the interfaces. Note that the coupling based on the Stokes flux at the interfaces provide a much more accurate solution, as already noticed in [FGQ99] for other problems.

In Fig. 2 (bottom) we show the number of D/N iterations needed to converge, at each time step, for the NS/OS coupling with either Oseen or Stokes flux across the interfaces.

In Fig. 3 we report the relative errors between the NS/OS ($\mathbf{u}_{NS/OS}$) and the full Navier-Stokes ($\mathbf{u}_{NS}$) solution for the two different decompositions illustrated in Fig. 4. We denote by Ω_0 the domain of the left decomposition of Fig. 4 in which Navier-Stokes equations are solved, and we define the error as:

$$e_{H^1(\Omega_0)} = \frac{\|\mathbf{u}_{NS} - \mathbf{u}_{NS/OS}\|_{H^1(\Omega_0)}}{\|\mathbf{u}_{NS}\|_{H^1(\Omega_0)}}.$$

As expected, the second partition, featuring Navier-Stokes subdomain larger than in the first one, provides more accurate results.

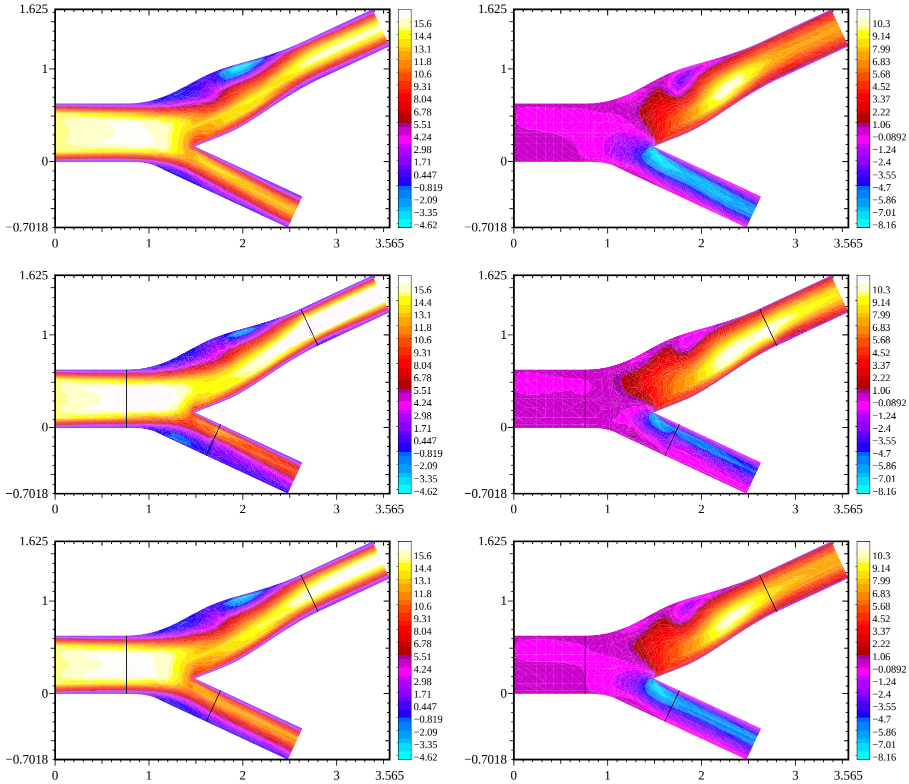

Figure 1: First (left) and second (rigth) components of the velocity for the full Navier-Stokes solution (top), the NS/OS coupling with Oseen flux at the interfaces (intermediate), the NS/OS coupling with Stokes flux at the interfaces (bottom). The results refer to $t = .3$ when the difference between the full Navier-Stokes solution and the coupled NS/OS solution is maximum.

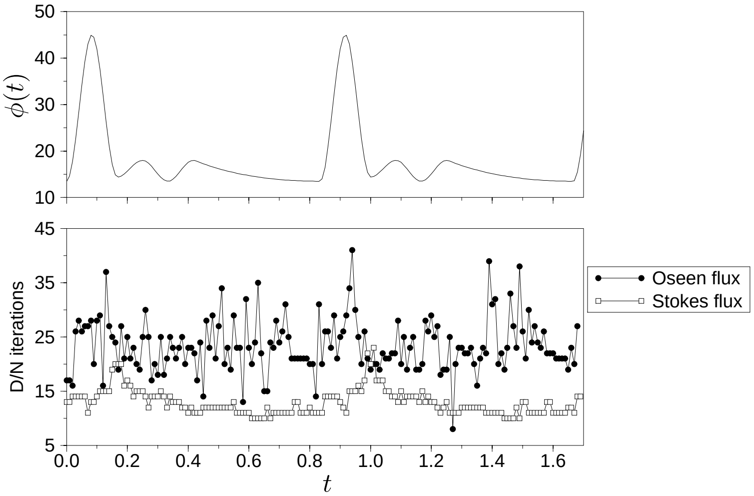

Figure 2: The fully developed time-dependent velocity profile (top) and the D/N iterations for the coupling with either Oseen or Stokes flux.

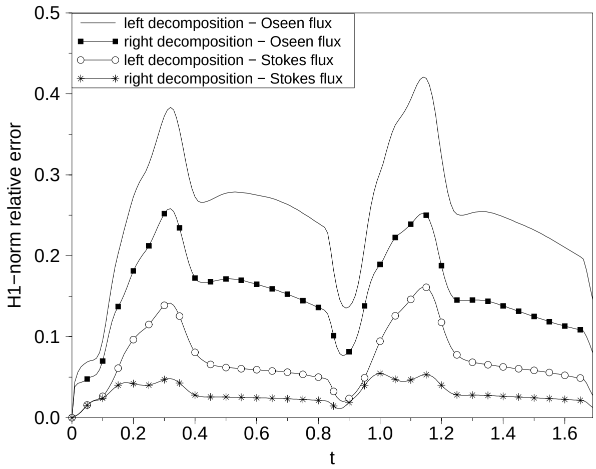

Figure 3: The errors $e_{H^1(\Omega_0)}$ between the NS/OS coupling and the full Navier-Stokes solution, with either Oseen or Stokes flux across the interfaces.

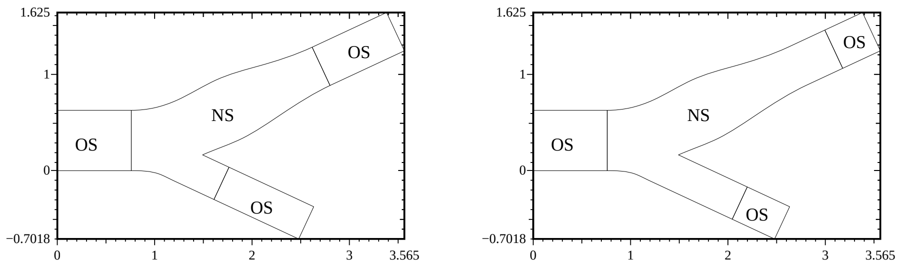

Figure 4: The two decompositions used for the error analysis of Fig. 3.

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