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15. Direct Mehtod of Lines for Solving an Elliptic Transmission Problem

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Introduction

The object of this paper is to present the numerical algorithm to obtain a finite difference solution for an elliptic transmission problem by use of the direct method of lines ([Nak65],[KK98], [KK99]). Let Π be a rectangular domain in $\mathbb{R}^2$, Ω_1 be an open subset of Π and $\Omega_2 = \Pi \setminus \overline{\Omega_1}$, $\Gamma = \partial\Omega_1$ (see Figure 1). Then the elliptic transmission problem is formulated as follows. And it is well known that (3) and (4) are called the conditions of transmission (cf. [DL90], [Lio71]).

Problem I Find $(u_1, u_2) \in H^1(\Omega_1) \times H^1(\Omega_2)$ such that

$$-\epsilon_1 \triangle u_1 = f_1 \quad \text{in} \quad \Omega_1 \,, \tag{1}$$

$$-\epsilon_2 \triangle u_2 = f_2 \quad \text{in} \quad \Omega_2 \,, \tag{2}$$

$$u_1 = u_2 \quad \text{on} \quad \Gamma \,, \tag{3}$$

$$\epsilon_1 \frac{\partial u_1}{\partial \nu} = \epsilon_2 \frac{\partial u_2}{\partial \nu} \quad \text{on} \quad \Gamma \,, \tag{4}$$

$$u_2 = g \quad \text{on} \quad \partial\Pi \,. \tag{5}$$

Hhere ϵ_1 and ϵ_2 are positive constants, $\{f_1, f_2\} \in L^2(\Omega_1) \times L^2(\Omega_2)$, $g \in H^{1/2}(\partial\Pi)$ and ν is the unit normal vector on Γ directed from Ω_1 to Ω_2 .

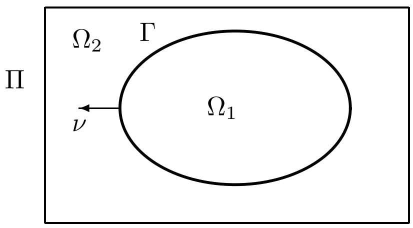

Figure 1: Interface Γ and unit normal ν

Equations (1)-(5) of this type are arisen in various contexts. One of such examples can be found in the context of electricity. In fact, let $\{\epsilon_1, \epsilon_2\}$ denote dielectric constant, $\{u_1, u_2\}$ be potential of the electric field and $\{f_1, f_2\}$ be charge density in the dielectric material $\{\Omega_1, \Omega_2\}$ respectively. Then the conditions (3) and (4) mean that the tangential component of the electric field and the normal component of electric

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flux density are continuous across Γ respectively. Moreover if g=0, (5) represents that $\mathbb{R}^2 \setminus \Pi$ is occupied by a perfect conductor.

The problem of transmission type has been studied from the viewpoint of both theoretical and numerical researchs. And the method of the auxilliary domain plays the important role in the field of numerical analysis. In this paper we present another point of view to solve it numerically. That is to use the method of the successive eliminations of lines and to solve directly the kernel of the Steklov-Poincaré operator T , which is defined as the linear operator from the Dirichlet data on Γ to the Neumann data on Γ

$$T : H^{1/2}(\Gamma) \ni w \to \epsilon_1 \frac{\partial u_1}{\partial \nu} - \epsilon_2 \frac{\partial u_2}{\partial \nu} \in H^{-1/2}(\Gamma),$$

in the sense of the finite difference. We remark here that the discretized equations of Problem I is reduced to solve the linear system of equations defined on Γ (i.e., the kernel of the Steklov-Poincaré operator T) and another parts of unknowns are automatically decided by the algebraic computation using the explicit formula of the approximate solutions stated in the section 4.

Now considering the kernel of the Steklov-Poincaré operator T, Problem I is rewritten by Problem II. In fact, two formulations are equivalent by use of the distribution theoretical approach and Green's formula. Hence from now on, we consider the construction of the solution for Problem II in the sense of the finite difference.

