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17. New Interface Conditions for Non-overlapping Domain Decomposition Iterative Procedures

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Introduction

A Seidel-type interface condition is considered for non-overlapping domain decomposition iterative methods. With a suitable pseudo-energy defined on interfaces, the convergence speed of the iterative scheme is shown to be as twice fast as that of the Jacobi scheme. Our analysis is entirely independent of the governing model problems of a specific type of partial differential equations, but depends only on the scheme of updating interface data. By this, our analysis covers Seidel-type schemes for a general class of problems, such as elliptic, Helmholtz, Maxwell, and elasticity problems, etc. In order to avoid the sequential nature of Seidel schemes and to implement them on parallel computers, *red-black Gauss-Seidel* schemes are also considered with equivalent efficiency to Seidel schemes.

Concerning domain decomposition iterative methods, P.-L. Lions [Lio88, Lio90] investigated the convergence properties by taking a suitable pseudo-energy with which he was able to show iterative solutions converge. This idea has been applied to a more difficult Helmholtz problem by Després [Des91, BD97]. An improved variant of Lions's method is proposed by Q. Deng and its convergence is analyzed in the Sobolev H^1 norm [Den97]. Exploiting the structure of mixed finite element, Douglas et al. obtained a more precise convergence rate by a spectral radius estimation of the iterative solution operator [DPRW93]. More efficient iterative schemes, such as Seidel-type and under-relaxation type domain decomposition iterative methods for elliptic, Helmholtz and electromagnetic problems have been considered in [CGJ98, CDJP97, DM97, Fen97, Gha97], and Seidel-type approaches based on nonconforming finite elements [DSSY99] were used in [HKS99, Kwo99, KS99] with estimations of spectral radii obtained. In this paper we show that the Seidel-schemes are exactly twice faster than the corresponding Jacobi-schemes.

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Domain decomposition iterative procedure

A model problem

Let Ω be a domain in $\mathbf{R}^N, N = 2, 3$, with the boundary $\Gamma = \partial\Omega$. Let us first consider the following model problem:

$$-\nabla \cdot (A\nabla u) + Bu = f \quad \text{in } \Omega,\ \nu \cdot A\nabla u + \alpha u = g \quad \text{on } \Gamma, \tag{1}$$

where ν is the unit outward normal vector to $\partial\Omega$. The coefficients $A = A(x), B = B(x) = B_R + iB_I$, and $\alpha = \alpha(x) = \alpha_R + i\alpha_I$ are assumed to satisfy

$$\begin{aligned} &0 < A_0|\xi|^2 \le A_{jk}(x)\xi_k\bar{\xi_j} \le A_1|\xi|^2 < \infty, \\ &|B(x)| < B_1 < \infty, \qquad |\alpha(x)| < B_2 < \infty. \end{aligned}$$

Notice that (1) covers the case of Helmholtz equation and (1) may be regarded as a general form of first-order absorbing boundary condition.

Non-overlapping domain decomposition iterative procedure

Let $\{\Omega_j : j = 1, \cdots, J\}$ be a non-overlapping decomposition of Ω such that

$$\bar{\Omega} = \cup_{j=1}^J \bar{\Omega}_j,\ \Omega_j \cap \Omega_k = \emptyset,\ j \ne k,$$

and set

$$\Gamma_j = \partial\Omega \cap \partial\Omega_j, \quad \Gamma_{jk} = \Gamma_{kj} = \partial\Omega_j \cap \partial\Omega_k.$$

Denote by $v_j := v|_{\Omega_j}$ the restriction of a function v to Ω_j for all j, and set

$$\begin{aligned} V_j = H^1(\Omega_j)\ \forall j; \quad V = \{v \big|\ v|_{\Omega_j} \in V_j,\ \forall j\}; \\ \Lambda = \{w \ \big|\ w|_{\Gamma_{jk}} = Tr_{\Gamma_{jk}}(w_j) \in H^{-1/2}(\Gamma_{jk})\ \forall k\ \forall j\}, \end{aligned}$$

where $H^s(\Omega), H^s(\Omega_j), s \in \mathbf{R}$, are the usual complex-valued Sobolev spaces and $Tr_{\Gamma_{jk}}$ is the trace operator to Γ_{jk}.

