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35. Two level Domain Decomposition for Multi-clusters

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Introduction

We discuss the design of parallel algorithms to solve elliptic problems on multi-clusters computers. Multi-clusters can be seen as two-level parallel architecture machines, since communication between clusters are usually much slower than communication or access to memory within each of the clusters. We introduce special algorithms that use two levels of parallelism and match the multi-cluster architecture. Efficient parallel algorithms that rely on fast uniform communication have been extensively developed in the past: we intend to use them for parallel computation within the clusters. On top of these local parallel algorithms, new *robust and parallel* algorithms are needed that can work with few clusters linked by a slow communication network. We present a two level domain decomposition algorithm that uses Aitken or Steffensen acceleration procedure combined to Schwarz for the outer loop and standard parallel domain decomposition for the inner loop. We demonstrate finally the interest of our algorithm for metacomputing.

We consider the design of parallel algorithms for multi-cluster architecture with few heterogeneous clusters linked by an affordable network of order 10Mb/s bandwidth. Each cluster can be a shared multiprocessors machine or an MIMD computer with a fast internal Network. The elapse time to access memory from a given processor to a given data on such architecture is then strongly dependent on the location of the datas. Fast scalable parallel algorithm for the Laplace problem with domain decomposition and/or multigrid on a uniform MIMD architecture have usually very poor efficiency on multi-cluster machine with slow inter-cluster network.

On the contrary a numerically unefficient iterative domain decomposition algorithm such as the classical additive Schwarz procedure for the Laplace problem, is easy to implement, robust and scalable on multi-cluster architecture. So our goal is the design of an acceleration procedure for iterative domain decomposition analogous to additive Schwarz that increases the numerical efficiency of the basic underlined algorithm but stay easy to implement, robust and scalable on multi-clusters. The common procedure to accelerate additive Schwarz method is the introduction of a coarse-grid operator [LSFQ97]. The resulting modified Schwarz algorithms becomes numerically efficient but the coarse grid computation might be a bottle neck for the parallel processing. We adopt here a different point of view and try to extract from a finite sequence of the interfaces generated by the Schwarz iterative procedure or analogous relaxation method, an accurate prediction of the interface's limit. We will show in simple case as finite difference approximation of Elliptic operator with con-

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stant coefficient on regular grids, that we can obtain a fast *direct* solver so called Aitken-Schwarz procedure. In more complex situation, we shall derive a fast iterative solver by alternating few Schwarz iterations with Aitken acceleration [SB80]. We will call this methodology Steffensen-Schwarz following the spirit of the Steffensen method in non-linear context [Hen64]. The main advantage of our approach is that the new algorithm requires only the coding of an independent subroutine that processes the sequences of interfaces generated by the basic domain decomposition method. In addition, we will show that this subroutine does not require too many communications and performs efficiently on multi-clusters with slow inter-cluster network. We will report in particular on a successful metacomputing experiment with distanced parallel computers.

The plan of this article is as follows. Next section presents a new family of domain decomposition algorithms in the one dimensional case. Then we generalize the method to multidimensional elliptic operator with strip domain decomposition, before presenting in an another section some extension of the results to linear elliptic operator with varying coefficients and non linear elliptic operators. Some results on large scale parallel computing are reported in the last section before our conclusion.

Basic idea in one D

two subdomains with Dirichlet-Dirichlet BC

Let us consider a linear problem

$$L[U] = f \ in \ \Omega, \ U_{|\partial\Omega} = 0. \tag{1}$$

L can be the continuous problem or the discrete one. We restrict ourselves to two subdomains and start with the additive Schwarz algorithm. For simplicity of the description of the method, we assume implicitly in the following notations that the homogeneous Dirichlet boundary condition in (1) is satisfied by all intermediate subproblems.

$$L[u_1^{n+1}] = f \ in \ \Omega_1, \ u_{1|\Gamma_1}^{n+1} = u_{2|\Gamma_1}^{n}, \tag{2}$$

$$L[u_2^{n+1}] = f \ in \ \Omega_2, \ u_{2|\Gamma_2}^{n+1} = u_{1|\Gamma_2}^{n}. \tag{3}$$

We observe that the operator T,

$$u_{i|\Gamma_i}^{n} - U_{\Gamma_i} \rightarrow u_{i|\Gamma_i}^{n+2} - U_{\Gamma_i} \tag{4}$$

is *linear*.

Let us consider first the one-dimensional case $\Omega = (0,1)$: the sequence $u_{i|\Gamma_i}^{2n}$ is a sequence of real numbers. Note that as long as the operator T is linear, the sequence $u_{i|\Gamma_i}^{n+2}$ has pure linear convergence (or divergence); that is, it satisfies the identity $u_{i|\Gamma_i}^{n+2} - U_{|\Gamma_i} = \delta(u_{i|\Gamma_i}^{n} - U_{|\Gamma_i})$, where δ is the amplification factor of the sequence. Let us assume $\delta \neq 1$. The Aitken acceleration procedure gives the *exact* limit of the sequence

on the interface Γ_i based on three successive Schwarz iterates $u^j_{i|\Gamma_i}$, $j = 1, 2, 3$, and the initial condition $u^0_{i|\Gamma_i}$, namely,

$$u^{\infty}_{\Gamma_i} = \frac{u^0_{i|\Gamma_i} u^3_{i|\Gamma_i} - u^1_{i|\Gamma_i} u^2_{i|\Gamma_i}}{u^3_{i|\Gamma_i} - u^2_{i|\Gamma_i} - u^1_{i|\Gamma_i} + u^0_{i|\Gamma_i}}.$$

An additional solve of each subproblem (2,3) with boundary conditions $u^{\infty}_{\Gamma_i}$ gives the solution of (1). The Aitken acceleration thus transforms the additive Schwarz procedure into an *exact* solver regardless of the speed of convergence of the original Schwarz method.

