

## 34. Domain Decomposition for Kalman Filter Method and Its Application to Tidal Flow at Onjuku Coast

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### Introduction

Recently remarkable progress has been done in the civil engineering field and many hi-level public works have been carried out. With these progress, numerical analysis in the field of civil engineering also advances more and more, and it plays an important role in the building of many kinds of structures. To grasp actual phenomenon about civil engineering more faithfully and briefly by using numerical analysis, it is necessary to further make progress customary analysis procedure. We have been studying tidal

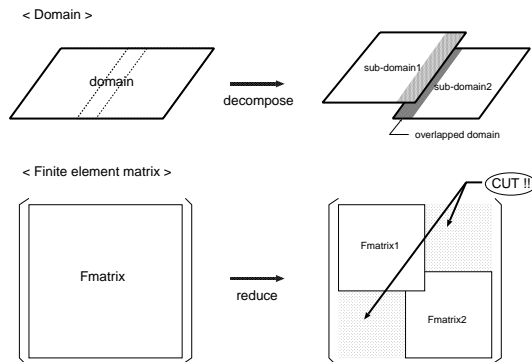


Figure 1: Image of domain decomposition

flow using Kalman filter method. But we have a serious problem that Kalman filter using FEM cannot be computed about problem with huge analytical domain. Then we decide to apply domain decomposition for Kalman filter with FEM. Using domain decomposition, finite element matrix becomes smaller and memory is economized. Computational time becomes very short and numerical examples with small mesh can be calculated. Kalman filter using FEM with many nodes could be calculated. Schwarz Alternating Procedure (SAP), one of domain decomposition technique, is used in this case. SAP is easy to treat and to apply for Kalman filter.

In this analysis, applying domain decomposition for actual problem of tidal flow, we verify effectiveness of this technique.

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## Finite Element Method

### Basic Equations

Linear shallow water equations, which consist of momentum equation and continuity equation, are shown as follows ;

$$\dot{u}_i + g\eta_{,i} = 0, \quad (1)$$

$$\dot{\eta} + hu_{i,i} = 0, \quad (2)$$

where  $u_i$  are velocities,  $\eta$  is water elevation,  $h$  is water depth and  $g$  is the gravity acceleration.

### Boundary Conditions

The boundary of water region consists of two parts  $S_1$  and  $S_2$ .  $S_1$  is the land boundary and  $S_2$  is the open boundary. The boundary conditions are shown as follows ;

$$u_n = ul + vm = \hat{u}_n \quad \text{on } S_1, \quad (3)$$

$$\eta = \hat{\eta} \quad \text{on } S_2, \quad (4)$$

where we use  $\hat{\phantom{x}}$  to represent the given values.

### Finite Element Equations

To obtain finite element equations, the shallow water equations can be discretized spatially and temporally. As regards spatial directions, Galerkin method with triangles finite element is used. As regards numerical integration in time, explicit method with BTM term is used. Then finite element equations can be obtained in the following form.

$$\bar{M}_{\alpha\beta} u_\beta^{n+1} = M_{\alpha\beta} u_\beta^n - \Delta t \{ g S_{\alpha\beta,x} \eta_\beta^n + \frac{\Delta t}{2} gh (H_{\alpha,x\beta,x} u_\beta^n + H_{\alpha,x\beta,y} v_\beta^n) \}, \quad (5)$$

$$\bar{M}_{\alpha\beta} v_\beta^{n+1} = M_{\alpha\beta} v_\beta^n - \Delta t \{ g S_{\alpha\beta,y} \eta_\beta^n + \frac{\Delta t}{2} gh (H_{\alpha,y\beta,x} u_\beta^n + H_{\alpha,y\beta,y} v_\beta^n) \}, \quad (6)$$

$$\bar{M}_{\alpha\beta} \eta_\beta^{n+1} = M_{\alpha\beta} \eta_\beta^n - \Delta t \{ h (S_{\alpha\beta,x} u_\beta^n + S_{\alpha\beta,y} v_\beta^n) + \frac{\Delta t}{2} gh (H_{\alpha,x\beta,x} \eta_\beta^n + H_{\alpha,y\beta,y} \eta_\beta^n) \}, \quad (7)$$

where BTM (Balancing Tensor Diffusivity) term is numerical viscosity.

