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16. Finite Difference Method with Fictitious Domain Applied to a Dirichlet Problem

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Introduction

In this paper we shall consider the construction of the solution by the method of lines coupled with a fictitious domain for the following Dirichlet problem (1) in a bounded domain Ω of $\mathbb{R}^2$.

Problem I. For given functions f and g, find u in $H^2(\Omega)$ such that

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here $f \in L^2(\Omega)$, $g \in H^{3/2}(\partial\Omega)$ and Ω is a bounded domain in $\mathbb{R}^2$ with the smooth boundary $\partial\Omega$ (see Figure 1).

The method of lines for solving Problem I works well if Ω is a rectangular domain since the finite difference solution is expressed explicitly by use of eigenvalues and eigenvectors for the finite difference scheme([BGN70], [Nak65]). But one says that this method seems difficult to be applied to the case where Ω is not a rectangular domain. However the solution algorithm using the fictitious domain and the domain decomposition has been developed recently ([AKP95], [GPP94], [HH99], [FKK95], [KK99], [MKM86]). Hence from this point of view we shall propose a numerical algorithm by the method of lines coupled with a fictitious domain in this paper.

First of all, we embed Ω in a rectangular domain Π whose boundary $\partial\Pi$ consists of straight lines parallel to axes and set $\Omega_1 = \Pi \setminus (\Omega \cup \partial\Omega)$ (see Figure 2). Then Π is called a fictitious domain.

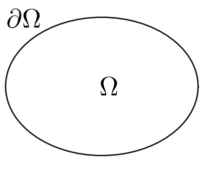

Figure 1: Figure 1

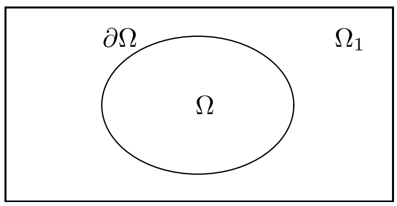

Figure 2: ($\Pi = \overline{\Omega} \cup \Omega_1$)

Hereafter we shall construct the numerical algorithm for solving Poisson's equation (1) in the fictitious domain Π. In §2, Problem I is reduced to a fictitious domain

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formulation by use of the distribution theoretical approach. In §3, we shall discuss characterizations of the solution for the fictitious domain formulation. In §4, a numerical algorithm of the direct method of lines will be proposed and the results of numerical computations will be shown.

A fictitious domain formulation of Problem I.

Using the trace operator γ in Sobolev space and distribution theoretical argument, we deduce a fictitious domain formulation from Problem I. It is well-known that there exists a function $G \in H^2(\Omega)$ such that $\gamma G = g$ on $\partial\Omega$ because of $g \in H^{3/2}(\partial\Omega)$. Then putting $u = v + G$, Problem I is reduced to

Problem II. Find $v \in H^2(\Omega)$ such that

$$\begin{cases} -\Delta v &= f + \Delta G \quad \text{in } \Omega, \\ \quad v &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2}$$

Remark 1 *Set $u = v + G$ where v is a solution of Problem II. Then u is a unique solution of Problem I. In this case, it is important to be independent of a choice of G in Problem II (see p.232 in [Miz73]). And this fact will be used in §4.*

We now define a function $\widetilde{v}$ as follows: for any function $v \in L^2(\Omega)$,

$$\begin{cases} \widetilde{v}(x) &= v(x) \quad (x \in \Omega) \\ \widetilde{v}(x) &= 0 \quad (x \in \mathbb{R}^2 \backslash \Omega). \end{cases} \tag{3}$$

Then for $v \in H_0^1(\Omega)$, $\widetilde{v}$ belongs to $H^1(\mathbb{R}^2)$ and the equality

$$\frac{\partial}{\partial x_i}\widetilde{v} = \widetilde{\frac{\partial v}{\partial x_i}} \tag{4}$$

holds(see p.187-189 in [Miz73]). Moreover operating Δ to $\widetilde{v}$, we have the following lemma which was shown by Kawarada([Kaw89]).

Lemma 1 *Let $v \in H^2(\Omega) \cap H_0^1(\Omega)$. Then*

$$\widetilde{\Delta v} = \Delta \widetilde{v} + \frac{\partial v}{\partial n} \cdot \delta(\partial\Omega) \quad \text{in the sense of distribution in } \mathbb{R}^2, \tag{5}$$

holds where n is the unit normal vector at $\partial\Omega$, directed towards the outer of Ω and $\delta(\partial\Omega)$ means the delta measure supported on $\partial\Omega$.