With the previous algorithm, we do need 3 solves of each subproblem to apply the Aitken acceleration and an additional solve of each subproblem to get the solution. We can derive a more numerically efficient algorithm that requires 3 solves of each subproblems in the following way: we have

$$u^{n+1}_{1|\Gamma_2} - U_{|\Gamma_2} = \delta_1 (u^n_{2|\Gamma_1} - U_{|\Gamma_1}), \tag{5}$$

$$u^{n+1}_{2|\Gamma_1} - U_{|\Gamma_1} = \delta_2 (u^n_{1|\Gamma_2} - U_{|\Gamma_2}), \tag{6}$$

where δ_1 (resp δ_2) is the damping factor associated to the operator L in subdomain Ω_1 (resp Ω_2) [Gar96]. Consequently

$$u^2_{1|\Gamma_2} - u^1_{1|\Gamma_2} = \delta_1 (u^1_{2|\Gamma_1} - u^0_{2|\Gamma_1}),$$

$$u^2_{2|\Gamma_1} - u^1_{2|\Gamma_1} = \delta_2 (u^1_{1|\Gamma_2} - u^0_{1|\Gamma_2}),$$

So except if the initial boundary conditions $u^0_{2|\Gamma_1}$ or $u^0_{1|\Gamma_2}$ matches with the exact solution U at the interfaces Γ_i , the amplification factors δ_1 and δ_2 can be computed from (5) and (6). Then if $\delta_1 \delta_2 \neq 1$ the limit $U_{|\Gamma_i}, i = 1, 2$ is obtained as the solution of the linear system (5, 6).

We observe that δ_1, δ_2 are dependent only on the operator and the partitioning of the domain. δ_1 for example can be computed before hand as follows. Let $v_{1/2}$ be the solution of

$$L[v_{1/2}] = 0 \; in \; \Omega_{1/2}, \; v_{|\Gamma_{1/2}} = 1. \tag{7}$$

We have $\delta_{1/2} = v_{|\Gamma_{2/1}}$. When $\delta_{1/2}$ is a priori known, we need only *one* Schwarz iterate to accelerate the interface and an additional solves for each subproblems. This is a total of two solves per subdomain. This feature is particularly attractive when the elliptic problem (1) has to be solved many times.

two subdomains with Dirichlet-Newman BC

It is interesting that the same idea applies to other well-known iterative procedures such as the Dirichlet-Newman iterative procedure that has the advantage of using non overlapping partitioning but the disadvantage of possible divergence. The relaxation procedure of the Funaro-Quarteroni algorithm [FQZ88], can fix this convergence

problem when the relaxation parameter is chosen correctly. However the Aitken acceleration procedure may solve the artificial interface problem whether the original Dirichlet-Neumann iterative procedure converges or diverges, as long as the sequence of solution at the interface behaves linearly! To be more specific let us consider for example the Helmholtz problem:

$$-\frac{d^2}{dx^2}U + \mu U = f \; in \; \Omega, \; U = 0 \; on \; \partial\Omega.$$

The domain Ω is split into two non overlapping subdomains that share the interface Γ. We consider the iterative procedure,

$$-\frac{d^2}{dx^2}u_1^n + \mu u_1^n = f \; in \; \Omega_1, \; u_1^n = u_2^n \; on \; \Gamma, \tag{8}$$

$$-\frac{d^2}{dx^2}u_2^n + \mu u_2^n = f \; in \; \Omega_2, \; \frac{\partial u_2^n}{\partial x} = \frac{\partial u_1^n}{\partial x} \; on \; \Gamma. \tag{9}$$

We approximate this problem with 2^d order finite differences for (8) and one side first order finite differences for the boundary condition in (9). The computation of each subproblems (8) and (9) is a priori a sequential process. The sequence of real numbers u_1^n generated by this algorithm has linear convergence to U_Γ or linear divergence depending on the interface location Γ, that is $u_{1|\Gamma}^{n+1} - U_{|\Gamma} = \delta(u_{1|\Gamma}^n - U_{|\Gamma})$, where δ is the amplification factor of the sequence. Once again the Aitken acceleration procedure gives the *exact* limit of this sequence no matter the value of $\delta \neq 1$, with

$$u_\Gamma^\infty = \frac{u_{1|\Gamma}^0 u_{1|\Gamma}^2 - u_{1|\Gamma}^1 u_{1|\Gamma}^1}{u_{1|\Gamma}^2 - 2u_{1|\Gamma}^1 + u_{1|\Gamma}^0}.$$

So far, we have restricted ourselves to domain decomposition with two subdomains. Next we will introduce a generalized Aitken acceleration technique that can be applied to an arbitrary number $q > 2$ of subdomains.

