*12th International Conference on Domain Decomposition Methods*
Editors: Tony Chan, Takashi Kako, Hideo Kawarada, Olivier Pironneau, 

# 38. A Fictitious Domain Decomposition Method for High Frequency Acoustic Scattering Problems

U. Hetmaniuk[1], C. Farhat[2]

## Introduction

It is well known that most PDE problems defined over an axisymmetric domain can be efficiently solved by a Fourier based solution method. However, for many applications, the underlying computational domain is not entirely axisymmetric, but has one or several major axisymmetric subdomains. For such problems, an axisymmetric analysis method is not applicable, and a straightforward one can be inefficient because it does not exploit the geometrical properties of the axisymmetric components. The objective of this paper is to fill this existing gap, and propose a computationally efficient method for solving problems on a class of partially axisymmetric domains [FUR99].

We illustrate our method for a submarine problem. Indeed, a submarine can be represented as the assembly of a major cylindrical component, and a few minor "features" that are however essential for the application itself. Our approach is presented here in the context of the finite element solution of the three-dimensional exterior Helmholtz problem in the high frequency regime. This problem is challenging because it leads to large-scale computations. For example, at a wave length equal to the length of a submarine divided by 360, the finite element discretization of such a problem requires hundreds of millions of grid points.

The proposed methodology is based on a fictitious domain approach (for example, see [DGH$^+$92]) where the original exterior Helmholtz problem is extended into an axisymmetric exterior problem, and where parts of the genuine boundary conditions are enforced through the utilization of Lagrange multipliers. The axisymmetry of the enlarged domain is then exploited by expanding the solution into a Fourier series. The Fourier modes of the solution are computed by solving a series of bidimensional problems coupled altogether by the Lagrange multipliers. The associated constrained problem is treated by extension of the FETI-H method [FML00], and a special coarse problem is constructed for accelerating the convergence of the corresponding interface problem. The resulting fictitious domain decomposition method is a fast solver because it transforms a 3D problem into a series of 2D ones.

For simplicity but without any loss of generality, we consider in this paper the case of a scatterer with a single component and one arbitrarily shaped feature. The generalization to an arbitrary number of axisymmetric components and features is straightforward.

[1]University of Colorado at Boulder, hetmaniu@hawkeye.colorado.edu
[2]University of Colorado at Boulder, charbel@boulder.colorado.edu

# Mathematical formulation

## Extension of the solution to a fictitious domain

We consider an impenetrable obstacle $\Omega$ composed of two substructures

$$\bar{\Omega} = \bar{C} \cup \bar{W}$$

where $C$ and $W$ are two disjoint open sets and $C$ is axisymmetric, as illustrated on Fig. 1.

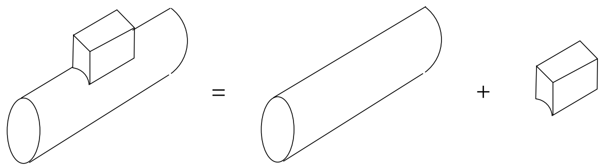


Figure 1: Physical decomposition of the scatterer

$B_R$ is the ball of radius $R$ centered at the center of geometry of $\Omega$, $n$ is the outward normal to $\partial B_R$ and $\frac{\partial}{\partial n}$ is the normal derivative operator. We define the following exterior domains and their intersection with $B_R$.

$$\left\{ \begin{array}{l} \Omega_e = \mathbb{R}^3 \backslash \bar{\Omega} \\ \Omega_{e,R} = \Omega_e \cap B_R \end{array} \right. \qquad \left\{ \begin{array}{l} C_e = \mathbb{R}^3 \backslash \bar{C} \\ C_{e,R} = C_e \cap B_R \end{array} \right.$$

The surface $\Gamma$ is defined as the intersection of $\partial W$ with $C_{e,R}$.

The focus model problem is given by

Find $u \in H^1(\Omega_{e,R})$ such that

$$\left\{ \begin{array}{rclll} \Delta u + k^2 u & = & f & \text{in} & \Omega_{e,R} \\ u & = & 0 & \text{on} & \partial\Omega \\ \frac{\partial u}{\partial n} & = & iku & \text{on} & \partial B_R \end{array} \right. \tag{1}$$

where $u$ is the acoustic scattered field and $f$ belongs to $L^2(\Omega_{e,R})$.

In this paper we consider only a spherical artificial boundary with a first-order approximation of the Sommerfeld condition. But any other axisymmetric boundary or absorbing condition could be used to ensure that the waves are outgoing.