By use of (3)-(5) and the definition of the weak derivative in the sense of the distribution, we have

Theorem 1 *Problem II is equivalent to the following Problem III. i.e.,*

Problem III. *Find $\widetilde{v} \in H_0^1(\Pi)$ and $w \in L^2(\partial\Omega)$ such that*

$$-\Delta \widetilde{v} \;=\; \widetilde{f + \Delta G} + w\ \delta(\partial\Omega) \ \text{ in } D'(\Pi) \tag{6}$$

Remark 2 $\widetilde{v} \in H_0^1(\Pi)$ *means that* $v \in H_0^1(\Omega)$ *and* $v \equiv 0$ *in* Ω_1.

Corollary 1 *The solution* $\{\widetilde{v}, w\}$ *of (6) has the following relation:*

$$w = \frac{\partial v}{\partial n} \quad on \quad \partial\Omega.$$

Remark 3 *We call (6) a fictitious domain formulation of Problem II and this formulation is essential for our discussions. It will be used to construct the finite difference solution of Problem I in §4.*

Characterization of the solution of the fictitious domain formulation (6)

Before proceeding to the construction of the finite difference scheme under the fictitious domain formulation, we state the relation between (6) and the auxiliary domain method ([Lio73]).

Proposition 1 *The following statements are equivalent to each other.*

(i) There exists a unique solution $v \in H^2(\Omega)$ *satisfying Problem II. And set* $w = \frac{\partial v}{\partial n}$.

(ii) There exists a solution $\{\widetilde{v}, w\} \in H_0^1(\Pi) \times L^2(\partial\Omega)$ *in Problem III which satisfies*

$$-\ \Delta\widetilde{v} \ = \ \widetilde{f + \Delta G} + w\ \delta(\partial\Omega) \ in \ \ D'(\Pi).$$

(iii) There exists a solution $\{v_0, v_1, w\} \in H_0^1(\Omega) \times H_0^1(\Omega_1) \times L^2(\partial\Omega)$ *such that*

$$\begin{cases} -\ \Delta v_0 = \ f + \Delta G \quad\quad in \ \ \Omega\ , \\ -\ \Delta v_1 = \ 0 \quad\quad in \ \ \Omega_1\ , \\ \ \ v_0 = v_1 = 0 \quad\quad on \ \ \partial\Omega\ , \\ \ \ \dfrac{\partial v_0}{\partial n} = w \quad\quad on \ \ \partial\Omega\ , \\ \ \ v_1 \ = \ 0 \quad\quad on \ \ \partial\Pi. \end{cases}$$

(iv) There exists a solution $\{v, w\} \in V \times L^2(\partial\Omega)$ *satisfying*

$$\int_{\Pi} \nabla v \cdot \nabla\varphi \ dx = \int_{\Omega} (f + \Delta G)\varphi + \int_{\partial\Omega} \omega\varphi \ d\Gamma \quad for \ any \ \varphi \in V \tag{7}$$

where $V = H^1(\Pi) \cap H_0^1(\Omega_1)$.

Remark 4 ω *in the form (7) is usually called a Lagrange multiplier .*

Proposition 2 *The solution* $\widetilde{v}$ *of Problem III with* $g = 0$ *is the limit function of the approximate solutions* $\{v_0^\epsilon, v_1^\epsilon\}$ *as* $\epsilon \to 0$*:*

$$\begin{cases} -\ \Delta v_0^\epsilon = \ f \quad\quad in \ \ \Omega\ , \\ -\epsilon^{2\alpha}\Delta v_1^\epsilon + \epsilon^{-2\beta}v_1^\epsilon = \ 0 \quad\quad in \ \ \Omega_1\ , \\ \ \ v_0^\epsilon = v_1^\epsilon \quad\quad on \ \ \partial\Omega\ , \\ \ \ \dfrac{\partial v_0^\epsilon}{\partial n} = \epsilon^{2\alpha}\dfrac{\partial v_1^\epsilon}{\partial n} \quad\quad on \ \ \partial\Omega\ , \\ \ \ v_1^\epsilon \ = \ 0 \quad\quad on \ \ \partial\Pi \end{cases}$$

for any α, β *satisfying* $0 < \alpha < \beta$.

Proof In fact, it is known that for α and β ($0 < \alpha < \beta$),
$v_0^\epsilon \to v_0$ in $H^1(\Omega)$, $v_1^\epsilon \to 0$ in $L^2(\Omega_1)$, and v_0 is the solution of Problem II (see Theorem 10.1, pp. 78-82 in [Lio73]). Hence setting $w = \frac{\partial v_0}{\partial n}$, $\widetilde{v_0}$ is exactly the solution of Problem III by (iii) of Propostion 1. ■

Numerical Algorithm of the fictitious domain formulation (6) by use of the direct method of lines

In this section, we shall propose a numerical algorithm of the direct method of lines by use of the fictitious domain formulation (6).