## Kalman Filter

### Basic Equations

The basic equations of Kalman filter are shown as follows;

$$x_{k+1} = F_k x_k + G_k w_k, \tag{8}$$

$$y_k = H_k x_k + v_k. \tag{9}$$

Eq.8 and Eq.9 are called system equation and observation equation respectively. Where  $x_k$  is the state vector,  $y_k$  is observed vector,  $F_k$  is state transition matrix that is called as finite element matrix,  $H_k$  is the observation matrix which has information about placement of observation points,  $G_k$  is driving matrix,  $w_k$  is the system noise and  $v_k$  is the observation noise.  $w_k$  and  $v_k$  are the white noise which are independent each other.

## Domain Decomposition Technique

### An Introduction to Schwarz Alternating Procedure(SAP)

Schwarz Alternating Procedure(SAP) is one of many Domain Decomposition Techniques. The decomposed domain with overlapping area is performed in this method. In order to define the SAP algorithm, computational domain  $\Omega$  is decomposed into some overlapping subdomains. It is assumed that  $\Omega$  is partitioned into  $J \geq 2$  intersecting subdomains which can be written as follow;

$$\Omega = \bigcup_{k=1}^J \Omega^k, \tag{10}$$

where  $\Omega^k$  is subdomain. The overlapping area is shown by  $\Omega^k \cap \Omega^{k+1}$  for  $k = 1, 2, \dots, (J - 1)$ . And the boundaries of  $\Omega^k$  are defined as;

$$\Gamma^k = \partial\Omega^k \cap \Gamma, \quad k = 1, 2, \dots, J \tag{11}$$

where  $\partial\Omega^k$  is whole boundary of subdomain  $\Omega^k$ ,  $\Gamma^k$  is internal boundary which is written in another way from Eq.(9);

$$\Gamma^k = \partial\Omega^k \setminus (\partial\Omega^k \cap \partial\Omega^{k+1}), \quad k = 1, 2, \dots, J,$$

and  $\Gamma$  is a set of internal boundary  $\Gamma^k$ . This is written as follow;

$$\Gamma = \bigcup_{k=1}^J \Gamma^k. \tag{12}$$

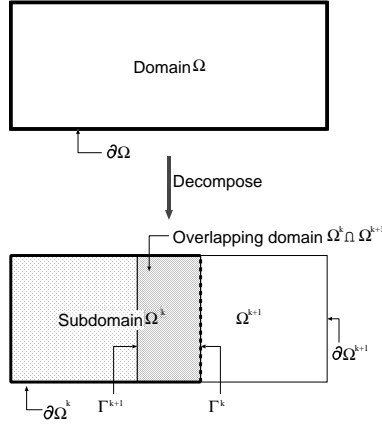


Figure 2: Model of Decomposed Domain

### Application for Finite Element Method

If Domain Decomposition Technique with SAP, which was explained above, was applied for Finite Element Method, basic equation could be written as follows;

$$\begin{aligned} \dot{u}_i^k + g\eta_{,i}^k &= 0, & \dot{\eta}^k + hu_{,i,i}^k &= 0 & \text{in } \Omega^k \\ u_n^k &= u^k l + v^k m = \hat{u}_n^k & & & \text{on } \partial\Omega_1^k, \\ \eta &= \hat{\eta}^k & & & \text{on } \partial\Omega_2^k, \\ k &= 1, 2, \dots, J. \end{aligned}$$

In order to obtain finite element equations about this case, these equations are discretized in each domain. And using finite element equations that are obtained by the discretization, we carry out numerical analysis.

### Numerical Example

#### Case 1

Computational domain is Onjuku Coast in Chiba Prefecture. This domain is divided into 300 nodes and 526 elements. In this analysis, two types of examples computed and these results are compared. One is the case where the analytical domain is decomposed into 2 sub-domains and the other is that not using SAP. Time increment  $\Delta t$  is 0.0020, system noise  $Q$  is 0.020 and observation noise  $R$  is 0.0010. Calculation is carried out by using observation data from July 16, 1997 to July 20 at 5 observation points.

In this case, computational domain decomposed into 2 subdomains with overlapping area is used. Sub-domain 1 has 169 nodes and sub-domain 2 has 173 nodes.