In order to obtain an axisymmetric computational domain, we embed the original domain $\Omega_{e,R}$ into $C_{e,R}$ which satisfies

$$\bar{C}_{e,R} = \bar{\Omega}_{e,R} \cup \bar{W}$$

We extend $u$ from $\Omega_{e,R}$ to the enlarged domain $C_{e,R}$ to a function (still denoted by $u$ for simplicity) with $H^1(C_{e,R})$ regularity. This regularity requirement implies the continuity of the trace of $u$ across the surface $\Gamma$.

Solving problem (1) is *equivalent* to solving the following problem

Find $u \in V = \{v \in H^1(C_{e,R}) \quad | \quad v = 0 \text{ on } \Gamma\}$ such that

$$\begin{cases} \Delta u + k^2 u &= \tilde{f} & \text{in} & C_{e,R} \\ u &= 0 & \text{on} & \partial C \\ \frac{\partial u}{\partial n} &= iku & \text{on} & \partial B_R \end{cases} \tag{2}$$

in the sense that the solution of problem (2) restricted to $\Omega_{e,R}$ satisfies the boundary value problem (1), and $\tilde{f}$ is an $L^2$-extension of $f$, for example, by 0.

We include the boundary condition on $\partial C$ into the definition of the functional space

$$Y = \{v \in H^1(C_{e,R}) \quad | \quad v = 0 \text{ on } \partial C\}$$

We can rewrite problem (2) into the following saddle-point problem :

$$\text{Find } (u, \mu) \in Y \times H^{-1/2}(\Gamma) \text{ such that}$$

$$\begin{cases} \int_{C_{e,r}} \nabla u.\nabla v - k^2 uvdx + \int_{\partial B_R} ikuvd\sigma = \int_{C_{e,R}} fvdx + \int_\Gamma \mu vd\sigma, \quad \forall v \in W \\ \\ \int_\Gamma \zeta.ud\sigma = 0, \quad \forall \zeta \in H^{-1/2}(\Gamma) \end{cases} \tag{3}$$

## Domain decomposition

For high-frequency acoustic scattering problems, numerical discretization leads to large-scale systems of equations. Thus a domain decomposition technique is useful for solving these systems. For the sake of clarity, but without any loss of generality, we present our domain decomposition method for the case of two subdomains.

We describe the axisymmetric domain $C_{e,R}$ in cylindrical coordinates $(r, \theta, z)$. $C_{e,R}$ is generated by rotation around the $z$-axis of a meridian plane $c_{e,R}$. We partition $c_{e,R}$ into two non-overlapping subdomains $c^1$ and $c^2$. The decomposition of $c_{e,R}$ induces a partition of $C_{e,R}$

$$\bar{C}_{e,R} = \bar{C}^1_{e,R} \cup \bar{C}^2_{e,R}$$

where $C^1_{e,R}$ (resp. $C^2_{e,R}$) is generated by the rotation of $c^1$ (resp. $c^2$) around the $z$-axis.

Let $u^s$ denote the restriction to $C^s_{e,R}$ of the solution of problem (2), for $s = 1, 2$. The interface between $C^1_{e,R}$ and $C^2_{e,R}$ is denoted $\Sigma_I$, which is axisymmetric. Now, we are looking for the functions $u^s$ in the following functional spaces

$$V_s = \{v \in H^1(C^s_{e,R}) \quad | \quad v = 0 \text{ on } \Gamma \cap C^s_{e,R}\}$$

for $s = 1, 2$.

For solving problem (2) on a partitioned domain, we adopt the FETI-H method [FML00] which introduces the two following problems

$$\text{Find } (u^1, u^2) \in V_1 \times V_2 \text{ such that}$$

$$\left\{\begin{array}{rcll} \Delta u^1 + k^2 u^1 & = & \tilde{f}_{|C^1_{e,R}} & \text{in} \quad C^1_{e,R} \\ u^1 & = & 0 & \text{on} \quad \partial C \cap \partial C^1_{e,R} \\ \frac{\partial u^1}{\partial n} & = & iku^1 & \text{on} \quad \partial B_R \cap \partial C^1_{e,R} \\ \frac{\partial u^1}{\partial \nu^1} + iku^1 & = & \lambda & \text{on} \quad \Sigma_I \end{array}\right.$$
$$\left\{\begin{array}{rcll} \Delta u^2 + k^2 u^2 & = & \tilde{f}_{|C^2_{e,R}} & \text{in} \quad C^2_{e,R} \\ u^2 & = & 0 & \text{on} \quad \partial C \cap \partial C^2_{e,R} \\ \frac{\partial u^2}{\partial n} & = & iku^2 & \text{on} \quad \partial B_R \cap \partial C^2_{e,R} \\ -\frac{\partial u^2}{\partial \nu^2} + iku^2 & = & \lambda & \text{on} \quad \Sigma_I \end{array}\right. \tag{4}$$

with the constraint

$$u^1 - u^2 = 0 \text{ on } \Sigma_I \tag{5}$$

Here, $\nu^s$ denotes here the unit outward normal on the interface boundary between $C^1_{e,R}$ and $C^2_{e,R}$, and $\lambda$ is a Lagrange multiplier field for enforcing the continuity at the interface of the solution.