Discretization of the fictitious domain formulation (6)

We first assume the fictitious domain Π given by

$$\Pi = \{ \ (x, y) \mid 0 < x < 1, \ 0 < y < 1 \ \}, \tag{8}$$

which consists of $\overline{\Omega}$ and Ω_1 where $\Omega_1 = \Pi \setminus \overline{\Omega}$ (see Figure 2).

While the set of grid points, $\overline{\Pi_h}$, is of the form

$$\overline{\Pi_h} = \{ \ (x_i, y_j) \mid 0 \le i \le m, \ 0 \le j \le m \ \},$$

here $x_i = ih, y_j = jh$ for a suitable spacing $h = 1/m$ and $P(i,j) = (x_i, y_j)$.

With each grid point (x_i, y_j) of Π_h, we associate the cross line with center (x_i, y_j):

$$M\Big((x_i, y_j)\Big) = \{ \ (x_i + s, y_j), s \in (-\frac{h}{2}, \frac{h}{2})\} \cup \{ \ (x_i, y_j + s), s \in (-\frac{h}{2}, \frac{h}{2})\}.$$

We then define

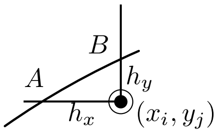
$$\begin{aligned}
\Omega_h^0 &= \{ \ (x_i, y_j) \ : \ (x_i, y_j) \in \overline{\Pi_h}, \ M((x_i, y_j)) \subset \Omega \ \}, \\
\Omega_h^1 &= \{ \ (x_i, y_j) \ : \ (x_i, y_j) \in \overline{\Pi_h}, \ M((x_i, y_j)) \subset \Omega_1 \ \}, \\
\partial\Omega_h^0 &= \{ \ (x_i, y_j) \ : \ (x_i, y_j) \in \overline{\Pi_h}, M((x_i, y_j)) \cap \partial\Omega \ne \phi \ \}, \\
\partial\Omega_h^1 &= \{ \ (x_i, y_j) \ : \ (x_i, y_j) \in \overline{\Pi_h}, \ M((x_i, y_j)) \cap \partial\Omega_1 \ne \phi \ \}.
\end{aligned}$$

We then define for each i, j ($0 \le i, j \le m$),

$$F_{i,j} = F(x_i, y_j) = \begin{cases} f(x_i, y_j) & \text{for } P(i,j) \in \Omega_h^0 \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

$$G_{i,j} = G(x_i, y_j) = \begin{cases} \overline{g}(x_i, y_j) & \text{for } P(i,j) \in \partial\Omega_h^0 \\ 0 & \text{otherwise} \ . \end{cases}$$

Here $\overline{g}(x_i, y_j)$ is defined as follows:



$$\overline{g}(x_i,y_j)=\begin{cases} g(x_i,y_j) \ \text{ if } h_x=0 \text{ or } h_y=0, \\ g(A) \ \text{ if } 0<h_x\le h/2 \text{ and } h_y>h/2, \\ g(B) \ \text{ if } h_x>h/2 \text{ and } 0<h_y\le h/2, \\ \dfrac{g(A)\times h_y+g(B)\times h_x}{h_x+h_y} \ \text{ if } 0<h_x,h_y<h/2. \end{cases}$$

Then the finite difference approximation of (6) can be formulated as follows.

Find $v(=\{v_{i,j}\})$ and $w(=\{w_{i,j}\})$ such that

$$-(\Delta_h v)(x_i,y_j)=F_{i,j}+(\Delta_h G)(x_i,y_j)+\frac{\sqrt{2}}{h}\ w_{i,j}\ \delta(P_{i,j}) \ \text{ for all } (x_i,y_j)\in\Pi_h \quad (9)$$

where $\delta(P_{i,j})=1$ if $P_{i,j}\in\partial\Omega_h^0, \delta(P_{i,j})=0$ if $P_{i,j}\notin\partial\Omega_h^0$ and the finite difference operator $-\Delta_h$, approximating the Laplace operator $-\Delta$ is of the form

$$-(\Delta_h v)(x_i,y_j)=\frac{1}{h^2}[v_{i+1,j}+v_{i-1,j}+v_{i,j+1}+v_{i,j-1}-4v_{i,j}]$$

and $v_{i,j}=v(x_i,y_j)$ as usual.

