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14. The Coupling of Natural BEM and Composite Grid FEM

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Introduction

The coupling of boundary elements and finite elements is of great importance for the numerical treatment of boundary value problems posed on unbounded domains. It permits us to combine the advantages of boundary elements for treating domains extended to infinity with those of finite elements in treating the complicated bounded domains.

The standard procedure of coupling the boundary element and finite element methods is described as follows. First, the (unbounded) domain is divided into two subregions, a bounded inner region and an unbounded outer one, by introducing an auxiliary common boundary. Next, the problem is reduced to an equivalent one in the bounded region. There are many ways to accomplish this reduction (refer to [Cos87], [FY83], [GHW94], [HZ94], [JN80], [Med98] and [ESH79]). The FEM-BEM coupled method can be viewed as a domain decomposition method to solve unbounded domain problems.

The natural boundary reduction method proposed by [FY83] has obvious advantages over the usual boundary reduction methods: the coupled bilinear form preserve automatically the symmetry and coerciveness of the original bilinear form,so not only the analysis of the discrete problem is simplified, but also the optimal error estimates and the numerical stability are restored (see [FY83] and [Yu93]).

It is well known that the analytic solution of the Dirichlet exterior problem is in general singular at the corner points. The fast adaptive composite grid (iteration) method advanced by McCormick (refer to [BPWX91], [MT86] and [McC89]) is very effective in dealing with this kind of local singularity. However, it can not be applied directly to the case of unbounded domain.

In the present paper we combine the composite grid method with the coupling method of natural boundary element and finite element to handle the corner singularity of the Dirichlet exterior problems. Under suitable assumptions we obtain the optimal error estimates of the corresponding approximate solutions. The underlying linear system is expensive to solve directly due to the complicated structure (which is neither sparse nor band). Instead, we introduce two iterative methods to solve this coupled system: (1) a combination algorithm between the inexact two-level multiplicative Schwarz method and the steepest descent method; (2) the preconditioning

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conjugate gradient (PCG) method by constructing a kind of simple preconditioner for the coupled “stiffness” matrix. Both the two algorithms have the fast convergence speed independent of the (coarse and fine) mesh sizes, which has been proved in [HY99b] and [HY99a]. We give numerical examples to illustrate our theoretical results.

The FEM-BEM coupling

We consider the following model exterior Dirichlet problem in two dimensions:

$$-\Delta u = f \quad in \ \ \Omega^c = \mathbf{R}^2 \backslash (\Omega \cup \Gamma), \tag{1}$$

$$u = g \qquad on \ \ \partial\Omega \tag{2}$$

with the asymptotic condition:

$$u(x,y) \ \ is \ \ bounded \ \ as \ \ r = \sqrt{x^2+y^2} \to \infty.$$

Where Ω is a Lipschitz bounded domain, f and g are given functions satisfying $f \in L^2(\Omega^c)$ and $g \in H^{\frac{1}{2}}(\partial\Omega)$.

The variational form of the boundary value problem (1) is: to find $u \in \bar{H}^1(\Omega^c)$, such that

$$D(u,v) = (f,v), \quad \forall\ v \in \bar{H}^1_0(\Omega^c), \tag{3}$$

where

$$\bar{H}^1(\Omega^c) = \{v : \frac{v}{\sqrt{(r^2+1)} \cdot \ln(r^2+2)}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \in L^2(\Omega^c)\},$$

$$\bar{H}^1_0(\Omega^c) = \{v : v \in H^1(\Omega^c), v|_{\partial\Omega} = 0\},$$

$$D(u,v) = (\nabla u, \nabla v), \ \ \forall\ u,v \in \bar{H}^1(\Omega^c),$$

with $(\cdot,\cdot)$ be the L^2 innerproduct on Ω^c.

Let Ω_0 is a circle disc (with the radius R) containing Ω and having a boundary Γ. Set $\Omega_1 = \Omega^c \cap \Omega_0$ and $\Omega_2 = \Omega_0^c = \mathbf{R}^2 \backslash \Omega_0$. We assume that the ratio of the area of Ω_1 over the area of Ω is not small.