Similarly to the previous section, we can introduce a saddle-point problem with the boundary condition on $\partial C$ inside a functional space and two Lagrange multipliers: $\lambda$ for the continuity at the interface of the solution, $\mu$ for enforcing the genuine boundary condition on $\Gamma$.

# A Fourier based finite element discretization

Each function $u^s$ is $2\pi$-periodical with respect to the cylindrical coordinate $\theta$. Hence, it can be expanded in a Fourier series with respect to $\theta$ as follows

$$u^s(r,\theta,z) = \sum_{n=-\infty}^{\infty} u^s_n(r,z) e^{in\theta} \tag{6}$$

The Fourier coefficients of $u^s$ are now functions of $(r,z)$ defined on $c^s$.

Discretizing the two-dimensional subdomains $c^s$ by finite elements and truncating the Fourier expansions leads to the following discrete expression of $u^1$ and $u^2$

$$\left\{\begin{array}{rcl} u^1(r,\theta,z) & = & \sum_{n=-n_\theta}^{n_\theta} \sum_{j=1}^{n^1_{cyl}} u^1_{n,j} X^1_j(r,z) e^{in\theta} \\ & & \\ u^2(r,\theta,z) & = & \sum_{n=-n_\theta}^{n_\theta} \sum_{j=1}^{n^2_{cyl}} u^2_{n,j} X^2_j(r,z) e^{in\theta} \end{array}\right. \tag{7}$$

where $n_\theta$ denotes the selected number of Fourier modes, $X^s_j(r,z)$ denote the shape functions associated with the chosen two-dimensional finite element discretization in $c^s$ and $u^s_{n,j}$ denote the corresponding nodal values.

We enforce all the constraints pointwise with discrete Lagrange multipliers, assuming the subdomains have matching discrete interfaces.

This discretization leads to the following algebraic system

$$\left\{\begin{array}{rcl} (\boldsymbol{K^1_{n_\theta}} - \boldsymbol{k^2 M^1_{n_\theta}} - \boldsymbol{ikM^1_{S,n_\theta}} + \boldsymbol{ikB^{1^T}_{n_\theta}} M_{bb} \boldsymbol{B^1_{n_\theta}})\boldsymbol{u^1_{n_\theta}} + \boldsymbol{B^{1^T}_{n_\theta}}\boldsymbol{\lambda} + \boldsymbol{C^{1^T}_{n_\theta}}\boldsymbol{\mu} & = & \boldsymbol{F^1_{n_\theta}} \\ (\boldsymbol{K^2_{n_\theta}} - \boldsymbol{k^2 M^2_{n_\theta}} - \boldsymbol{ikM^2_{S,n_\theta}} - \boldsymbol{ikB^{2^T}_{n_\theta}} M_{bb} \boldsymbol{B^2_{n_\theta}})\boldsymbol{u^2_{n_\theta}} + \boldsymbol{B^{2^T}_{n_\theta}}\boldsymbol{\lambda} + \boldsymbol{C^{2^T}_{n_\theta}}\boldsymbol{\mu} & = & \boldsymbol{F^2_{n_\theta}} \\ \boldsymbol{B^1_{n_\theta}}\boldsymbol{u^1_{n_\theta}} + \boldsymbol{B^2_{n_\theta}}\boldsymbol{u^2_{n_\theta}} & = & \boldsymbol{0} \\ \boldsymbol{C^1_{n_\theta}}\boldsymbol{u^1_{n_\theta}} + \boldsymbol{C^2_{n_\theta}}\boldsymbol{u^2_{n_\theta}} & = & \boldsymbol{0} \end{array}\right. \tag{8}$$

