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3. Dual and Dual-Primal FETI Methods for Elliptic Problems with Discontinuous Coefficients in Three Dimensions

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Introduction

The Finite Element Tearing and Interconnecting (FETI) methods were first introduced by Farhat and Roux [FMR94]. An important advance, making the rate of convergence of the iteration less sensitive to the number of unknowns of the local problems, was made by Farhat, Mandel, and Roux a few years later [FMR94]. For a detailed introduction, see [FR94] and we also refer to our own papers for many additional references. Our own work, cf. [KW01, KW00b], owes a great deal to the pioneering theoretical work by Mandel and Tezaur [MT96, MT00].

The principal purpose of this paper is to survey some recent results developed by the authors. We introduce new one-parameter families of one-level FETI as well as of dual–primal FETI preconditioners which have a rate of convergence which is bounded independently of possible jumps of the coefficients of an elliptic model problem often considered in the theory of Neumann–Neumann and other iterative substructuring algorithms; see, e.g., [DW95, DSW94, MB96] and the references therein. Our new results become possible because of special scalings. One of them, for the preconditioner, is closely related to an important algorithmic idea used in the best of the Neumann–Neumann methods. The other scaling affects the choice of the projection which is used in each step of the one–level FETI iteration, whether preconditioned or not. For a certain choice of the two scalings, our preconditioner for the one–level FETI methods results in a method that is identical to one recently tested successfully for very difficult and large problems by Bhardwaj et al. [BDF$^+$00]. The scaling used in the preconditioner was originally introduced by Rixen and Farhat; see [RF99]. We note that, by now, many variants of the FETI algorithms have been designed and that a number of them have been tested extensively; see in particular [RFTM99]. Some of our results have also already been extended to Maxwell's equation in two dimensions by Toselli and Klawonn [TK99].

Recently, Farhat et al. [FLLT$^+$99] introduced a dual–primal FETI algorithm suitable for second order elliptic problems in the plane and for plate problems. A convergence analysis in the case of benign coefficients is given by Mandel and Tezaur [MT00]. Numerical experiments show a poor performance for this algorithm in three

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dimensions; cf. [FLLT+99]. Recent experiments with alternative algorithms are reported in [FLP00, Pie00]. We give a brief description of our own recent work in the final section; see [KW00b] for many more details.

The remainder of this paper is organized as follows. In the next, the second section, we introduce our elliptic problems and the basic geometry of the decomposition. In the following section, we give a short introduction to one–level FETI methods. In the fourth section, we introduce our family of preconditioners and formulate one of our main results; our results could also be extended to certain other elliptic problems as in [KW00a]. Finally, we present results on a new dual–primal FETI method for problems with discontinuous coefficient in three dimensions; see [KW00b].

A model problem, finite elements, and geometry

Let $\Omega \subset \mathbb{R}^3$, be a bounded, polyhedral region, let $\partial\Omega_D \subset \partial\Omega$ be a closed set of positive measure, and let $\partial\Omega_N := \partial\Omega \setminus \partial\Omega_D$ be its complement. We impose homogeneous Dirichlet and general Neumann boundary conditions, respectively, on these two subsets and introduce the Sobolev space $H_0^1(\Omega, \partial\Omega_D) := \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega_D\}$.

For simplicity, we will only consider a piecewise linear, conforming finite element approximation of the following scalar, second order model problem:

Find $u \in H_0^1(\Omega, \partial\Omega_D)$, such that

$$a(u, v) = f(v) \quad \forall v \in H_0^1(\Omega, \partial\Omega_D), \tag{1}$$

where

$$a(u, v) := \int_\Omega \rho(x) \nabla u \cdot \nabla v dx, \;\; f(v) := \int_\Omega f v dx \; + \; \int_{\partial\Omega_N} g_N v ds, \tag{2}$$

where g_N is the Neumann boundary data defined on $\partial\Omega_N$; it provides a contribution to the load vector of the finite element problem. The coefficient $\rho(x) > 0$ for $x \in \Omega$.