more than 2 subdomains case with Dirichlet-Dirichlet BC

Let $\Omega_i = (x_i^l, x_i^r), \; i = 1..q$ be a partition of Ω with $x_2^l < x_1^r < x_3^l < x_2^r, ..., x_q^l < x_{q-1}^r$. We consider the additive Schwarz algorithm

for $i = 1..q$, *do*
$L[u_i^{n+1}] = f \; in \; \Omega_i, \; u_i^{n+1}(x_i^l) = u_{i-1}^n(x_i^l), \; u_i^{n+1}(x_i^r) = u_{i+1}^n(x_i^r),$
enddo

Let us denote $u_i^{l,n+1} = u_i^{n+1}(x_i^l)$, $u_i^{r,n+1} = u_i^{n+1}(x_i^r)$ and $\tilde{u}^n$ (respt $\tilde{u}$) be the n iterated (respt exact) solution restricted at the interface, i.e

$$\tilde{u}^n = (u_2^{l,n}, u_1^{r,n}, u_3^{l,n}, u_2^{r,n}, ..., u_q^{l,n}, u_{q-1}^{r,n})$$

The operator $\tilde{u}^n \to \tilde{u}^{n+1}$ is linear. Let us denote P its matrix. P has the following pentadiagonal structure:

$$\begin{vmatrix} 0 & \delta_1^r & 0 & 0 & .... & & & \\ \delta_2^{l,l} & 0 & 0 & \delta_2^{l,r} & ... & & & \\ \delta_2^{r,l} & 0 & 0 & \delta_2^{r,r} & ... & & & \\ & & & & & & & \\ & & & ... & \delta_{q-1}^{l,l} & 0 & 0 & \delta_{q-1}^{l,r} \\ & & & ... & \delta_{q-1}^{r,l} & 0 & 0 & \delta_{q-1}^{r,r} \\ & & & ... & 0 & 0 & \delta_q^r & 0 \end{vmatrix}$$

δ_1^r and δ_q^r can be computed as in the two subdomain cases.

The subblocks $P_i = \begin{vmatrix} \delta_i^{l,l} & \delta_i^{l,r} \\ \delta_i^{r,l} & \delta_i^{r,r} \end{vmatrix}$ $i = 2..q-1$ can be computed with 3 Schwarz iterates as follows.
We have $(u_{i-1}^{r,n+1} - \tilde{u}_{i-1}^r, u_{i+1}^{l,n+1} - \tilde{u}_{i+1}^l)^t = P_i(u_i^{l,n} - \tilde{u}_i^l, u_i^{r,n} - \tilde{u}_i^r)^t$. Therefore

$$\begin{pmatrix} u_{i-1}^{r,n+3} - u_{i-1}^{r,n+2} & u_{i-1}^{r,n+2} - u_{i-1}^{r,n+1} \\ u_{i+1}^{l,n+3} - u_{i+1}^{l,n+2} & u_{i+1}^{l,n+2} - u_{i+1}^{l,n+1} \end{pmatrix} = P_i \begin{pmatrix} u_i^{l,n+2} - u_i^{l,n+1} & u_i^{l,n} - u_i^{l,n} \\ u_i^{r,n+2} - u_i^{r,n+1} & u_i^{r,n} - u_i^{r,n} \end{pmatrix} \quad (10)$$

In practice the last matrix on right hand side of the previous equation is non singular and P_i can be computed, but it cannot be guaranty. However, one can always compute before hand the coefficients of P_i as follows. Let v be the solution of

$$L[v] = 0 \; in \; \Omega_i, \; v(x_i^l) = 1, \; v(x_i^r) = 0, \quad (11)$$

and w be the solution of

$$L[w] = 0 \; in \; \Omega_i, \; w(x_i^l) = 0, \; w(x_i^r) = 1. \quad (12)$$

We have then $\delta_i^{l,l} = v(x_{i-1}^r)$, $\delta_i^{l,r} = v(x_{i+1}^l)$ $\delta_i^{r,l} = w(x_{i-1}^r)$ and $\delta_i^{r,r} = w(x_{i+1}^l)$. We observe that this computation of the subblocks P_i can be done in parallel.

In addition, for the Helmotz operator $L[u] = u'' - \lambda u$, or generally speaking elliptic problems with constant coefficients, the matrix P is known analytically.

From the equality

$$\tilde{u}^{n+1} - \tilde{u} = P(\tilde{u}^n - \tilde{u}),$$

one writes the generalized Aitken acceleration as follows:

$$\tilde{u}^\infty = (Id - P)^{-1}(\tilde{u}^{n+1} - P\tilde{u}^n). \quad (13)$$

If the additive Schwarz method converges, then $||P|| < 1$ and $Id - P$ is non singular.
The algorithm is then

- step1 : compute analytically or numerically in parallel each subblocks P_i from each subproblems (11,12).
- step2: apply one additive Schwarz iterate.
- step3: apply generalized Aitken acceleration on the interfaces based on (13) with $n = 0$.
- step4: compute in parallel the solution for each subdomain.

Algorithm I

From the point of view of parallelism step1 and step4 does not requires any communication. step2 requires local communication between subdomains that overlap. Step3 on the contrary requires global communication. We will see in the next section, how theses basic ideas can be extended on multidimensional elliptic operators and how to minimize the global communications involved in step3.