$\boldsymbol{K^s_{n_\theta}}$, $\boldsymbol{M^s_{n_\theta}}$ are the so-called stiffness and mass matrices for the substructure $C^s_{e,R}$. Matrix $\boldsymbol{M^s_{S,n_\theta}}$ is induced by the Sommerfeld radiation condition and is non-zero only at the degrees of freedom lying on the outer boundary of the domain. Matrix $\boldsymbol{M_{bb}}$ is an interface mass matrix introduced in the FETI-H method for local damping, in order to avoid local resonance. The vectors $\boldsymbol{u^s_{n_\theta}}$ and $\boldsymbol{F^s_{n_\theta}}$ are respectively the vectors of Fourier coefficients of the solution and the load on substructure $C^s_{e,R}$, and $\boldsymbol{\lambda}$ is the vector of Lagrange multipliers for enforcing the continuity at the interface of the Fourier coefficients. The matrices $\boldsymbol{B^s_{n_\theta}}$ depend on the shape functions $X^s_j(r,z)$ and on the discretization of the Lagrange multiplier field $\lambda$. With our assumptions, each $\boldsymbol{B^s_{n_\theta}}$ becomes a Boolean substructure connectivity matrix. $\boldsymbol{\mu}$ is the vector of Lagrange multipliers for enforcing pointwise the constraints on $\Gamma$. The matrices $\boldsymbol{C^s_{n_\theta}}$ depend on the discretization of $u^s$ and of the Lagrange multiplier field $\mu$. For each node $k$ lying on $\Gamma \cap c^s$ and for which the second cylindrical coordinate in $C^s_{e,R}$ is denoted by $\theta^k$, a constraint equation can be written as follows

$$\sum_{n=-n_\theta}^{n=n_\theta} u^s_{n,k} e^{in\theta^k} = 0 \tag{9}$$

The system of equations (8) has the pattern of the FETI-H equations with a set of multipoint constraints (MPCs). Therefore, it is most efficiently solved by the numerically scalable FETI-H solver [FML00] coupled with an appropriate treatment of the MPCs [FLR98].

By gathering the Lagrange multipliers $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ together and also the matrices $\boldsymbol{B^s_{n_\theta}}$, $\boldsymbol{C^s_{n_\theta}}$ together, we can define an extended dual interface problem. We solve this dual problem with the FETI-H solver, where at each iteration the MPCs are exactly satisfied and where the Krylov space for the search directions is enriched by the range of a coarse matrix $Q$ [FML00] based now on the Fourier coefficients of planar waves.

The generalization to an arbitrary number of subdomains is straightforward. One needs only to follow the methodology defined in [FML00] for signing efficiently all the interfaces of the subdomains.

## Numerical experiments

We illustrate our embedding method with the resolution of the Helmholtz equation on the exterior domain of an obstacle. The structure is composed of a large cylindrical component and a conical tower of 45 degrees.

The problem is formulated as follows

$$\left\{\begin{array}{rcll} \Delta u + k^2 u & = & 0 & \text{in} \quad \Omega_{e,R} \\ u & = & 1 & \text{on} \quad \partial\Omega \\ \frac{\partial u}{\partial n} & = & 0 & \text{on} \quad S_R \end{array}\right. \qquad \left\{\begin{array}{l} \text{Diameter of the cylinder : } a = 1 \\ \text{Length of the cylinder : } L = 10 \\ \text{Wavenumber : } kL = 10 \\ \text{Wavelength : } \lambda = 2\pi \\ \text{Mesh size : } h = \lambda/25 \\ \text{Distance } S_R \text{ - obstacle : } 0.5\lambda \end{array}\right. \tag{10}$$

For this computation, $S_R$ has a cylindrical shape.

We discretize the domain $\Omega_{e,R}$ by 343,680 8-noded brick elements. We compute a reference solution by performing a global finite element analysis with Q1 functions, using the classical FETI-H method.

We compute a solution obtained by our methodology with 40 Fourier modes and 172 Lagrange multipliers for enforcing part of the boundary condition. The two-dimensional mesh for computing the Fourier coefficients is made of 1,072 Q1 elements.

As shown on Fig. 2 and Fig. 3, the results obtained by the fictitious method are in excellent agreement with those obtained by a global analysis method.