We decompose Ω into non-overlapping subdomains $\Omega_i, i = 1, \ldots, N$, also known as substructures, and each of which is the union of shape-regular elements with the finite element nodes on the boundaries of neighboring subdomains matching across the interface $\Gamma := \overline{\left(\bigcup_{i=1}^N \partial\Omega_i\right)} \setminus \partial\Omega$. The interface Γ is decomposed into subdomain faces, regarded as open sets, which are shared by two subregions, edges which are shared by more than two subregions and the vertices which form the endpoints of edges. We denote faces of Ω_i by $\mathcal{F}^{ij}$, edges by $\mathcal{E}^{ik}$, and vertices by $\mathcal{V}^{i\ell}$.

We denote the standard finite element space of continuous, piecewise linear functions on Ω_i by $W^h(\Omega_i)$. For simplicity, we assume that the triangulation of each subdomain is quasi uniform. The diameter of Ω_i is H_i, or generically, H. We denote the corresponding finite element trace spaces by $W_i := W^h(\partial\Omega_i), i = 1, \ldots, N$, and by $W := \prod_{i=1}^N W_i$ the associated product space. We will often consider elements of W which are discontinuous across the interface.

The finite element approximation of the elliptic problem is continuous across Γ and we denote the corresponding subspace of W by $\widehat{W}$. We note that while the stiffness matrix K and Schur complement S which correspond to the product space W generally are singular those of $\widehat{W}$ are not.

For the dual–primal FETI methods, we will also use an additional, intermediate subspace $\widetilde{W}$ of W for which a relatively small number of continuity constraints are enforced across the interface throughout the iteration. In our dual–primal FETI methods, the selection of these constraints will be closely related to the coarse spaces of certain primal iterative substructuring methods. One of the benefits of working in $\widetilde{W}$, rather than in W, is that certain related Schur complements $\widetilde{S}$ and S_{Δ} are positive definite.

We assume that possible jumps of $\rho(x)$ are aligned with the subdomain boundaries and, for simplicity, that on each subregion Ω_i, $\rho(x)$ has the constant value $\rho_i > 0$. Our bilinear form and load vector can then be written, in terms of contributions from individual subregions, as

$$a(u,v) = \sum_{i=1}^{N} \rho_i \int_{\Omega_i} \nabla u \cdot \nabla v dx, \quad f(v) = \sum_{i=1}^{N} \Big(\int_{\Omega_i} f v dx \; + \; \int_{\partial\Omega_i \cap \partial\Omega_N} g_N v ds \Big). \tag{3}$$

In our theoretical analysis, we assume that the subregions Ω_i are tetrahedra or hexahedra and that they are shape regular, i.e., not very thin. We also make a number of technical assumptions on the intersection of the boundary of the substructures and $\partial\Omega_D$; see [KW01]. We assume that H_i and H_j are comparable if the subdomains Ω_i and Ω_j are neighbors. The sets of nodes in Ω_i, on $\partial\Omega_i$, and on Γ are denoted by $\Omega_{i,h}, \partial\Omega_{i,h}$, and Γ_h, respectively.

As in previous work on Neumann–Neumann algorithms, a crucial role is played by *the weighted counting functions* $\mu_i \in \widehat{W}$, which are associated with the individual subdomain boundaries $\partial\Omega_i$; cf., e.g., [DSW96, DW95]. In this paper they will be used primarily in the definition of certain diagonal scaling matrices. These functions are defined, for $\gamma \in [1/2, \infty)$, and for $x \in \Gamma_h \cup \partial\Omega_h$, by a sum of contributions from Ω_i, and its relevant next neighbors

$$\mu_i(x) = \begin{cases} \sum_{j \in \mathcal{N}_x} \rho_j^{\gamma}(x) & x \in \partial\Omega_{i,h} \cap \partial\Omega_{j,h}, \\ \rho_i^{\gamma}(x) & x \in \partial\Omega_{i,h} \cap (\partial\Omega_h \setminus \Gamma_h), \\ 0 & x \in (\Gamma_h \cup \partial\Omega_h) \setminus \partial\Omega_{i,h}. \end{cases} \tag{4}$$

Here, $\mathcal{N}_x$ is the set of indices of the subregions which have x on its boundary. We note that any node of Γ_h belongs either to two faces, more than two edges, or to the vertices of several substructures.