Multidimensional elliptic operator

general formal framework

Next, let us consider the multidimensional case with the discretized version of the problem (1). We restrict ourselves for simplicity to the two overlapping subdomain case and the additive Schwarz algorithm (2, 3). Let us denote $E_i^h, i = 1, 2$ some finite vector space used to approximate the solution restricted to the artificial interface $\Gamma_i, i = 1, 2$. Let $b_i^j, j = 1..N$ be a set of basis functions for this vector space and P be the corresponding matrix of the linear operator T

$$u^n_{i|\Gamma_i} - U_{\Gamma_i} \rightarrow u^{n+2}_{i|\Gamma_i} - U_{\Gamma_i}.$$

We denote by $u^n_{i,j}, j = 1, .., N$ the components of $u^n_{i|\Gamma_i}$, and we have then

$$(u^{n+2}_{i,j} - U_{j|\Gamma_i})_{j=1,..,N} = P(u^n_{i,j} - U_{j|\Gamma_i})_{j=1,..,N}.$$

let us suppose that the interface sequence is such that the matrix $(u^{2(j+1)}_{k,i} - u^{2j}_{k,i})_{i=1,..,N,j=0,..,N-1}$ is non singular. Let Id be the matrix for the identity operator. We introduce a generalized Aitken acceleration with the following formula: first

$$P = (u^{2(j+1)}_{k,i} - u^{2j}_{k,i})_{i=1,..,N,j=1,..,N}(u^{2(j+1)}_{k,i} - u^{2j}_{k,i})^{-1}_{i=1,..,N,j=0,..,N-1}, k = 1, 2,$$

and second, if $Id - P$ is non singular, the trace of the exact solution $(u_{k,i})_{i=1,..,N}$ on interface $\Gamma_k, k = 1, 2$ is the solution of the linear system

$$(Id - P)(u^\infty_{k,i})_{i=1,..,N} = (u^{2N+2}_{k,i})_{i=1,..,N} - P\,(u^{2N}_{k,i})_{i=1,..,N}.$$

If this generalized Aitken procedure works, it should be a priori independently of the spectral radius of P, that is, the convergence of the underlined Schwarz additive iterative procedure is not needed. In conclusion, $2N + 1$ Schwarz iterates produce a priori enough data to compute via this generalized Aitken acceleration the interface value $U_{|\Gamma_k}, k = 1, .., 2$. This computation is amenable to $N + 1$ Schwarz iterates, if one accelerates the sequence of coupled interfaces corresponding to the linear mapping

$$(u^n_{1|\Gamma_1} - U_{\Gamma_1}, u^n_{2|\Gamma_2} - U_{\Gamma_2}) \rightarrow (u^{n+1}_{1|\Gamma_1} - U_{\Gamma_1}, u^{n+1}_{2|\Gamma_2} - U_{\Gamma_2}).$$

However, we can expect that the matrix $(u^{2(j+1)}_{k,i} - u^{2j}_{k,i})_{i=1,..,N,j=0,..,N-1}$ is ill-conditioned and that the computed value of P is very sensitive to the data. In addition N or $2N$ Schwarz iterates is too many iterates to be considered as an efficient procedure.

Nevertheless, we have numerical evidence that this procedure can perform on two dimensional linear elliptic problems with stiff coefficients [GTD99]

We are currently investigating diverse strategies to make this algorithm useful and efficient in the framework of unstructured grid but we will restrict ourselves in this paper to the case of regular grids for which sine or cosine expansion of the traces generated by additive Schwarz is a natural tool.

Aitken-Schwarz method for Elliptic Operator

Let us consider first the Poisson problem $u_{xx} + u_{yy} = f$ in the square $(0, \pi)^2$ with Dirichlet boundary conditions. We partition the domain into an arbitrary number nd of overlapping strips: $\Omega = \bigcup_{j=1..nd} \Omega_j$. We introduce the regular discretization in the y direction $y_i = (i-1)h$, $h = \frac{1}{N-1}$, and central second-order finite differences of the u_{yy} derivative. Let us denote by $\hat{u}_i$ (resp. $\hat{f}_i$) the coefficient of the sine expansion of u (resp. f). The Poisson problem decomposes then into N independents semi-discretized equation corresponding to sinus waves $sin(iy), i = 1..N$,

$$\hat{u}_{i,xx} - 4/h^2 \; sin^2(i\frac{h}{2}) \; \hat{u}_i = \hat{f}_i, \tag{14}$$

The matrix P for the set of basis functions $b_i = sin(i\frac{y}{\pi})$ is therefore *diagonal*. The Aitken Schwarz algorithm is very similar to the algorithm derived in the one dimensional case. In particular the coefficients of each wave number of the trace of the solutions generated by the Schwarz algorithm has its own linear rate of convergence, the high frequencies terms being damped the fastest. The algorithm writes:

- step1 : compute analytically or numerically in parallel each subblocks P_i from each subproblems (11,12) and each operator $L_i[v] = v_{xx} - 4/h^2 \; sin^2(i\frac{h}{2}) \; v$.
- step2: apply one additive Schwarz iterate to the Poisson problem with block solver of choice i.e multigrids, FFT etc...
- step3:
  - compute the sine expansion $\hat{u}^n_{j|\Gamma_i}, n = 0, 1$ of the traces on the artificial interface $\Gamma_i, i = 1..nd$ for the initial boundary condition $u^0_{|\Gamma_i}$ and the solution given by one Schwarz iterate $u^1_{|\Gamma_i}$.
  - apply generalized Aitken acceleration based on (13) with $n = 0$ *separately* to each wave coefficients in order to get $\hat{u}^\infty_{j|\Gamma_i}$.
  - recompose the trace $u^\infty_{j|\Gamma_i}$ in physical space.
- step4: compute in parallel the solution in each subdomains Ω_j, with new inner BCs and block solver of choice.