Let $G(p,p')$ denote the Green function of the Laplace operator on the domain Ω_2. Set

$$\frac{\partial}{\partial n} G(p,p') = G_n^{(2)}(p,p'), \ \ p,p' \in \Gamma,$$

and

$$-\int_\Gamma \frac{\partial^2}{\partial n \partial n'} G(p,p') \cdot u(p')dp' = K_2 u(p), \ \ p \in \Gamma.$$

where n and n' denote respectively the exterior normal vectors of Γ (which is regarded as the boundary of Ω_2) at the points p and p'.

Define the bilinear form

$$D_1(u,v) = \int_{\Omega_1} \nabla u \cdot \nabla v ds, \quad u,v \in H^1(\Omega_1)$$

and the Sobolev spaces

$$H_g^1(\Omega_1) = \{v : v \in H^1(\Omega_1), v|_{\partial\Omega} = g\}$$

and

$$H_0^1(\Omega_1) = \{v : v \in H^1(\Omega_1), v|_{\partial\Omega} = 0\}.$$

Let $< \cdot, \cdot >_\Gamma$ denote the L^2 innerproduct on Γ. Then, it can be verified by the Green formular that (3) is equivalent to the coupling variational problem (see [Yu93]): to find $u \in H_g^1(\Omega_1)$ such that

$$D_1(u, v) + < K_2 u, v >_\Gamma = \int\int_{\Omega_1} f v dx dy - < w_f, v >_\Gamma, \ \forall \ v \in H_0^1(\Omega_1), \tag{4}$$

where

$$w_f(p) = \int\int_{\Omega_2} f(p') G_n^{(2)}(p, p') dp', \ p \in \Gamma.$$

The coupling bilinear form

$$A(u, v) = D_1(u, v) + < K_2 u, v >_\Gamma$$

is symmetric, bounded and coercive in $H_0^1(\Omega_1)$, so (4) has unique solution $u \in H_g^1(\Omega_1)$.

Composite grid discretization

Without loss of generality, we assume that: (i) the domain Ω is a polygon; (ii) $g \equiv 0$. Let the auxiliary boundary Γ be divided into m circular arcs with the same length. Moreover, let the domain Ω_1 be divided into some quasi-uniform triangular or quarilateral elements with the diameter H ($\approx 2\pi R/m$), such that the finite element nodes on Γ coincide with the m dividing points on Γ. The corresponding piecewise linear finite element space is denoted by $S_H(\Omega_1) \subset H_g^1(\Omega_1) = H_0^1(\Omega_1)$. Because the analytic solution u is in general singular nearby the concave angle points of Ω_1, even if the given functions f and g are smooth enough on their definition domains Ω^c and $\partial\Omega$, the finite-dimensional subspace $S_H(\Omega_1)$ can not provide a "good" approximation of u unless the mesh size H is very small. Let Ω_3 is a subdomain of Ω_1, such that $\overline{\Omega}_3$ containes the concave angle points of Ω_1. We assume that Ω_3 is just the union set of some elements of Ω_1. Set

$$H_0^1(\Omega_3) = \{v : v \in H^1(\Omega_1), supp \ v \subset \Omega_3\}.$$

We make a refining division to Ω_3, such that the diameter of finer elements is $h < H$. Let $S_h^0(\Omega_3) \subset H_0^1(\Omega_3)$ be the corresponding piecewise linear finite element space. We define the composite grid space $S_{h,H} \subset H_g^1(\Omega_1) = H_0^1(\Omega_1)$ by $S_{h,H} = S_H(\Omega_1) + S_h^0(\Omega_3)$.

The discrete variational problem of (4) is: to find $u_{h,H} \in S_{h,H}$ such that

$$A(u_{h,H}, v) = \int\int_{\Omega_1} f v dx dy - < w_f, v >_\Gamma, \ \forall \ v \in S_{h,H} \cap H_0^1(\Omega_1). \tag{5}$$

For this approximation, we have the following error estimates (which have been proved in [HY99b] or [HY99a]).