The pseudo inverses $\mu_i^{\dagger}$ are defined, for $x \in \Gamma_h \cup \partial\Omega_h$, by

$$\mu_i^{\dagger}(x) = \begin{cases} \mu_i^{-1}(x) & \text{if } \mu_i(x) \neq 0, \\ 0 & \text{if } \mu_i(x) = 0. \end{cases}$$

A review of one–level FETI methods

In this section, we give a brief review of the original FETI method of Farhat and Roux [FMR94, FR94] and the variant with a Dirichlet preconditioner introduced in Farhat, Mandel, and Roux [FMR94]. The more general projection operators, described in this section, were first introduced for heterogeneous problems in [FR94] and they have been tested in very large scale numerical experiments; see [BDF+00].

For a chosen finite element method and for each subdomain Ω_i, we assemble the local stiffness matrix $K^{(i)}$ and the local load vector corresponding to a single, appropriate term in the sums of (3). Any nodal variable, not associated with Γ_h, is called interior and it only belongs to one substructure. The interior variables of any subdomain can be eliminated by a step of block Gaussian elimination; this work can clearly be parallelized across the subdomains. The resulting matrices are the Schur complements

$$S^{(i)} = K_{\Gamma\Gamma}^{(i)} - K_{\Gamma I}^{(i)} (K_{II}^{(i)})^{-1} K_{I\Gamma}^{(i)}, \quad i = 1, \ldots, N.$$

Here, Γ and I represent the interface and interior, respectively. We note that the $S^{(i)}$ are only needed in terms of matrix-vector products and that therefore the elements of these matrices need not be explicitly computed.

The values of the right hand vectors also change when the interior variables are eliminated. We denote the resulting vectors, representing the modified load originating in Ω_i, by f_i and the local vectors of interface nodal values by u_i.

We can now reformulate the finite element problem, reduced to the interface Γ, as a minimization problem with constraints given by the requirement of continuity across Γ :

Find $u \in W$, such that

$$\left. \begin{array}{c} J(u) := \frac{1}{2}\langle Su, u\rangle - \langle f, u\rangle \rightarrow \ \min \\ Bu = 0 \end{array} \right\} \tag{5}$$

where $u = [u_1 \ldots u_N]^t, f = [f_1 \ldots f_N]^t$, and $S = diag_{i=1}^N (S^{(i)})$ is block–diagonal. The matrix $B = [B^{(1)}, \ldots , B^{(N)}]$ is constructed from $\{0, 1, -1\}$ such that the values of the solution u, associated with more than one subdomain, coincide when $Bu = 0$. We note that the choice of B is far from unique. The local Schur complements $S^{(i)}$ are positive semidefinite and they are singular for any subregion with a boundary which does not intersect $\partial\Omega_D$. The problem (5) is uniquely solvable if and only if $ker\,(S) \cap ker\,(B) = \{0\}$, i.e., if and only if S is invertible on $ker\,(B)$.

By introducing a vector of Lagrange multipliers λ, to enforce the constraints $Bu = 0$, we obtain a saddle point formulation of (5):

Find $(u, \lambda) \in W \times U$, such that

$$\left. \begin{array}{ccccc} Su & + & B^t\lambda & = & f \\ Bu & & & = & 0 \end{array} \right\}. \tag{6}$$

We note that the solution λ of (6) is unique only up to an additive vector of $ker\,(B^t)$. The space of Lagrange multipliers U is therefore chosen as $range\,(B)$.

We will also use a full column rank matrix built from all of the null space elements of S; these elements are associated with individual subdomains (the rigid body motions in the case of elasticity),

$$R = [R^{(1)} \ldots R^{(N)}].$$

Thus, $range\,(R) = ker\,(S)$. We note that no subdomain with a boundary which intersects $\partial\Omega_D$ contributes to R.

The solution of the first equation in (6) exists if and only if $f - B^t\lambda \in range\,(S)$; this constraint will lead to the introduction of a projection P. We obtain,

$$u = S^{\dagger}(f - B^t\lambda) + R\alpha \text{ if } f - B^t\lambda \perp ker\,(S),$$

where $S^\dagger$ is a pseudoinverse of S. The value of α can be determined easily once λ has been found.