Algorithm II

This algorithm has a very high potential of parallelism. step 1 and 4 are fully parallel. Step 2 requires only local communication and scale well with the number of processors. Step 3 requires global communication of interfaces in Fourier space. But high frequency have very fast decay and little influence on the final solution. Therefore one can restrict adaptively the Aitken acceleration process of step3 to a subset

$\hat{u}_j^n, j = 1..M$, with $M < N$, and minimize the amount of global communications. In addition the arithmetic complexity of step3 that is the kernel of the method is negligible compare to step2. Further, this procedure works independently of the discretization and grids in x direction as long as the block solvers for each subproblems are exact. The same idea can be applied to Elliptic problems with constant coefficients or x dependent coefficients since the matrix P in such cases stays diagonal. Let us notice that for Elliptic problem with homogeneous Neumann BC instead of Dirichlet BC, one has to accelerate the cosine expansion of the interface's sequence. For Elliptic problem with non homogeneous BC, it is convenient to work on a shifted sequence that satisfies the homogeneous BC.

To exemplify the Aitken Schwarz procedure with a slightly more difficult case, let us consider the transmission problem:

$$-\mu_1 \Delta u_1 + u_1 = f \ \ in \ \ (0, \frac{\pi}{2}) \times (0, \pi) \tag{15}$$

$$-\mu_2 \Delta u_2 + u_2 = f \ \ in \ \ (\frac{\pi}{2}, 1) \times (0, \pi) \tag{16}$$

with homogeneous Dirichlet boundary conditions. μ_1 and μ_2 are positive constants. Let us discretize this simple problem with second order central differences and iterate with a Dirichlet-Neumann domain decomposition. For $\mu_1 = 1$ and $\mu_2 = 8$ this procedure is *linearly* divergent, but the following Aitken acceleration applied to the sine expansion of the trace of the solution $u_1(., y)$ at $x = \frac{\pi}{2}$,

$$\hat{u}_k^\infty = \hat{u}_k^0 - \frac{(\hat{u}_k^1 - \hat{u}_k^0)^2}{\hat{u}_k^2 - 2\hat{u}_k^1 + \hat{u}_k^0},$$

generates the sine expansion of the exact interface solution modulo the residual error of each subdomain solve. Fig 1 reports on the numerical result obtained with matlab for a small test case i.e 25 by 25 grid points. This example is interesting because the convergence history has not the classical behavior that one may expect!.

Let us now describe briefly some key aspect of the stability of the Aitken Schwarz algorithm.

sensitivity analysis

It is interesting to understand how behaves the Aitken-Schwarz method if one use inexact block solver or approximation of the matrix of operator T. This is obviously related to the stability of the acceleration procedure with respect to perturbation of P or perturbation of $\tilde{u}^n$. Let us summarize briefly the results we found for discrete linear elliptic operators that satisfies a maximum principle. Extension of the results and details of the analysis will be available in a forthcoming paper.

We assume for simplicity a uniform strip domain decomposition and writes

$$\begin{pmatrix} \delta_1 & 0\ 0\ \delta_2 \\ \delta_2 & 0\ 0\ \delta_1 \end{pmatrix} \tag{17}$$

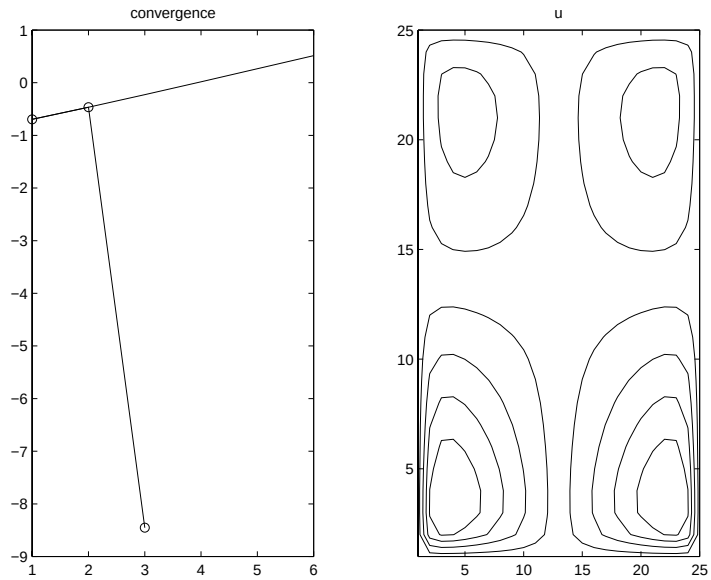

Figure 1: Dirichlet-Newman Algorithm for a Transmission Problem. Solid line (resp. -o- line) gives the log_{10} (error in maximum norm) on the discrete solution additive with basic procedure (resp. new method)

the generic subblock of P for a given wave number k.