Then the *domain decomposition iterative procedure* for solving (1) is as follows.

1. **Initialization Step.** An initial approximation $u^0 \in V$.

2. **Iterative Step.** For $n = 1, 2, \cdots$, solve iteratively the subdomain problems for $u_j^n, j = 1, \cdots, J$:

$$-\nabla \cdot (A\nabla u_j^n) + Bu_j^n = f_j \quad \text{in } \Omega_j, \tag{2}$$

$$\nu_j \cdot A\nabla u_j^n + \alpha u_j^n = g_j \quad \text{on } \Gamma_j, \tag{3}$$

with the interface conditions

$$\nu_j \cdot A\nabla u_j^n + \beta u_j^n = -\nu_k \cdot A\nabla u_k^{n-1} + \beta u_k^{n-1} \quad \text{on } \Gamma_{jk},\ \forall k, \tag{4}$$

where ν_j is the unit outward normal vector to $\partial\Omega_j$, and β is a matching parameter such that $\beta|_{\Gamma_{jk}} = \beta|_{\Gamma_{kj}}\ \forall k\, \forall j$.

The weak problem for (2) is then to find $u^n \in V$ such that

$$a_j(u_j^n, \varphi) + \sum_k \langle \beta u_j^n, \varphi \rangle_{\Gamma_{jk}} = F_j(\varphi) + \sum_k \langle -\nu_k \cdot A\nabla u_k^{n-1} + \beta u_k^{n-1}, \varphi \rangle_{\Gamma_{kj}}, \quad \varphi \in V_j, \tag{5}$$

where

$$\begin{aligned} a_j(u_j, \varphi) &:= (A\nabla u_j, \nabla\varphi)_j + \big(Bu_j, \varphi\big)_j + \langle \alpha u_j, \varphi \rangle_{\Gamma_j}, \\ F_j(\varphi) &:= (f_j, \varphi) + \langle g_j, \varphi \rangle_{\Gamma_j}, \end{aligned}$$

with $(\cdot,\cdot)_j$ and $\langle \cdot,\cdot \rangle_{\Gamma_{jk}}$ being the $L^2(\Omega_j)$ and $L^2(\Gamma_{jk})$ inner products, respectively.

For each n, denote by $\lambda^n \in \Lambda$ the oblique normal traces:

$$\lambda_{jk}^n := \nu_j \cdot A\nabla u_j^n, \quad \Gamma_{jk}\, \forall k.$$

Then the interface condition (4) can be equivalently written in the form

$$\lambda_{jk}^n + \beta u_j^n = -\lambda_{kj}^{n-1} + \beta u_k^{n-1}, \quad \Gamma_{jk}\, \forall k, \tag{6}$$

and the weak formulation (5) takes the form

$$a_j(u_j^n, \varphi) + \sum_k \langle \beta u_j^n, \varphi \rangle_{\Gamma_{jk}} = F_j(\varphi) + \sum_k \langle \beta u_k^{n-1} + \beta u_k^{n-1}, \varphi \rangle_{\Gamma_{jk}}, \quad \varphi \in V_j. \tag{7}$$

Each **Iterative Step** consists of the following two substeps:
Substep 2a. Solve the subdomain problems (7) for $u^n \in V$;
Substep 2b. Update $\lambda^n \in \Lambda$ by (6).

The updating procedure (6) may be regarded as a *Jacobi-type scheme* with which subdomain problems (7) for all j can be easily parallelizable. Can we have a *Seidel type (Gauss-Seidel or red-black Gausss-Seidel type)* scheme for the updating procedure which guarantees faster convergence than the Jacobi-type scheme? The answer is *affirmatively* given. It will be clear from Remark 1 that Gauss-Seidel schemes will be as twice fast as the corresponding Jacobi ones, and from the next section that, by exploiting the red-black procedure, Gauss-Seidel schemes will guarantee such fast convergence when implemented in parallel.