Theorem 2 *There exists a unique solution* $\{v_{i,j}\}$ *and* $\{w_{i,j}\}$ *of (9).*

Proof In fact, (9) is rewritten as follows. Find $v(=\{v_{i,j}\})$ and $w(=\{w_{i,j}\})$ such that

$$-(\Delta_h v)(x_i,y_j)=F_{i,j}+(\Delta_h G)(x_i,y_j) \ \text{ for all } (x_i,y_j)\in\Omega_h^0 \quad (10)$$

$$v(x_i,y_j)=0 \quad \text{ for all } (x_i,y_j)\in\partial\Omega_h^0, \quad (11)$$

$$-(\Delta_h v)(x_i,y_j)=0 \ \text{ for all } (x_i,y_j)\in\Omega_h^1 \quad (12)$$

$$v(x_i,y_j)=0 \quad \text{ for all } (x_i,y_j)\in\partial\Omega_h^1, \quad (13)$$

$$-(\Delta_h v)(x_i,y_j)=\frac{\sqrt{2}}{h}\ w_{i,j}\ \delta(P_{i,m+j}) \ \text{ for all } (x_i,y_j)\in\partial\Omega_h^0. \quad (14)$$

Then it follwos from these forms and the standard arguments in the finite difference method ([Joh67]) that the Dirichlet problems (10)- (11) and (12)-(13) have unique solutions $v_{i,j}$ on $\Pi_h\backslash\partial\Omega_h^0$, from which and (14) , $v_{i,j}$ on $\partial\Omega_h^0$ are uniquely determined. ■

Now in order to give the matrix expression of (9), we shall introduce the following notations. For each $i(1 \le i \le m-1)$,

$$V_i = (v_{i,1}, \cdots, v_{i,m-1})^T, \tag{15}$$

$$W_i = (\xi_{i,1}, \cdots, \xi_{i,m-1})^T, \tag{16}$$

$$Z_i = h^2 \ (F_{i,1} + (\Delta_h)G(x_i, y_1), \cdots, F_{i,m-1} + (\Delta_h)G(x_i, y_{m-1})\ )^T \tag{17}$$

Here we set $\xi_{i,j} = \sqrt{2}\ h\ w_{i,j}\ \ (1 \le i, j \le m-1)$.

Remark 5 *If $P(i,j) \in \partial\Omega_h^0$, then $v_{i,j} = 0$ and $w_{i,j} \neq 0$. If $P(i,j) \notin \partial\Omega_h^0$, then $w_{i,j} = 0$.*

We introduce the concept of the support of vectors which is used in our numerical algorithm.

Definition 1. *The support for an (m-1)-vector $V_i(= \{v_{i,j}\})$ is defined by*

$$supp(V_i) = \{j \mid v_{i,j} \neq 0\}.$$

Then Remark 5 shows that $supp(V_i) \cap supp(W_i) = \phi$ and $\xi_{i,j} = 0$ if $j \notin supp(W_i)$.

Using the above notations, the discrete equation for (6) is to

find $\{V_i\ ,\ W_i\}\ (1 \le i \le m-1)$ such that

$$A\ V_i = \ V_{i-1} + V_{i+1} + W_i + Z_i \quad (1 \le i \le m-1) \tag{18}$$

where $V_0 = 0$, $V_m = 0$ and A is $(m-1) \times (m-1)$ matrix as follows;

$$A = \begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & -1 \\ & & & -1 & 4 \end{pmatrix}. \tag{19}$$

Numerical algorithm by use of the direct method of lines

From now on we shall construct a numerical algorithm for (18). Using successive eliminations by lines, we have

Theorem 3 *For each k $(1 \le k \le m-1)$,*

$$PV_k = \sum_{i=1}^{k-1} D_m^{-1} D_{m-k} D_i PW_i + \sum_{i=k}^{m-1} D_m^{-1} D_k D_{m-i} PW_i + G_k \tag{20}$$

holds where $G_k = \sum_{i=1}^{k-1} D_m^{-1} D_{m-k} D_i PZ_i + \sum_{i=k}^{m-1} D_m^{-1} D_k D_{m-i} PZ_i$

and the diagonal matrix D_k and the orthogonal matrix P are determined by

$$D_k = \begin{pmatrix} a_1^k & & & \mathbf{0} \\ & a_2^k & & \\ & & \ddots & \\ \mathbf{0} & & & a_{m-1}^k \end{pmatrix}$$