Let $\tilde{P}$ be an approximation of P. The relative error on the artificial interface vector $\tilde{u}$ is then bounded by

$$2\frac{||(Id-P)^{-1}||^2||(P-\tilde{P})||}{1-||(Id-P)^{-1}(P-\tilde{P})||}+||(Id-P)^{-1}(P-\tilde{P})||.$$

Since the operator L satisfied a maximum principle, this corresponds to the global error. A straightforward application of this estimate is the minimization of the communication constraint in step 3 of Aitken-Schwarz'Algorithm, if one neglects interactions between subdomains that are not neighbors. It is equivalent to approximate P with the following matrix $\tilde{P}$ for acceleration:

$$\begin{vmatrix} 0 & \delta_1 & 0 & 0 & .... & & & & \\ \delta_1 & 0 & 0 & 0 & ... & & & & \\ 0 & 0 & 0 & \delta_1 & ... & & & & \\ & & & & & & & & \\ & & & ... & \delta_1 & 0 & 0 & 0 & \\ & & & ... & 0 & 0 & 0 & \delta_1 & \\ & & & ... & 0 & 0 & \delta_1 & 0 & \end{vmatrix}$$

The error on the corresponding predicted wave amplitude of the interface given by the incomplete Aitken acceleration is then bounded by $(2\delta_2\frac{1+\delta_1}{1-\delta_1}+\delta_2)/(1-\delta_1^2)$. It is clear that δ_1 and δ_2 decrease as the corresponding frequency increases. One can therefore

decouple adaptively the computation depending on the wave number, preserving the overall accuracy of the method.

One can also analyze the impact of inexact sub-block solver. Let us restrict ourselves to the Poisson problem in two space dimensions with five point schemes. If P_i is computed either analytically or independently with high accuracy, the numerical error is then bounded by $\frac{\eta}{h}$ where η stands for the maximum error in each inexact block solves and h for the time step. If P is computed numerically from Schwarz iterates with inexact sub-block solve the situation is more complicated. The acceleration procedure is much more sensitive and we get an upper bound of order $\frac{\eta}{h^3}$.

Because the accuracy of the Aitken-Schwarz procedure deteriorates with the uncomplete construction of the matrix P or the inexact sub-block solve, it is natural to apply the same acceleration procedure in a loop until appropriate convergence. We name this procedure a Steffensen-Schwarz algorithm and we are going to show that this algorithm is suitable to solve elliptic problems far more complicated than the Poisson problem.

Steffensen-Schwarz method for linear and non linear elliptic operator

Let us consider first the Linear case $L = -\Delta u + a(x,y)u$, with a varying smooth coefficient a. In all numerical experiments, thereafter, we will consider strip domain decomposition with *minimum overlap*, i.e one mesh overlap.

Linear Elliptic operator

For simplicity of the presentation, we consider (4) with only two overlapping subdomains. The elementary methods described for the Poisson problem in Section *Multidimensional elliptic operator* fails to be an exact solver if the grid has a non constant space step in the y direction or if the operator has coefficients depending on the x *and* y variable, because P is no longer diagonal but rather a dense matrix!. However if one approximates the coefficients a by its Z truncated Cosine expansions as follows,

$$a(x,y) \approx \Sigma_{k=1..Z} \hat{a}_k(x) cos((k-1)y),$$

matrix P is then a sparse matrix of bandwidth $2Z+1$. Our heuristic strategy is therefore to try to rebuild from the sequence of $2Z+1$ consecutive interfaces generated by Schwarz, a band approximation P_Z of P. We look then for P_Z such that,

$$(\hat{u}_i^{2Z+2} - \hat{u}_i^{2Z+1}, ..., \hat{u}_i^3 - \hat{u}_i^2, \hat{u}_i^2 - \hat{u}_i^1) = (P_{i,i-Z}, ..., P_{i,i+Z}) \times S_B, \tag{18}$$

where S_B is the following subblock

$$\begin{pmatrix} \hat{u}_{k-Z}^{2Z+1} - \hat{u}_{k-Z}^{2Z} & ...\hat{u}_{k-Z}^1 - \hat{u}_{k-Z}^0 \\ . & ... \\ . & ... \\ \hat{u}_{k+Z}^{2Z+1} - \hat{u}_{k+Z}^{2Z} & ...\hat{u}_{k+Z}^1 - \hat{u}_{k+Z}^0 \end{pmatrix} \tag{19}$$

provided by the Schwarz iterative process. (18) holds for $Z < i \leq N - Z$. Similar equation can be written with appropriate reduced dimension for the end terms of the diagonal of P_Z that is when $i \leq Z$ or $i > N - Z$. If S_B is non singular, the k^{ieme} row of P_Z is well defined. Otherwise, we have to decrease Z for this specific row until the subblock is non singular. In practice the conditioning of the subblock deteriorates when the frequency increases but only low frequencies needed to be accelerated since high frequencies are damped very fast by the Schwarz method itself.

Fig 2a and Fig 2b give numerical illustration of the method for different coefficient functions $a(x, y)$ and different choices for the bandwidths. Convergence curves are commented with + sign for Z=1, o sign for Z=2 and v sign for Z=3. We have chosen coefficients $a = 1 + y$ and $a = 1. + exp(sin(y)$ that have cosine expansion with growing speed of convergence. Our numerical experiment seems to confirm that the faster the cosine expansion of $a(x, .)$ converges, the faster converges the Steffensen approximation with the diagonal approximation $Z = 1$ of P. On the contrary the $Z = 3$ approximation improves best the convergence compare to the algorithm with $Z = 1$, when the convergence of the Fourier expansion of a is slow -see Fig 2a.