Theorem 1 *Assume that* $f \in L^2(\Omega^c)$ *and* $g \in H^{\frac{1}{2}}(\partial\Omega)$*. Then, there is a decomposition* $u = \hat{u} + \tilde{u}$*, such that* $\hat{u} \in H^2(\Omega_1) \cap H_0^1(\Omega_1)$ *and* $\tilde{u} \in H_0^1(\Omega_3) \cap H^{1+\alpha}(\Omega_3)$ *with* $0 < \alpha < 1$*. Moreover, we have*

$$(\|u_{h,H} - u\|_{1,\Omega_1}^2 + \|u_{h,H} - u\|_{\frac{1}{2},\Gamma}^2)^{\frac{1}{2}} \le C(h^\alpha \|\tilde{u}\|_{1+\alpha,\Omega_3} + H\|\hat{u}\|_{2,\Omega_1}) \tag{6}$$

and

$$\|u_{h,H} - u\|_{0,\Omega_1} \le C(h^{2\alpha}\|\tilde{u}\|_{1+\alpha,\Omega_3} + H^2\|\hat{u}\|_{2,\Omega_1}). \tag{7}$$

Remark 1 *The above theorem indicates that the fine mesh size* h *and the coarse mesh size* H *should satisfy* $h^\alpha \approx H$*.*

It is clear that the stiffness matrix of the bilinear form $A(\cdot,\cdot)$ is neither sparse nor band. Thus, it is expensive to solve the discrete problem (5) in the direct way.

A iteration algorithm of the discrete problem

In this section, we introduce an iteration algorithm to solve (5).

For ease of notation, we set

$$V = S_{h,H},\ V_1 = S_h^0(\Omega_3) \text{ and } V_2 = S_H(\Omega_1).$$

At first, we describe a version of the composite grid iteration algorithm (refer to [MT86] and [McC89]), which is applied to solving (5).

The standard algorithm Let $u_0 \in V$ be a initial approximation. When we have gotten $u_n \in V$, we look for $u_{n+1} \in V$ as follows:

1^o Solving $u^1 \in V_1$ by

$$A(u^1, v_1) = \Phi(v_1) - A(u_n, v_1),\ \forall v_1 \in V_1,$$

namely,

$$D_1(u^1, v_1) = (f, v_1) - D_1(u_n, v_1),\ \forall v_1 \in V_1.$$

Set

$$u_{n+\frac{1}{2}} = u_n + u^1;$$

2^o Solving $u^2 \in V_2$ by

$$D_1(u^2, v_2) = \Phi(v_2) - A(u_{n+\frac{1}{2}}, v_2),\ \forall v_2 \in V_2.$$

Set

$$u_{n+1} = u_{n+\frac{1}{2}} + \theta u^2,$$

where $\theta > 0$ is a relaxation parameter (remaining to be determined).

We define the projection-like operator $Q_H : V \to V_2$

$$D_1(Q_H\varphi, \psi) = A(\varphi, \psi),\ \varphi \in V,\ \forall \psi \in V_2.$$

Here, we have used the fact that $D_1(\cdot,\cdot)$ is symmetric and positive definite in V_2. Let $e_n = u_{h,H} - u_n$ denote the error function. It can be verified directly that the error propagation relation is

$$e_{n+1} = (I - \theta Q_H)(I - P_h)e_n.$$

It can be shown (refer to [HY99b]) that there is a constant $\tilde{C} > 1$, such that

$$D_1(\varphi,\varphi) \le A(\varphi,\varphi) \le \tilde{C} D_1(\varphi,\varphi),\ \forall \varphi \in V.$$

Thus, from the convergence theory of the multiplicative Schwarz iteration (see [SBG96] and [Xu92]), we know that the above iteration algorithm is convergent, provided the relaxation parameter θ is chosen as $0 < \theta < 2/\tilde{C}$. However, there is no simple way to estimate the value of the constant $\tilde{C}$.

We discuss how to choose the relaxation parameter θ when we do not know the value of the constant $\tilde{C}$.