Problem II Find $u \in H^1(\Pi)$ such that

$$\begin{cases} -\text{div}\,(a(x)\,\nabla u) = f & \text{in} \quad D'(\Pi)\ , \\ u = g & \text{on} \quad \partial\Pi\ . \end{cases} \tag{6}$$

Here $a(x) = \epsilon_1\,\chi_{\Omega_1}(x) + \epsilon_2\,\chi_{\Omega_2}(x)$, $f(x) = f_1(x)\,\chi_{\Omega_1}(x) + f_2(x)\,\chi_{\Omega_2}(x)$ and $\chi_\Omega(x)$ is defined by

$$\chi_\Omega(x) = \begin{cases} 1 & if\ x \in \Omega \\ 0 & if\ x \notin \Omega \end{cases}$$

for any subset Ω of Π.

The contents of this paper are as follows. In the second section, we introduce a small perturbation on Γ for the numerical computation, which is defined by $\frac{1}{2}(\ f_1(x) + f_2(x)\ )\ \delta(x-\Gamma)$, in the discretized formulation of Problem II. Roughly speaking, it implies that $T(u_h) = O(h)$ for any mesh size h . In the third section, we prepare the representation formula of the solution for a system of linear equations. This is the background in the numerical algorithm we propose here. In the fourth section, the kernel of the Steklov-Poincaré operator T and the explicit formula of the approximate solutions will be presented using the results in the third section. In the fifth section, two numerical results will be shown by use of the explicit formula in the fourth section.

Finite difference approximation for Problem II

We partition the region Π into rectangles by vertical $m-1$ lines and horizontal $n-1$ lines. We denote mesh size for x direction as Δx and for y direction as Δy. Moreover by Γ_Δ we denote the set of all mesh points (which are interior of Π) such that from each point the horizontal distance to Γ is less than $\Delta x/2$ or the vertical distance to Γ is less then $\Delta y/2$ (see Figure 2). We designate the point in Γ_Δ as artifitial interface mesh point. By Π_Δ we denote the set of all interior mesh points which do not belong to Γ_Δ.

In order to denote a discretized model for Problem II we prepare some notations. We assume that the boundary data g is continuous on $\partial\Pi$ and the charge densities f_1, f_2 are coutinuous on $\overline{\Omega}_1$, $\overline{\Omega}_2$ respectively. Let denote u_{ij} as approximate value of the solution u at mesh point P_{ij}. We denote $P_{i+1/2,j}$ as the center of the points P_{ij} and $P_{i+1,j}$ and denote $P_{i,j+1/2}$ as the center of the points P_{ij} and $P_{i,j+1}$. For every mesh piont $P_{ij} \in \Gamma_\Delta$ we denote P_{ij}^Γ as the nearest point form P_{ij} among the points which are on the intersection of mesh lines and Γ (see Figure 3).

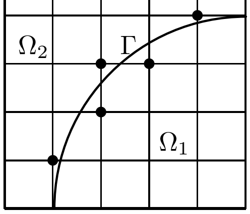

Figure 2: Mesh point near Γ

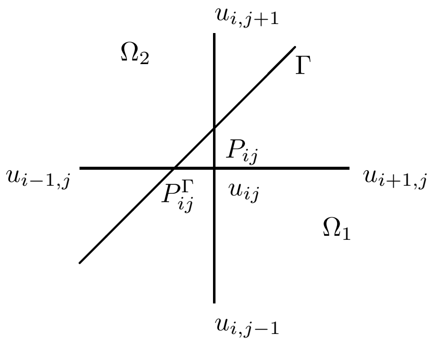

Figure 3: The point P_{ij}^Γ

Let us define the function $\epsilon(P)$ and the elements f_{ij} as the following :

$$\epsilon(P) = \begin{cases} \epsilon_1 & \text{if } P \in \Omega_1, \\ (\epsilon_1 + \epsilon_2)/2 & \text{if } P \in \Gamma, \\ \epsilon_2 & \text{if } P \in \Omega_2, \end{cases} \qquad f_{ij} = \begin{cases} f(P_{ij}) & \text{if } P_{ij} \in \Pi_\Delta, \\ (f^1(P_{ij}^\Gamma) + f^2(P_{ij}^\Gamma))/2 & \text{if } P_{ij} \in \Gamma_\Delta. \end{cases}$$