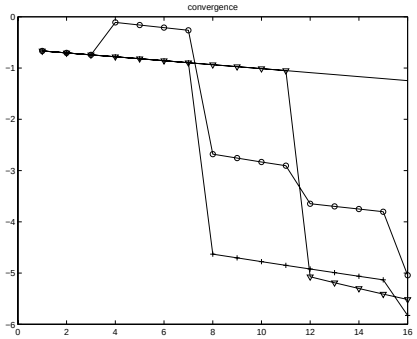

Fig 2a: $a(x, y) = 1. + y$

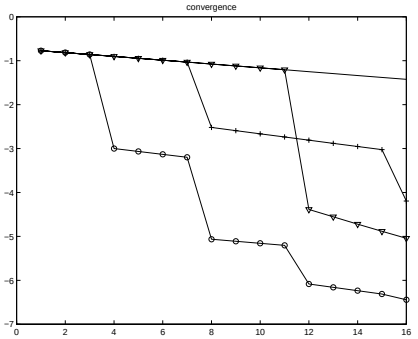

Fig 2b: $a(x, y) = 1. + exp(sin(y))$

Fig 3a and Fig 3b report on similar results but for the Poisson problem on an irregular domain that is a square except on one side, that is replaced by a reentry corner.

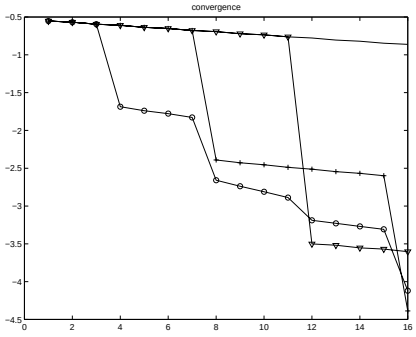

Fig 3a:Convergence history for irregular geometry

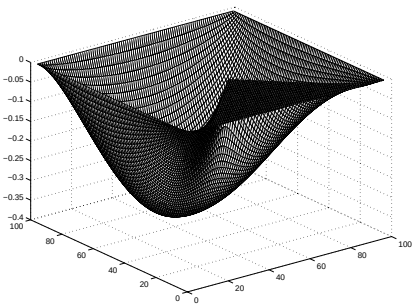

Fig 3b:Solution of the problem

the non linear case

We consider a one dimensional nonlinear problem that is a simplified model of a semiconductor device [Sel84]. The model writes

$$\Delta u = e^u - e^{-u} + f,\ in(0,d), \tag{20}$$

$$f = tanh(20(\frac{x}{d} - \frac{1}{2})),\ x \in (0,d), \tag{21}$$

$$u(0) = asinh(\frac{f(0)}{2}) + u_o, u(d) = asinh(\frac{f(d)}{2}) \tag{22}$$

The problem is discretized by means of second-order central finite differences. We apply Steffensen-Schwarz method with two subdomains and minimum overlap. In particular, we solve a non linear problem in each subblock at each iteration step of additive Schwarz. Fig 4a reports on the numerical results with 80 grid points. The convergence history shows that the closer the iterate gets to the final solution, the better is the result of the Aitken acceleration. This Newton like property of convergence of the algorithm can be actually proven using the monotonicity of the discrete non linear operator -see also [Hen64].

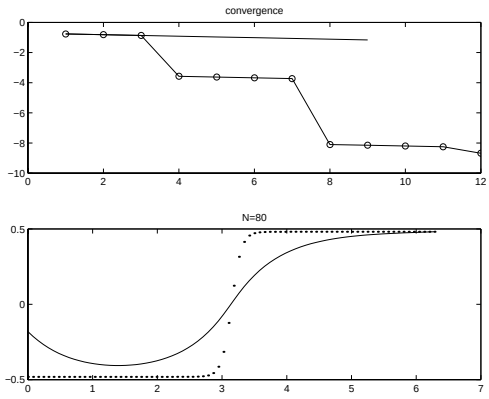

Fig 4a: One D semi conductor problem

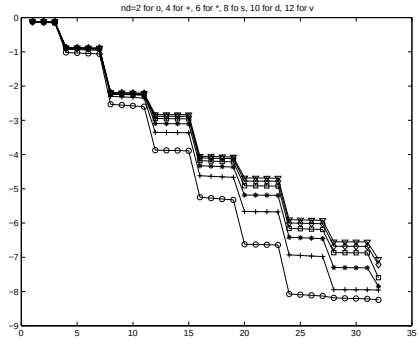

Fig 4b: Convergence for the two-D Bratu problem

We consider second the *Bratu* problem [Wie96],

$$-\Delta u = \lambda e^u,\ in\ \Omega = (0,1)^2, \tag{23}$$

$$u_{|\partial\Omega} = 0 \tag{24}$$

This problem has a smooth solution for $\lambda \in (0, 6.81)$. We have experimented the Steffensen Schwarz algorithm for the classical five points finite difference scheme with strip domain decomposition, an arbitrary number of subdomains and $\lambda = 6$. Our numerical experiments have shown that the Steffensen-Schwarz algorithm with diagonal approximation of P is best. Let us notice that $u(x_i, .)$ restricted to artificial interfaces of strip domain decomposition is a continuous periodic function of period 1; non homogeneous boundary conditions might then lead to a different choice for Z.

Fig 4b shows the solution and the convergence of our methods with a grid of approximatively fixed size 60×60 and an increasing number of subdomains from 2

to 12. It can be seen that unfortunately the one dimensional quadratic convergence property is lost in multidimensional problems, because the linear approximation of the operator has coefficients depending on space. As a matter of fact each step between two plateau in the convergence history has about the same size. However, it is most interesting to notice that the number of Steffensen-Schwarz iterates required to reach a given level of accuracy depends slightly on the number of subdomains. The total number of Schwarz iterates to reach an error less than 10^{-7} in maximum norm is 24 with 3 subdomains, and 32 with 12 subdomains.