Set $e_0(\theta^0_{-1}) = u_{h,H} - u_0$. If we have determined value of positive number θ^0_{n-1}, then we set

$$e_{n+1}(\theta_n) = (I - \theta_n Q_H)(I - P_h)e_n(\theta^0_{n-1}),\ n = 0, 1, \cdots .$$

Let $\|\cdot\|$ denote the norm generated by the innerproduct $[\cdot,\cdot] = A(\cdot,\cdot)$. We define the function of θ_n by

$$F(\theta_n) = \|e_{n+1}(\theta_n)\|^2,\ n = 0, 1, \cdots .$$

Our idea is to select properly a positive number θ^0_n, such that

$$F(\theta^0_n) = \min_{\theta_n} F(\theta_n),\ n = 0, 1, \cdots . \tag{8}$$

Without loss of generality, we assume that $g_n = (I - P_h)e_n(\theta^0_{n-1}) \neq 0$ (otherwise, $u_{n+\frac{1}{2}} = u_{h,H}$). Since there is a decomposition $g_n = v^1_n + v^2_n$, with $v^1_n \in V_1$ and $v^2_n \in V_2$, we have

$$\begin{aligned} \|g_n\|^2 &= [g_n, v^1_n + v^2_n] && (9)\\ &= D_1(g_n, v^1_n) + [g_n, v^2_n] && (10)\\ &= D_1(Q_H g_n, v^2_n). && (11) \end{aligned}$$

Hence $Q_H g_n \neq 0$. Therefore, it follows from (4.1) that

$$F'(\theta_n) = 0.$$

Thus, we obtain

$$\theta^0_n = \frac{[g_n, Q_H g_n]}{\|Q_H g_n\|^2}.$$

We must illustrate how to calculate these positive numbers θ^0_n. In fact, $Q_H g_n$ can be obtained directly by the step 1 and step 2 in the above standard algorithm, namely, $Q_H g_n = u^2$. Furthermore, we have

$$[g_n, Q_H g_n] = D_1(Q_H g_n, Q_H g_n) = |u_2|^2_{1,\Omega_1}.$$

Now, we can describe an new algorithm.

Schwarz-steepest descent algorithm Let $u_0 \in V$ be a initial approximation. When we have gotten $u_n \in V$, we look for $u_{n+1} \in V$ as follows:

1^o Solving $u^1 \in V_1$ by

$$D_1(u^1, v_1) = (f, v_1) - D_1(u_n, v_1), \ \forall v_1 \in V_1,$$

and set

$$u_{n+\frac{1}{2}} = u_n + u^1;$$

2^o Solving $u^2 \in V_2$ by

$$D_1(u^2, v_2) = \Phi(v_2) - A(u_{n+\frac{1}{2}}, v_2), \ \forall v_2 \in V_2.$$

3^o Computing norms $|u^2|^2_{1,\Omega_1}$ and $\|u^2\|^2$, and set

$$u_{n+1} = u_{n+\frac{1}{2}} + \theta^0_n u^2,$$

with $\theta^0_n = \frac{|u^2|^2_{1,\Omega_1}}{\|u^2\|^2}$.

For the above algorithm, we have the following convergence result (see [HY99b]).

Theorem 2 *There is a constant C independent of h and H, such that*

$$\|e_{n+1}(\theta^0_n)\|^2 \le (1 - \frac{1}{C})\|e_n(\theta^0_{n-1})\|^2, \ n \ge 1. \quad (12)$$

Remark 2 *If we set $\theta^0_n = 1$, which corresponds to the standard two-level multiplicative Schwarz algorithm, this algorithm may be divergent.*

A preconditioner for the discrete system

Because the stiffness matrix associated with the discrete problem (5) is symmetric and positive definite, this linear system can also be solved by the PCG method.

Now we construct a kind of preconditioner for this bilinear form.

For convenience' sake, we define the operators $A, \bar{A} : V \to V$ by

$$(A\varphi, \psi) = D_1(\varphi, \psi) + < K_2\varphi, \psi >_\Gamma, \ \forall \varphi, \ \psi \in V$$

and

$$(\bar{A}\varphi, \psi) = D_1(\varphi, \psi), \varphi \in S_{h,H}, \ \forall \psi \in V.$$

Let $A_1 : V_1 \to V_1$ and $A_2 : V_2 \to V_2$ denote the restrictions of the operator $\bar{A}$, which satisfy

$$(A_1\varphi_1, \psi_1) = (\bar{A}\varphi_1, \psi_1), \ \varphi_1 \in V_1, \ \forall \psi_1 \in V_1$$

and

$$(A_2\varphi_2, \psi_2) = (\bar{A}\varphi_2, \psi_2), \ \varphi_2 \in V_2, \ \forall \psi_2 \in V_2.$$

It is clear that the operators A_1 and A_2 are symmetric and positive definite with respect to the L^2 innerproduct.