By use of above notations we define a discretized model for Problem II at a mesh piont P_{ij} by the following form :

Discretized formula for Problem II at P_{ij}

$$\begin{aligned} -\frac{1}{\Delta x}\left[\epsilon(P_{i+1/2,j})\frac{u_{i+1,j} - u_{ij}}{\Delta x} - \epsilon(P_{i-1/2,j})\frac{u_{ij} - u_{i-1,j}}{\Delta x}\right] \\ - \frac{1}{\Delta y}\left[\epsilon(P_{i,j+1/2})\frac{u_{i,j+1} - u_{ij}}{\Delta y} - \epsilon(P_{i,j-1/2})\frac{u_{ij} - u_{i,j-1}}{\Delta y}\right] = f_{ij}. \end{aligned} \tag{7}$$

Now we put that

$$b_{ij}^W = \epsilon(P_{i-1/2,j}),\ b_{ij}^E = \epsilon(P_{i+1/2,j}),\ c_{ij}^S = \epsilon(P_{i,j-1/2}),\ c_{ij}^N = \epsilon(P_{i,j+1/2}), \tag{8}$$

then we can rewrite the equation (7) to the following form :

$$-tc_{ij}^{S}u_{i,j-1}+\big(b_{ij}^{W}+b_{ij}^{E}+t(c_{ij}^{S}+c_{ij}^{N})\big)u_{ij}-tc_{ij}^{N}u_{i,j+1} \\ =b_{ij}^{W}u_{i-1,j}+b_{ij}^{E}u_{i+1,j}+(\Delta x)^2 f_{ij}, \tag{9}$$

where $t=(\Delta x)^2/(\Delta y)^2$. The coefficients $b_{ij}^{W},\dots,c_{ij}^{E}$ have the following properties.

$$b_{ij}^{E}=b_{i+1,j}^{W},\quad c_{ij}^{N}=c_{i,j+1}^{S} \qquad \text{for all mesh points ,} \tag{10}$$

$$b_{ij}^{W}=b_{ij}^{E}=\epsilon(P_{ij})=c_{ij}^{S}=c_{ij}^{N} \qquad \text{if } P_{ij}\in\Pi_{\Delta}. \tag{11}$$

These properties are obvious from definitions (8).

Now for $i=1,2,\dots,m-1$, we denote U_i as unknown columun vector $[u_{ij}]_{1\le j\le n-1}$ and define coefficient matrices A_i^{ϵ}, B_i^{W} and B_i^{E} by the following forms.

$$A_i^{\epsilon}=\begin{bmatrix} a_{i,1}^{\epsilon} & -tc_{i,1}^{N} & 0 & \cdots & \cdots \\ -tc_{i,2}^{S} & a_{i,2}^{\epsilon} & -tc_{i,2}^{N} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & 0 & -tc_{i,n-1}^{S} & a_{i,n-1}^{\epsilon} \end{bmatrix}, \tag{12}$$

$$B_i^{W}=\mathrm{diag}[b_{ij}^{W}]_{1\le j\le n-1},\quad B_i^{E}=\mathrm{diag}[b_{ij}^{E}]_{1\le j\le n-1}, \tag{13}$$

where $a_{ij}^{\epsilon}=b_{ij}^{W}+b_{ij}^{E}+t(c_{ij}^{S}+c_{ij}^{N})$ and A_i^{ϵ} is a tridiagonal matrix.

By use of these notations we can rewrite equations (9) to the following system of equations which is a discretized model for Problem II .

Problem III Find vectors U_i $(1\le i\le m-1)$ such that

$$A_i^{\epsilon}U_i=B_i^{W}U_{i-1}+B_i^{E}U_{i+1}+F_i \quad (1\le i\le m-1), \tag{14}$$

where $U_0=0$, $U_m=0$ and F_i $(1\le i\le m-1)$ are known vectors constructed form the functions f and g.