We are going now to return to parallel efficiency of this new domain decomposition domain that was our motivation.

Application to distributed computing

We report on performance of Aitken Schwarz algorithm for three dimensional Poisson problem. Each subblock will be solved on a parallel system itself with a "classical" parallel algorithm. We will therefore referee to subblocks as macro subblocks since there are also decomposed into subdomains. To be more precise our Aitken Schwarz code is part of a 3 dimensional Navier Stokes code and is used to solve simultaneously 3 Laplace problems for each component of the flow speed [TD93]. The Aitken-Schwarz method in three D is similar to the two D algorithm II, except that we use two dimensional FFT for interfaces. In addition, the matrix $P_{i,j}$ corresponding to each couples of sine waves $[sin(iy), sin(jz)]$ can be precomputed analytically. Each macro subblock is solved with a parallel algorithm that combines multigrid and Schur dual complement method (**MCSD**). This parallel macro-block solver is very efficient and scalable on large MIMD system with uniform network.

We first compare the Aitken Schwarz method with MCSD on a SGI Origin 2000 system thanks to the **Centre Informatique National de l'Enseignement Supérieur** support.

Table 1 gives elapse time for 3 Laplace solve with 8388608 grid points. The "one Macro-Subdomain row" corresponds to MCSD algorithm. The three next rows correspond to the combination of Aitken Schwarz for the macro domain decomposition and MCSD in each macro subdomain. Our actual implementation of Aitken Schwarz is not optimum, since we use blocking communications, redundant interface treatment, and gather of all the interfaces. However we see that this new method can compete with our former optimized implementation of MCSD technique.

On large MIMD machine the salient feature of our multilevel domain decomposition is not used because we have not been able to allocate the processes in order to get the best performance of SGI network. In metacomputing experiments, we obtain our main result: table 2 shows that Aitken Schwarz performs 10 times better than MCSD when one use two clusters linked by a 10Mb/s network. In this experiment we have used two different generations of Compaq clusters with one or two 4 ev5 hypernodes called 4100 Dec alpha servers and dual ev6 hypernode called DS20. The elapsed time in this table are given for 3 Laplace solves and a total of 197000 unknowns. Table 3 shows that our Aitken Schwarz gives also very good results for a slow non dedicated network i.e 2Mb/s that is the France Telecom regular link between University Lyon1-Claude Bernard and Ecole Normale Supérieure of Lyon (**ENSL**) 10 kilometers away. The

total number of unknowns in this last experiment is 288000, and we use in addition to Dec alpha4100 and DS20 alpha servers (respectively CDCSP-MOBY and CDCSP DS20), the sun Enterprise 10000 parallel computer of the Pole of Numerical Simulation and Modeling of ENSL (PSMN-SDF1). Let us mention, that in this last case, it is hopeless to use MCSD.

Number of Macro Subdomains	Time in second	Error in Maximum norm	FFT	gather interface
1	46.5	1.3 E-12		
2	71.5	2.0 E-12	2.0s	0.3s
4	56.1	1.0 E-12	4.6s	6.5s
8	59.6	2.6 E-12	11.0s	13.0s

Table 1: *Performance of the analytical Aitken Schwarz algorithm on SGI system.*

Cluster 1 3 or 4 processors	Cluster 2 2 or 3 or 4 processors	Elapse time in second	Bandwidth of network
4 ev5 CDCSP-MOBY	4 ev5 CDCSP-MOBY	28.4s	100 Mb/s
4 ev5 CDCSP-MOBY	2 ev6 CDCSP-DS20	29.4s	10 Mb/s
2 ev5 CDCSP-MOBY 1 ev6 CDCSP-DS20	2 ev5 CDCSP-MOBY 1 ev6 CDCSP-DS20	220.7s	10 Mb/s

Table 2: *Performance of the analytical Aitken Schwarz algorithm on intranet.*

We are currently running similar experiment with metacomputing between large parallel systems located in different countries in order to validate our approach on 3D large scale complex problems.

Conclusion

We have developed in this paper a new two levels domain decomposition method designed to work efficiently on multi-cluster architecture. We have combined fast parallel solvers such as Multigrids and Schur complement dual that are scalable and efficient inside the clusters and acceleration of robust solvers as additive Schwarz algorithm that does not require too many inter-cluster communications. We have shown that

Cluster 1 4 processors	Cluster 2 4 processors	Cluster 3 4 or 2 processors	Elapse time in second	Bandwidth of network
4 PSMN-SDF1	4 PSMN-SDF1	4 PSMN-SDF1	28.8 s	not available
4 ev5 CDCSP-MOBY	4 ev5 CDCSP-MOBY	4 ev5 CDCSP-MOBY	20.7s	100 Mb/s
4 PSMN-SDF1	4 ev5 CDCSP-MOBY	2 ev6 CDCSP-DS20	31.2s	2 Mb/s

Table 3: *Performance of the analytical Aitken Schwarz algorithm on City's Network.*

the basic idea of acceleration of relaxation domain decomposition method via Aitken transform is a possible efficient alternative to acceleration that use multilevel grid concepts for the efficient solution of Elliptic problem with regular grids and we hope to extend similar ideas in the context of unstructured meshes.

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