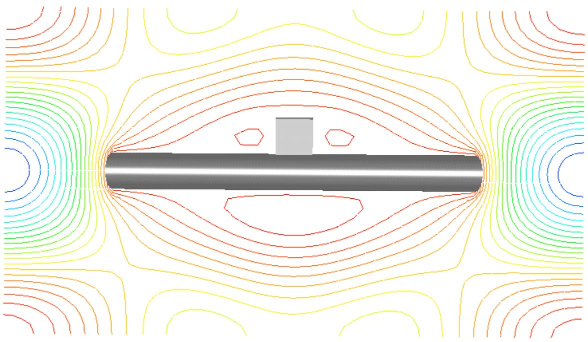

Figure 2: Isovalues of the reference solution

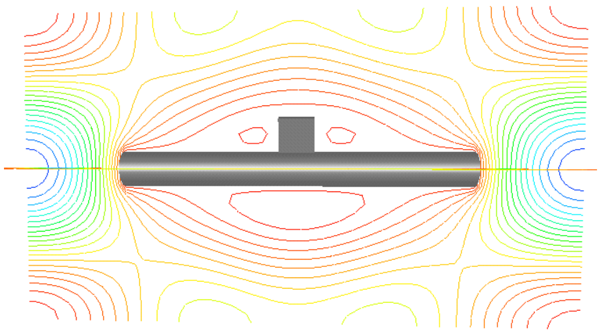

Figure 3: Isovalues of the solution with 40 modes

In all the cases, we use the following convergence criterion

$$\| \tilde{\boldsymbol{K}}\boldsymbol{u} - \boldsymbol{f} \| \leq \mathbf{10^{-6}} \| \boldsymbol{f} \|$$

where $\tilde{\boldsymbol{K}}$ denotes the generalized stiffness matrix of the system to be solved, $\boldsymbol{u}$ denotes either the nodal values of the 3D solution or the nodal values of the Fourier coefficients of the solution and $\boldsymbol{f}$ the corresponding right-hand side.

The performance results of the FETI-H method applied to the solution of the 3D computation are reported on the table 1. These results are achieved for 200 subdomains on a single processor Origin 2000 computer. The size of the coarse grid problem, on which the GCR solver iterates, is 1,577.

| Number of iterations | Total CPU time | Total memory cost |
|---|---|---|
| 130 | 2,548 s | 2,172 Mb |

Table 1: Performance results for the 3D computation on a single processor Origin 2000

The performance results of the method with fictitious domain are reported on the table 2, using 1, 3 and 5 subdomains for the axisymmetric component on a single processor. Note that for the case of one subdomain, the constrained problem is solved by a direct method.

| Nb. of subd. | Size of pb. coarse | Number of iterations | Total CPU time | Total memory cost |
|---|---|---|---|---|
| 1 | 172 | DIRECT | 145 s | 449 Mb |
| 3 | 216 | 10 | 232 s | 402 Mb |
| 5 | 260 | 12 | 253 s | 426 Mb |

Table 2: Performance results for the fictitious methodology

As expected, the fictitious domain method is an order magnitude faster and less memory intensive than the 3D domain decomposition based FETI-H method, because this fictitious domain transforms a 3D problem into a series of 2D ones. We also note that for the considered wavenumber $k$, the size of the 2D mesh is such that solving the 2D Fourier problems by a direct method is faster than solving them by a domain decomposition one. However, one can expect this trend to reverse for larger values of $ka$.

# Conclusion

In this paper, we have presented a fictitious domain decomposition method that allows exploiting a potential partial axisymmetry of a given computational domain. This in turn results in a dramatic reduction of the size of the system of equations to be solved, without a loss of accuracy. Therefore, this fictitious domain decomposition method enables the solution of high frequency 3D acoustic scattering problems on contemporary computational platforms.

## Acknowledgment

The authors acknowledge the support by the Office of Naval Research under Grant N-00014-95-1-0663. The first author also acknowledges the support by the Corps des Ponts et Chaussées, France.

## References

[DGH+92]Q. V. Dihn, R. Glowinski, J. He, V. Kwock, T. W. Pan, and J. Périaux. Lagrange multiplier approach to fictitious domain methods: Application to fluid dynamics and electromagnetics. In David E. Keyes, Tony F. Chan, Gérard A. Meurant, Jeffrey S. Scroggs, and Robert G. Voigt, editors, *Fifth International Symposium on Domain Decomposition Methods for Partial Differential Equations*, pages 151–194, Philadelphia, PA, 1992. SIAM.

[FLR98]C. Farhat, C. Lacour, and D. Rixen. Incorporation of linear multipoint constraints in substructure based iterative solvers. part 1 : A numerically scalable algorithm. *Int. J. Numer. Meth. Eng.*, 43:997–1016, 1998.

[FML00]C. Farhat, A. Macedo, and M. Lesoinne. A two-level domain decomposition method for the iterative solution of high-frequency exterior Helmholtz problems. *Numer. Math.*, 85(2):283–303, 2000.

[FUR99]C. Farhat, U.Hetmaniuk, and D. Rixen. An efficient substructuring method for analyzing structures with major axisymmetric components. In *AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference and Exhibit, 40th, St. Louis, MO, Apr 12-15 1999*, pages 832–837, 1999. AIAA Paper 99-1283.