Substituting u into the second equation of (6) gives

$$BS^\dagger B^t\lambda = BS^\dagger f + BR\alpha. \tag{7}$$

We now introduce a symmetric, positive definite matrix Q which induces an inner product on U; it is defined by $\langle\lambda,\mu\rangle_Q := \langle\lambda, Q\mu\rangle$. By considering the component which is $Q^{-1}-$orthogonal to $G := BR$, we find that

$$\left.\begin{array}{rcl} P^tF\lambda &=& P^td \\ G^t\lambda &=& e \end{array}\right\} \tag{8}$$

with $F := BS^\dagger B^t, d := BS^\dagger f, P := I - QG(G^tQG)^{-1}G^t$, and $e := R^tf$. We note that P is an orthogonal projection, from U onto $ker\,(G^t)$, in the $Q^{-1}-$inner product, i.e., the inner product defined by $\langle\lambda, Q^{-1}\mu\rangle$.

There are different good choices for Q. In the case of homogeneous coefficients, it is sufficient to use $Q = I$, while for problems with jumps in the coefficients, we have to make a more elaborate choice to make our proofs work satisfactorily. In our analysis, Q will be a diagonal scaling matrix or we will use the preconditioner; other alternatives are discussed in [BDF$^+$00, FR94].

By multiplying (7) by $(G^tQG)^{-1}G^tQ$, we find that $\alpha := (G^tQG)^{-1}G^tQ(F\lambda - d)$ which then fully determines the primal variables in terms of λ.

We introduce the space

$$V := \{\mu \in U\ :\ \langle\mu, Bz\rangle = 0 \quad \forall z \in ker\,(S)\} = ker\,(G^t) = range\,(P),$$

and a space that is isomorphic to its dual,

$$V' := \{\lambda \in U : \langle\lambda, Bz\rangle_Q = 0 \quad \forall z \in ker\,(S)\} = range\,(P^t).$$

As is usual in the literature on FETI methods, we can call V the space of admissible increments. The original FETI method is a conjugate gradient method in the space V applied to

$$P^tF\lambda = P^td, \qquad \lambda \in \lambda_0 + V, \tag{9}$$

with an initial approximation λ_0 chosen such that $G^t\lambda_0 = e$. The most basic FETI preconditioner, as introduced in Farhat, Mandel, and Roux [FMR94], is of the form

$$M^{-1} := BSB^t.$$

To apply M^{-1} to a vector, N independent Dirichlet problems have to be solved, one on each subregion; it is therefore called the Dirichlet preconditioner.

To keep the search directions of the resulting preconditioned conjugate gradient method in the space V, the application of the preconditioner M^{-1} is followed by an application of the projection P. Hence, the Dirichlet variant of the FETI method is the conjugate gradient algorithm applied to the equation

$$PM^{-1}P^tF\,\lambda = PM^{-1}P^t\,d, \qquad \lambda \in \lambda_0 + V. \tag{10}$$

We note that for $\lambda \in V$, $PM^{-1}P^tF\lambda = PM^{-1}P^tP^tFP\lambda$, and that we can therefore view the operator on left hand side of (10) as the product of two symmetric matrices.

It is well known that an appropriate norm of the iteration error of the conjugate gradient method will decrease at least by a factor

$$2(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1})^k,$$

in k steps. Here κ is the ratio of the largest and smallest eigenvalues of the iteration operator. The main task in the theory is therefore always to obtain a good bound for the condition number κ.

We note that several different possibilities of improving the FETI preconditioner M^{-1} have already been explored. Some interesting variants are discussed by Rixen and Farhat [RF99], in a framework of mechanically consistent preconditioners, in the case of redundant Lagrange multipliers; see also Klawonn and Widlund [KW01, section 5] for an analysis.

New one-level FETI preconditioners with non-redundant Lagrange multipliers

In this section, we outline some of our results on a family of new FETI preconditioners with an improved condition number estimate compared to that of Mandel and Tezaur [MT96]. Most importantly, we obtain a uniform bound for arbitrary positive values of the ρ_i if the scaling matrix Q, which enters the definition of P, is chosen carefully. In our proofs, we use several arguments developed in [MT96], but our presentation also differs considerably in several respects.

We now assume that B has full row rank, i.e., the constraints are linearly independent and there are no redundant Lagrange multipliers.