From the equations (10) we know that A_i^{ϵ} $(1\le i\le m-1)$ are symmetric matricies and $B_i^{E}=B_{i+1}^{W}$ $(1\le i\le m-2)$.

Construction of the solution for linear equations based on the direct method of lines

Before proceeding to solve Problem III, we shall state our result about linear equations for the unkown matrix $\{X_i\}$ as follows:

$$A\,X_i=X_{i-1}+X_{i+1}+Y_i \quad (1\le i\le m-1). \tag{15}$$

Here we make the following assumptions:

(H1) A is a square matrix of order N.

(H2) X_i $(0\le i\le m)$ and Y_i $(1\le i\le m-1)$ are $N\times M$ matrices which satisfy the system of equations (15).

Then we have the following representation for any one X_k.

Theorem 1 *We assume (H1),(H2). Then we have*

$$\begin{aligned} A_m X_k = A_{m-k} X_0 + A_{m-k} \sum_{i=1}^{k-1} A_i Y_i \\ + A_k \sum_{i=k}^{m-1} A_{m-i} Y_i + A_k X_m \qquad (1 \le k \le m-1) \end{aligned} \tag{16}$$

where the sequence of matrices $\{A_i\}$ *is defined by*

$$A_1 = I, \; A_2 = A, \; A_{i+1} = A A_i - A_{i-1} \quad (i = 2, 3, \dots). \tag{17}$$

If $A = [a_{ij}]$ is the $(n-1)-$ symmetrix tridiagonal matrix as follows:

$$a_{jj} = 2s \quad a_{j,j+1} = a_{j+1,j} = -t \quad \text{where} \quad s = 1 + t \tag{18}$$

then the representation (16) turns out to be a very simple form. In fact we can reduce A to a diagonal form by means of the orthgonal transfomation : $P = {}^t[P_1, P_2, \dots, P_{n-1}]$ where

$$p_{l,j} = \sqrt{\frac{2}{n}} \sin\left(\frac{l\, j\, \pi}{n}\right) \quad (1 \le l, j \le n-1). \tag{19}$$

and P has the following properties,

$${}^tP = P, \; P^2 = I. \tag{20}$$

Multiplying $P A_m^{-1}$ on the both sides of (16), we obtain the following result.

Proposition 1 *Assume* $X_0 = X_m = O$*, then we have*

$$P X_k = \sum_{i=1}^{k-1} D_{m-k,i} P Y_i + \sum_{i=k}^{m-1} D_{k,m-i} P Y_i \qquad (1 \le k \le m-1) \tag{21}$$

where for l *and* i $(1 \le l, i \le m-1)$

$$D_{l,i} = P A_m^{-1} A_l A_i P = \text{diag}\left[\frac{\sinh(l\, a_j)\, \sinh(i\, a_j)}{\sinh(m\, a_j)\, \sinh a_j}\right]_{1 \le j \le n-1}. \tag{22}$$

Explicit formula of the solution for Problem III

We return to Problem III. We can rewrite the system of equations (14) to the following new system of equations, which is more useful forms for the Method of Lines, by use of splitting unknown vectors. Then our last problem is reduced to the following.

Problem IV Find $\{V_i, W_i\}$ $(1 \le i \le m-1)$ such that

$$A V_i = V_{i-1} + V_{i+1} + F_i + B_i^W W_{i-1} - A_i^\epsilon W_i + B_i^E W_{i+1} \qquad (1 \le i \le m-1) \tag{23}$$

where $V_0 = V_m = W_0 = W_m = 0$ and the matrix A is given by the equation (18).

Now we will derive the system of equations (23) from the system (9). At first, for any column vector $V(=[v_j])$, we define a set of indices as that $\mathrm{supp}(V) = \{j \mid v_j \neq 0\}$.