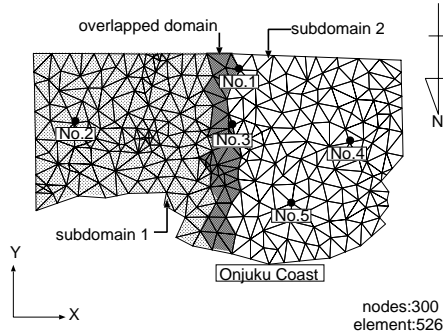


Figure 3: Computational domain of case 1

**Algorithm of Case 1**

Kalman filter with FEM applying SAP is calculated as following algorithm:

1. Kalman-gain is calculated about sub-domain 1 and converged.
2. Kalman-gain is calculated about sub-domain 2 and converged.
3. Using Kalman-gain converged at 1., optimal estimated value is computed about sub-domain1.
4. Using Kalman-gain converged at 2., optimal estimated value is computed about sub-domain2.
5. In overlapped area, results at 3. and 4. are compared.  
If difference between these results is not small enough, go to 3..
6. Results are written.

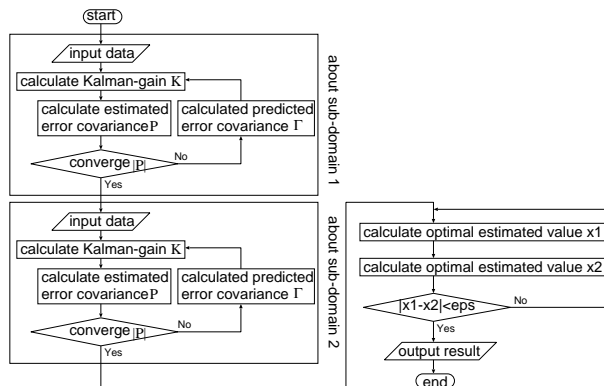


Figure 4: Algorithm of Kalman filter using SAP

Numerical Result

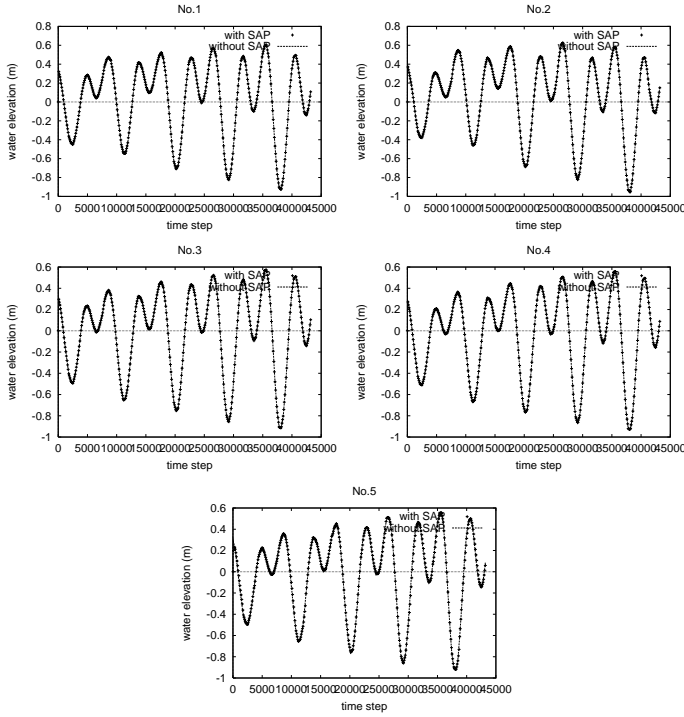


Figure 5: Comparison about water elevation

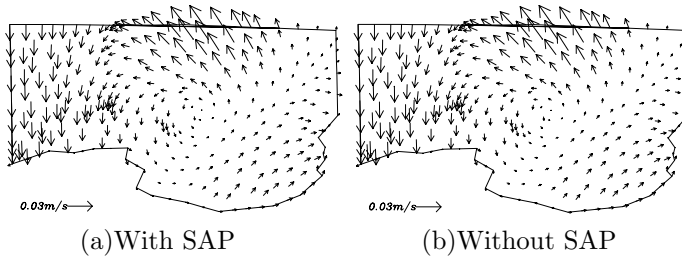


Figure 6: Velocity distribution (Residual flow)

Fig.5 shows comparing two results, one is using SAP and the other is not using SAP. According to this figure, it is recognized that difference between two values is very small. Fig.6 shows velocity distributions of residual flow. Fig.6(a) is result of analysis with SAP and Fig.6(b) is that without SAP. Comparing two figures, flow conditions are almost the same each other.