where the elements a_j^k $(1 \leq j \leq m-1)$ are determined exactly by

$$a_j^k = \frac{\sinh(ka_j)}{\sinh(a_j)}, \quad a_j = \cosh^{-1}(\frac{\lambda_j}{2}), \quad \lambda_j = 2\Big(2 - \cos(\frac{j}{m}\pi)\Big). \tag{21}$$

and the orthogonal matrix $P = [p_1, p_2, \cdots, p_{m-1}]$ consists of

$$p_j = \sqrt{\frac{2}{m}} \begin{pmatrix} \sin(\frac{j}{m}\pi) \\ \sin(\frac{2j}{m}\pi) \\ \cdot \\ \cdot \\ \cdot \\ \sin(\frac{(m-1)j}{m}\pi) \end{pmatrix} \qquad (1 \leq j \leq m-1). \tag{22}$$

Remark 6 *By use of the property of the orthogonal matrix P, PV_i and PW_i are expressed as follows:*

$$PV_i = \sum_{j \in supp(V_i)} v_{i,j}\ p_j, \quad PW_i = \sum_{j \in supp(W_i)} w_{i,j}\ p_j. \tag{23}$$

Finally we propose a numerical algorithm which is deduced from Theorem 3 and (23).

Numerical algorithm:

(1st step) Calculate $\{\xi_{i,j}\}(j \in supp(W_i),\ 1 \leq i \leq m-1)$ such that

$$\sum_{i=1}^{k-1} \sum_{j \in supp(W_i)} \Big(p_l \bullet D_m^{-1} D_{m-k} D_i\ p_j\Big)\ \xi_{i,j} + \sum_{i=k}^{m-1} \sum_{j \in supp(W_i)} \Big(p_l \bullet D_m^{-1} D_k D_{m-i}\ p_j\Big)\ \xi_{i,j}$$
$$= -p_l \bullet G_k \text{ for all } l \in supp(W_k).$$

(2nd step) Compute $\{v_{k,l}\}$ $(l \in supp(V_k), 1 \leq k \leq m-1)$ by

$$v_{k,l} = \sum_{i=1}^{k-1} \sum_{j \in supp(W_i)} \Big(p_l \bullet D_m^{-1} D_{m-k} D_i\ p_j\Big)\ \xi_{i,j} + \sum_{i=k}^{m-1} \sum_{j \in supp(W_i)} \Big(p_l \bullet D_m^{-1} D_k D_{m-i}\ p_j\Big)\ \xi_{i,j}$$
$$+\ p_l \bullet G_k \quad \text{for all} \quad l \in supp(V_k).$$

Here $\bullet$ means the inner product in $\mathbb{R}^{m-1}$.

Remark 7 *This is a generalization of the corresponding one in [KK99].*

Numerical experiments

Using the numerical algorithm of Theorem 3, we consider the Dirichlet problem:

$$\begin{cases} -\Delta u &= 0 \quad \text{in } \Omega \, , \\ \quad u &= U \quad \text{on } \partial\Omega \, . \end{cases}$$

Here

$U(x,y)$=sinh$(\pi x/2)$ sin$(\pi y/2)$ and $\Omega = \{(x,y) \mid \frac{(x-1/2)^2}{(1/4)^2} + \frac{(y-1/2)^2}{(1/8)^2} < 1\}$ that is the same geometry as one in [GPP94].

Then we get the following table of the choice of different mesh interval dh as for the maximum error (MaxEr) and the average error (AvEr) where $MaxEr = max\{|U(ih,jh) - U_{i,j}| \, ; \, P(i,j) \in \Omega_h^0\}$ and $AvEr = \sum_{i,j=1}^{m-1}\{\, |U(ih,jh) - U_{i,j}|; \, P(i,j) \in \Omega_h^0\}/N_h$ (N_h : the total number of the mesh points in Ω_h^0).

$dh = 1/n$	$MaxEr$	$AvEr$
$n = 16$	9.430569×10^{-3}	3.588982×10^{-3}
$n = 32$	5.492716×10^{-3}	1.065969×10^{-3}
$n = 64$	5.258107×10^{-3}	6.026563×10^{-4}
$n = 128$	2.969270×10^{-3}	3.058067×10^{-4}

Concluding remarks

We have presented the numerical algorithm of the direct method of lines coupled with the fictitious domain. This method which use the regular mesh is very simple and easy to perform the calculation, and yet the above maximum errors are same as one in the standard framework of the finite difference method in nonrectangular domain ([Joh67]). Therefore this argument shows that the finite difference method under the regular mesh is able to be applied to the case of general domains with the help of the fictitious domain.

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