Let us divide each unknown vector U_i into two parts.

$$U_i = U_i' + W_i, \qquad (1 \leq i \leq m-1) \tag{24}$$

where

$$\mathrm{supp}(U_i') \subseteq \{j \mid P_{ij} \in \Pi_\Delta\} \quad \text{and} \quad \mathrm{supp}(W_i) \subseteq \{j \mid P_{ij} \in \Gamma_\Delta\}. \tag{25}$$

If $j \in \mathrm{supp}(U_i')$ then by use of the above definition (25) and the relation (11), we obtain the eqaution that $B_i^W U_i' = B_i^E U_i'$. Then we define the new unknown vectors V_i:

$$V_i = B_i^W U_i' = B_i^E U_i' \qquad (1 \leq i \leq m-1). \tag{26}$$

From the definition of V_i and W_i we have the following relations.

$$\mathrm{supp}(V_i) \cap \mathrm{supp}(W_i) = \emptyset \qquad (1 \leq i \leq m-1). \tag{27}$$

Moreover it follows from (10) that

$$V_i = B_{i-1}^E U_i' = B_{i+1}^W U_i' \qquad \text{and} \qquad A_i^\epsilon U_i' = A\, V_i \qquad (1 \leq i \leq m-1), \tag{28}$$

where the matrix A is given by the equation (18). By use of the relations (28) we can rewrite the system of equations (14) to the new system of equations (23).

Applying Proposition 1 to our difference equations (23) we obtain the following expressions.

Proposition 2 *For each number k ($1 \leq k \leq m-1$),*

$$\begin{aligned} P\,V_k = &\sum_{i=1}^{k-1} D_{m-k,i}\, P\left[B_i^W\, W_{i-1} - A_i^\epsilon\, W_i + B_i^E\, W_{i+1}\right] \\ &+ \sum_{i=k}^{m-1} D_{k,m-i}\, P\left[B_i^W\, W_{i-1} - A_i^\epsilon\, W_i + B_i^E\, W_{i+1}\right] \\ &+ \left(\sum_{i=1}^{k-1} D_{m-k,i}\, P\, F_i + \sum_{i=k}^{m-1} D_{k,m-i}\, P\, F_i\right) \end{aligned} \tag{29}$$

From the equations (29), we can elimimate V_k and get the linear equations for $\{W_i\}$ by use of the support property (27) and the orthogonarity of $\{P_l\}$.

Theorem 2 (Equations for W_i) *For $l \in supp(W_k)$,*

$$\begin{aligned} &\sum_{i=1}^{k-1} {}^tP_l\, D_{m-k,i}\, P\left[-B_i^W\, W_{i-1} + A_i^\epsilon\, W_i - B_i^E\, W_{i+1}\right] \\ &\quad + \sum_{i=k}^{m-1} {}^tP_l\, D_{k,m-i}\, P\left[-B_i^W\, W_{i-1} + A_i^\epsilon\, W_i - B_i^E\, W_{i+1}\right] \\ &\quad = {}^tP_l \left(\sum_{i=1}^{k-1} D_{m-k,i}\, P\, F_i + \sum_{i=k}^{m-1} D_{k,m-i}\, P\, F_i\right). \end{aligned} \tag{30}$$

By use of the orthogonarity of $\{P_l\}$, we get the following Theorem.

Theorem 3 (Expressions for $v_{i,j}$) *For $l \in supp(V_k)$,*

$$
\begin{aligned}
v_{k,l} = &\sum_{i=1}^{k-1} {}^tP_l\, D_{m-k,i}\, P\left[B_i^W\, W_{i-1} - A_i^{\epsilon}\, W_i + B_i^E\, W_{i+1}\right] \\
&+ \sum_{i=k}^{m-1} {}^tP_l\, D_{k,m-i}\, P\left[B_i^W\, W_{i-1} - A_i^{\epsilon}\, W_i + B_i^E\, W_{i+1}\right] \\
&+ {}^tP_l \left(\sum_{i=1}^{k-1} D_{m-k,i}\, P\, F_i + \sum_{i=k}^{m-1} D_{k,m-i}\, P\, F_i\right).
\end{aligned}
\tag{31}
$$

Examples

Using Theorem 2 and 3, we show the numerical results for the elliptic transmission problem (1) under the following geometry.