We define the preconditioner of the operator A as

$$B = A_1^{-1}Q_1 + A_2^{-1}Q_2, \tag{13}$$

where $Q_1 : V \to V_1$ and $Q_2 : V \to V_2$ are the L^2 orthogonal projection operators.

The following result has been proved in [HY99a].

Theorem 3 *There exists a constant C independent of h and H, such that*

$$cond(BA) \le C. \tag{14}$$

Remark 3 *Since the operator K_2 in the second section can be expressed explicitly, we need not solve any (singular) integral equation. Instead, we need only to calculate some singular integrations (refer to [HY99b], [HY99a] and [Yu93]). Besides, only two subproblems with two standard bases are needed to be solved. These are the main merits of the algorithm introduced in this paper.*

Remark 4 *The preconditioning algorithm introduced in this section has faster convergence speed than the Schwarz algorithm introduced in the last section (see the next section). Moreover, it is additive, so the result can be extended directly to the case of inexact local solver. On the other hand, the stiffness matrix of (5) can not be obtained directly (refer to [MT86], [McC89] and [SBG96]), because $V_1 \cap V_2 \ne \emptyset$. For the Schwarz algorithm given in the last section, the global stiffness matrix of the bilinear form $A(\cdot,\cdot)$ need not be generated (therefore, no need to care about basis for V). Besides, this algorithm has minimal memory requirement. These are the merits of the Schwarz algorithm.*

Numerical examples

To illustrate the theoretical results stated in this paper, we consider

$$-\Delta u = f, \quad \in \ \Omega^c, \tag{15}$$

$$u = g, \qquad on\ \partial\Omega, \tag{16}$$

where $\Omega = [-1, 0] \times [-1, 0]$; f and g are given functions such that its exact sulution is $u(x, y) = \frac{(x^2+y^2)^{\frac{1}{3}}}{(x+\frac{1}{2})^2+(y+\frac{1}{2})^2}$.

$$(f(x,y) = -u(x,y)\{\frac{2/3}{(x^2+y^2)[(x+\frac{1}{2})^2+(y+\frac{1}{2})^2]} + \frac{8/3}{(x+\frac{1}{2})^2+(y+\frac{1}{2})^2} - \frac{8/9}{x^2+y^2}\})$$

It is clear that the analytic solution u is singular at the corner point (0,0) ($\alpha = \frac{2}{3}$). This problem is solved by the method introduced in the second section. Here, radius of the auxiliary circle Γ is $R = 2$. Moreover, the subdomain Ω_3 is chosen as the sector with radius 1. We use quasi-uniform triangular elements. The resulting linear system is solved by the Schwarz-steepest descent algorithm (or the PCG method with the preconditioner defined in the last section).

The error estimates (6) and (7) are confirmed by Table 1 (with the equivalent discrete norms).

Table 1

error estimates ($H = 4\pi/m, h = H/4$)

m	$\|u_H - u\|_{1,\Omega_1}$	$\|u_{h,H} - u\|_{1,\Omega_1}$	$\|u_H - u\|_{0,\Omega_1}$	$\|u_{h,H} - u\|_{0,\Omega_1}$
20	9.87D-1	7.25D-1	9.31D-1	4.66D-1
40	6.37D-1	3.64D-1	3.75D-1	1.20D-1
80	4.12D-1	1.83D-1	1.53D-1	3.14D-2
160	2.65D-1	9.24D-2	6.14D-2	8.07D-3 (or 8.09D-3)

The numbers of iteration are given in Table 2 (or Table 3), which can confirm Theorem 8 (or Theorem 13). Here, the domination error with the discrete l^2 norm is 5.0×10^{-5}.

Table 2

numbers of iteration

m	20	40	80	160
iter	21	22	21	22

Table 3

numbers of iteration

m	20	40	80	160
PCG	14	14	15	14

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