Our new preconditioner is defined, for any diagonal matrix D with positive elements, as

$$\widehat{M}^{-1} := (BD^{-1}B^t)^{-1}BD^{-1}SD^{-1}B^t(BD^{-1}B^t)^{-1}. \tag{11}$$

To obtain a method, which converges at a rate which is independent of the coefficient jumps, we now choose a special family of matrices D; a careful choice of the scaling Q, introduced in the definition of the operator P, will also be required. As in previous work on Neumann–Neumann algorithms, a crucial role is played by the weighted counting functions μ_i, associated with the individual $\partial\Omega_i$, and already introduced in (4). The diagonal matrix $D^{(i)}$ has the diagonal entry $\rho_i^{\gamma}(x)\mu_i^{\dagger}(x)$ corresponding to the point $x \in \partial\Omega_{i,h}$. Finally, we set $D := diag_{i=1}^N(D^{(i)})$. We note that this matrix is a block–diagonal matrix which operates on elements in the product space W.

We now give a condition number estimate for the preconditioned FETI operator $P\widehat{M}^{-1}P^tF$; cf. [KW01]. The result holds for $Q = \widehat{M}^{-1}$ and also for a special choice of B and a special diagonal Q; in the case of continuous coefficients, it is sufficient to choose Q as a multiple of the identity matrix for the next theorem to be valid.

Theorem 1 *The condition number of the FETI method, with the new preconditioner $\widehat{M}$, satisfies*

$$\kappa(P\widehat{M}^{-1}P^tF) \leq C\,(1+\log(H/h))^2.$$

Here, $\kappa(P\widehat{M}^{-1}P^tF)$ is the spectral condition number of $P\widehat{M}^{-1}P^tF$, and C is independent of h, H, γ, and the values of the ρ_i.

A New Dual–Primal FETI method

In previous studies of dual–primal FETI methods for problems in two dimensions, see Farhat, Lesoinne, Le Tallec, Pierson, and Rixen [FLLT$^+$99] and Mandel and Tezaur [MT00], the constraints on the degrees of freedom associated with the vertices of the substructures are enforced, i.e., the corresponding degrees of freedom have been added to the global set of variables, while all the constraints associated with the edge nodes are enforced only at the convergence of the iterative method. In each step of the iteration a fully assembled linear subsystem is solved. In a simple two–dimensional case, this subsystem corresponds to all the interior and cross point variables; these variables can be eliminated at a modest expense since we can first eliminate all the interior variables, in parallel across the subdomains, resulting in a Schur complement for the cross point variables which can be shown to be sparse. It has a dimension which equals the number of subdomain vertices which do not belong to $\partial\Omega_D$.

In their recent paper, Mandel and Tezaur [MT00] established a condition number bound of the form $C(1+\log(H/h))^2$ for the resulting FETI method equipped with a Dirichlet preconditioner which is very similar to those used for the older FETI methods and which is built from local solvers on the subregions with zero Dirichlet conditions at the vertices of the subregions. They also established a corresponding result for a fourth-order elliptic problem in the plane. Their elegant proof in [MT00] relies, for the second order equation, on linear algebra arguments and a lemma from a classical paper by Bramble, Pasciak, and Schatz [BPS86, Lemma 3.5].

The same algorithm is also defined for three dimensions but it does not perform well. This is undoubtedly related to the poor performance of many *vertex-based* iterative substructuring methods; see [DSW94, Section 6.1] and [KW00b]. Recently, Farhat et al. added constraints to this basic algorithm, see [FLP00], and improved the performance.

In our approach, we first carry out a change of variables prior to dividing the variables into a primal and a dual subspace. The number of constraints enforced in each iteration will now be larger, but we will still be able to work with a number of constraints which is uniformly bounded for each substructure.

One of our new algorithms is given in terms of a space $\widetilde{W} \subset W$ for which we have continuity at the subdomain vertices, and also common values of the averages over all edges and all faces of the interface. This space can naturally be written as a direct sum of two subspaces, corresponding to a primal and a dual part of the problem, i.e.,

$$\widetilde{W} = \widehat{W}_\Pi \oplus \widetilde{W}_\Delta.$$

The first subspace, $\widehat{W}_\Pi \subset \widehat{W}$, which together with the interior subspaces, defines the subsystem which is fully assembled, factored, and solved in each iteration step.