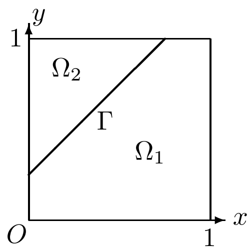

Figure 4: Example 1

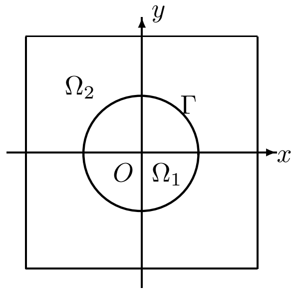

Figure 5: Example 2

Example 1:

Let $\Pi = (0,1) \times (0,1) = \Omega_1 \cup \Gamma \cup \Omega_2$, $\Gamma : x - y + 1/4 = 0$ as Figure 4. Set $\epsilon_1 = 1, \quad \epsilon_2 = 3$ in Problem I and $\Delta x = \Delta y = h$ in (7).

We then use test functions:

$$
u = \begin{cases} \sin(x - y + 1/4) + x + 1 & \text{in } \Omega_1, \\ (x - y + 1/4)^2 + x + 1 & \text{in } \Omega_2, \end{cases} \qquad f = \begin{cases} 2\,\epsilon_1 \sin(x - y + 1/4) & \text{in } \Omega_1, \\ -4\,\epsilon_2 & \text{in } \Omega_2. \end{cases}
$$

and get the Table 1 below.

Example 2:

Let $\Pi = (-0.5, -0.5) \times (0.5, 0.5) = \Omega_1 \cup \Gamma \cup \Omega_2$, $\Gamma : x^2 + y^2 = R^2$, $R = 1/4$ as Figure 5. Set $\epsilon_1 = 5, \quad \epsilon_2 = 3$ in Problem I and $\Delta x = \Delta y = h$ in (7).

We then use test functions:

$$u = \begin{cases} x^3 - y^3 & \text{in } \Omega_1, \\ (x^3 - y^3)(x^2 + y^2)/R^2 & \text{in } \Omega_2, \end{cases}$$

$$f = \begin{cases} -6\,\epsilon_1\,(x - y) & \text{in } \Omega_1, \\ -2\,\epsilon_2\,(11\,x^3 - 3\,x^2\,y + 3\,x\,y^2 - 11\,y^3)/R^2 & \text{in } \Omega_2. \end{cases}$$

and get the Table 2 below.

In the table 1 and 2, we use the following notations. The 'maximum point' (i, j) means the mesh point where the muximum error occurs and $\|u - u_h\|_\infty = \max_{i,j}\, |u(i\,h, j\,h) - u_{i,j}|$. Moreover 'ratio' means the percentage of the number of unknowns $\{w_{ij}\}$ for the system of linear equations in Theorem 2 to the total number of unknowns $\{u_{ij}\}$ in (7).

mesh size	ratio	maximum point	$\|u - u_h\|_\infty$
1/16	4.89%	(6, 10)	6.232701×10^{-5}
1/32	2.39%	(12, 20)	1.557993×10^{-5}
1/64	1.18%	(24, 40)	3.894824×10^{-6}
1/128	0.59%	(48, 80)	9.736942×10^{-7}

Table 1: Numerical result of example 1

mesh size	ratio	maximum point	$\|u - u_h\|_\infty$
1/16	10.67%	(4, 12)	7.426605×10^{-3}
1/32	4.58%	(9, 19)	2.119219×10^{-3}
1/64	2.32%	(28, 47)	8.459878×10^{-4}
1/128	1.12%	(58, 95)	4.221186×10^{-4}

Table 2: Numerical result of example 2

Remark

Both examples show that the 'ratio' is decreasing in proportion to mesh size. Hence our method seems to be advantageous in the situation where the mesh size is very small.

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