### Case 2

Computational domain is simple model of channel. This domain is divided into 33 nodes and 40 elements. Time increment  $\Delta t$  is 0.0020, system noise  $Q$  is 0.000001 and observation noise  $R$  is 0.50. Quasi-observation data is given for 3 observation points.

In this case, computational domain decomposed into 3 subdomains with overlapping area is used. Each sub-domain has 15 nodes.

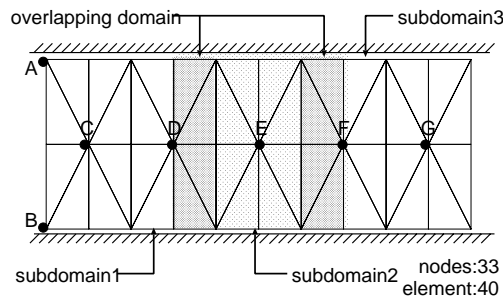


Figure 7: Computational domain of case 2

### Numerical Result

Fig.8 shows comparing two results, one is using SAP and the other is not using SAP. According to this figure, it is recognized that difference between two values is very small. Fig.9 shows velocity distributions of residual flow. Fig.9(a) is the result of analysis with SAP and Fig.9(b) is that without SAP. Comparing two figures, small difference exists but direction of velocities can be estimated.

### Conclusion

In this analysis, domain decomposition is applied for model problem of shallow water flow and actual tidal flow problem using Kalman filter.

In case 1, comparing results of two computations, two values almost agree (Fig.5 and Fig.6). These results show that algorithm of SAP with two sub-domains is established. And CPU time becomes 3 times shorter than that without SAP. In case2, time histories of results of water elevation also agree each other (Fig.8). Comparing velocity distributions using SAP and without using SAP shows that small difference exists. The estimation using SAP is smaller than that not using SAP. But direction of velocity is almost the same. If we refine the iteration method, more faithful analysis could be carried out.

Not only water elevation but also velocity distribution show that SAP with three sub-domains is not perfect. It seems that cause of this failure is overlapping areas are plural. If overlapping areas exist more than one, values obtained by calculation are not reflecting results of whole domain.

Algorithm of Kalman filter using SAP with three sub-domains has to be established. If we would succeed, CPU time becomes shorter.

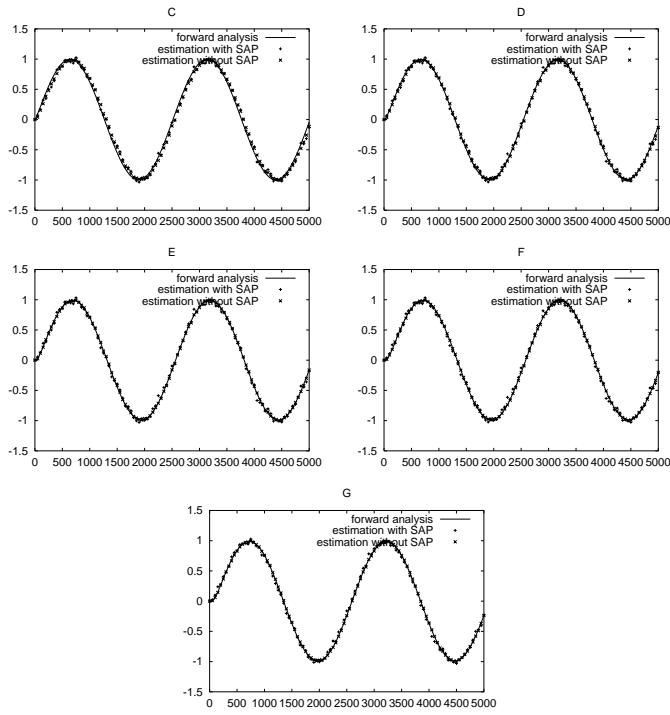


Figure 8: Comparison about water elevation

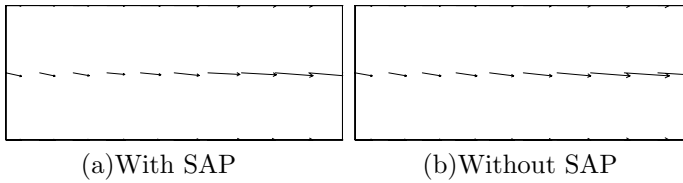


Figure 9: Velocity distribution (Residual flow)

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