It is defined as the range of the following interpolation operator I_B^h defined, for any $u_h \in \widetilde{W}$, by

$$I_B^h u_h(x) = \sum_{\mathcal{V}^{i\ell} \in \Gamma} u^h(\mathcal{V}^{i\ell}) \varphi_{\mathcal{V}^{i\ell}}(x) + \sum_{\mathcal{E}^{ik} \subset \Gamma} \bar{u}^h_{\mathcal{E}^{ik}} \theta_{\mathcal{E}^{ik}}(x) + \sum_{\mathcal{F}^{ij} \subset \Gamma} \bar{u}^h_{\mathcal{F}^{ij}} \theta_{\mathcal{F}^{ij}}(x). \qquad (12)$$

Here,

$$\bar{u}^h_{\mathcal{E}^{ik}} = \frac{\int_{\mathcal{E}^{ik}} I^h(\theta_{\mathcal{E}^{ik}} u_h) ds}{\int_{\mathcal{E}^{ik}} \theta_{\mathcal{E}^{ik}} ds} \quad and \quad \bar{u}^h_{\mathcal{F}^{ij}} = \frac{\int_{\mathcal{F}^{ij}} I^h(\theta_{\mathcal{F}^{ij}} u_h) dx}{\int_{\mathcal{F}^{ij}} \theta_{\mathcal{F}^{ik}} dx},$$

$\varphi_{\mathcal{V}^{i\ell}}$ are the standard nodal basis function, and $\theta_{\mathcal{E}^{ik}}$ and $\theta_{\mathcal{F}^{ij}}$ the discrete harmonic functions which equal 1 on $\mathcal{E}_h^{ik}$ and $\mathcal{F}_h^{ij}$, respectively, and vanish elsewhere on Γ_h. The operator I_B^h, introduced in [DSW94, p. 1690], has almost optimal stability properties. Let us note that several cheaper algorithm, based on different interpolation operators, are also discussed in [KW00b].

The subspace $\widehat{W}_\Pi$ is thus given in terms of the vertex variables, the averages of the values over the individual edges of the set of interface nodes Γ_h, and the averages over the individual faces of substructures.

We note that the dimension of this first subspace is relatively small; in the case of hexahedral substructures there are seven global variables for each interior substructure since there are eight vertices, each shared by eight hexahedra, twelve edges, each shared by four, and six faces each shared by a pair of substructures. We note that the count is smaller, relative to the number of substructures, in the case of tetrahedral subregions. We can demonstrate that the resulting system can be assembled and solved at an acceptable cost which only exceeds that for the more primitive algorithm in which we enforce only the vertex constraints in each step, by a constant factor. We note that we have also developed a second method with only four global variables per subdomain; our theoretical results for that method involves a third power of the logarithm. We have no doubts that a number of other promising alternatives could be developed given the rich choice of coarse spaces for the primal iterative substructuring methods.

The second subspace, denoted by $\widetilde{W}_\Delta$, is associated with the nodal points on the edges and faces of the interface Γ. It is the direct sum of local subspaces of $\widetilde{W}$. For each subdomain Ω_i, the local subspace consists of functions that vanish at the subdomain vertices and have zero average on each individual edge and face. They are extended by zero on all of the $\partial\Omega_j, j \neq i$; it is easy to see that these functions satisfies the continuity requirements associated with $\widetilde{W}$.

The linear systems solved in the preconditioning step of our FETI–DP algorithm, which is directly related to $\widetilde{W}_\Delta$, have zero Dirichlet boundary conditions at the vertices and also satisfy the constraints that the averages over individual edges and faces vanish. The nodal values represent the original nodal values minus the average over the edge or face to which the node belongs. This construction makes the local solvers well defined and the resulting set of variables represent a subspace complementary to the first subspace; together with the interior spaces they represent the variables of the entire linear space of the partially subassembled system.

We can now formulate one of our FETI–DP algorithms; for details on its implementation, we refer to Klawonn and Widlund [KW00b].

We first eliminate, after a partial change of variables, all unknowns of the first subspace as well as the interior variables, and obtain a Schur complement $\widetilde{S}$.

Analogously, we get from the load vectors associated with each subdomain a reduced right hand side $\tilde{f}_\Delta$. We can now reformulate the original finite element problem, reduced to the degrees of freedom of the second subspace $\widetilde{W}_\Delta$, as a minimization problem with constraints given by the requirement of continuity across Γ_h:

Find $u_\Delta \in \widetilde{W}_\Delta$, such that

$$\left.\begin{array}{c} J(u_\Delta) := \frac{1}{2}\langle \widetilde{S}u_\Delta, u_\Delta\rangle - \langle \tilde{f}_\Delta, u_d\rangle \to \min \\ B_\Delta u_\Delta = 0 \end{array}\right\}. \tag{13}$$

The matrix B_Δ is constructed from $\{0, 1, -1\}$ in the same fashion as B. Since we already have imposed a constraint on the averages over each edge and each face, we may drop one of the point constraints for each edge and each face when constructing the matrix B_Δ. By introducing a set of Lagrange multipliers $\lambda \in V := \mathit{range}\,(B_\Delta)$, to enforce the constraints $B_\Delta u_\Delta = 0$, we obtain a saddle point formulation of (13), which is similar to (6). We use that $\widetilde{S}$ is invertible and eliminate the subvector u_Δ, and obtain the following system for the dual variable:

$$F_\Delta \lambda = d_\Delta, \tag{14}$$

where

$$F_\Delta := B_\Delta \widetilde{S}^{-1} B_\Delta^t$$

and the right hand side

$$d_\Delta := B_\Delta \widetilde{S}^{-1} \tilde{f}_\Delta.$$

To define the FETI–DP Dirichlet preconditioner, we need to introduce an additional, third set of Schur complement matrices,

$$S_{\Delta\Delta}^{(i)} := K_{\Delta\Delta}^{(i)} - K_{\Delta I}^{(i)} (K_{II}^{(i)})^{-1} K_{I\Delta}^{(i)}, \quad i = 1, \dots, N,$$

which can also be obtained from $S^{(i)}$ by removing the rows and columns that correspond to the vertices and the edge and face averages, i.e., all the variables of the first subspace $\widehat{W}_\Pi$. Here, $K_{\Delta\Delta}^{(i)}$ is the principal minor of the stiffness matrix after the change of variables and it is related to the variables of $\widetilde{W}_\Delta$. The associated block–diagonal matrix is denoted by

$$S_{\Delta\Delta} := \mathit{diag}_{i=1}^N (S_{\Delta\Delta}^{(i)}).$$

We can compute the action of $S_{\Delta\Delta}$ on a vector from the second subspace $\widetilde{W}_\Delta$ by solving local problems with solutions that are constrained to vanish at the cross points and to have zero edge and face averages; these constraints can be enforced by using Lagrange multipliers or a partial change of basis.

As in the fourth section, cf. (11), we solve the dual system (14) using the preconditioned conjugate gradient algorithm with the preconditioner

$$M_B^{-1} := (B_\Delta D_\Delta^{-1} B_\Delta^t)^{-1} B_\Delta D_\Delta^{-1} S_{\Delta\Delta} D_\Delta^{-1} B_\Delta^t (B_\Delta D_\Delta^{-1} B_\Delta^t)^{-1}. \tag{15}$$

Here, D_Δ is a diagonal matrix with positive elements on the diagonal. It can be easily seen that $B_\Delta D_\Delta^{-1} B_\Delta^t$ is a block-diagonal matrix and thus its inverse can be computed at essentially no extra cost; the block sizes are n_x, where n_x is the number of Lagrange multipliers employed to enforce continuity at the point x. In order to obtain a method that converges at a rate which is independent of the coefficient jumps, we now choose a special family of matrices D_Δ, cf. also Klawonn and Widlund [KW01, sect. 4]. We first define the contributions of each subdomain boundary $\partial\Omega_i$ in terms of a diagonal matrix $D_\Delta^{(i)}$. For any point x on an edge or a face of Ω_i, there is an entry on the diagonal of $D_\Delta^{(i)}$ equal to $\rho_i^\gamma(x)\mu_i^\dagger(x)$. We now set

$$D_\Delta := diag_{j=1}^N(D_\Delta^{(j)}).$$

The dual–primal FETI method is now the standard preconditioned conjugate gradient algorithm for solving the preconditioned system

$$M_B^{-1}F_\Delta\lambda = M_B^{-1}d_\Delta.$$

A proof of the following theorem can be found in Klawonn and Widlund [KW00b].

Theorem 2 *The condition number of the FETI–DP method with the preconditioner M_B satisfies*

$$\kappa(M_B^{-1}F_\Delta) \leq C\,(1+\log(H/h))^2.$$

Here, C is independent of h, H, γ, and the values of the ρ_i.

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