Seidel-type Domain Decomposition Iterative Method

Gauss-Seidel iteration procedure

The *Seidel-type* domain decomposition iterative procedure is obtained by replacing the interface condition (4) by

$$\nu_j \cdot A\nabla u_j^n + \beta u_j^n = \begin{cases} -\nu_k \cdot A\nabla u_k^{n-1} + \beta u_k^{n-1}, & j < k, \\ -\nu_k \cdot A\nabla u_k^n + \beta u_k^n, & j > k, \end{cases} \quad \text{on } \Gamma_{jk},\ \forall k, \tag{8}$$

and hence (7) by

$$a_j(u_j^n,\nabla\varphi)_j+\sum_k\langle\beta u_j^n,\varphi\rangle_{\Gamma_{jk}} \tag{9}$$
$$=F_j(\varphi)+\begin{cases}\sum_k\langle-\lambda_{kj}^{n-1}+\beta u_k^{n-1},\varphi\rangle_{\Gamma_{jk}}, & j<k,\\ \sum_k\langle-\lambda_{kj}^{n}+\beta u_k^{n},\varphi\rangle_{\Gamma_{jk}}, & j>k.\end{cases}$$

Let $\tilde u_j=u|_{\Omega_j}$ and $\tilde\lambda_{jk}=-\nu_j\cdot A\nabla\tilde u_j|_{\Gamma_{jk}}$ so that $\tilde u_j$ and $\tilde\lambda_{jk}$ satisfy the local equations

$$\begin{aligned}a_j(\nabla\tilde u_j,\nabla\varphi)_j-\sum_k\langle\tilde\lambda_{jk},\varphi\rangle_{\Gamma_{jk}} &= F_j(\varphi),\ \varphi\in V_j,\\ \tilde\lambda_{jk} &= -\tilde\lambda_{kj}-\beta(\tilde u_j-\tilde u_k),\ \Gamma_{jk}\,\forall k.\end{aligned}$$

We will show the convergence of (u_j^n,λ_{jk}^n) to $(\tilde u_j,\tilde\lambda_{jk})$. Set

$$e_j^n=u_j^n-\tilde u_j,\qquad \mu_{jk}^n=\lambda_{jk}^n-\tilde\lambda_{jk}.$$

From (9) and (10), we have the error equations: for all j,

$$a_j(\nabla e_j^n,\nabla\varphi)-\sum_k\langle\mu_{jk}^n,\varphi\rangle_{\Gamma_{jk}}=0,\ \varphi\in V_j, \tag{10}$$

$$\mu_{jk}^n=\begin{cases}-\mu_{kj}^{n-1}-\beta(e_j^n-e_k^{n-1}), & j<k,\\ -\mu_{kj}^{n}-\beta(e_j^n-e_k^{n}), & j>k,\end{cases}\quad \text{on } \Gamma_{jk},\ \forall k. \tag{11}$$

The choice $v=e_j^n$ in (10) gives

$$a_j(\nabla e_j^n,\nabla e_j^n)-\sum_k\langle\mu_{jk}^n,e_j^n\rangle_{\Gamma_{jk}}=0. \tag{12}$$

We rewrite (11) as follows:

$$\begin{aligned}\mu_{jk}^n &= -\mu_{kj}^{n-1}-\beta(e_j^n-e_k^{n-1}), && j<k,\\ \mu_{jk}^n &= -\mu_{kj}^{n}-\beta(e_j^n-e_k^{n}), && j>k,\\ &= \mu_{jk}^{n-1}+\beta(e_k^n-e_j^{n-1})-\beta(e_j^n-e_k^n)\\ &= \mu_{jk}^{n-1}-\beta e_j^n+2\beta e_k^n-\beta e_j^{n-1}.\end{aligned} \tag{13}$$

This motivates us to define the *pseudo-energy* for the Seidel-type iterative procedure by

$$R^n:=R(e^n,\mu^n)=\sum_{j<k}\left|\mu_{jk}^n+\beta e_j^n\right|_{0,\Gamma_{jk}}+\sum_{j>k}\left|\mu_{jk}^n+\beta(e_j^n-2e_k^n)\right|_{0,\Gamma_{jk}}. \tag{14}$$

We observe that by (13), for $j>k$,

$$\mu_{jk}^n+\beta(e_j^n-2e_k^n)=-\mu_{kj}^n-\beta e_k^n,$$

which implies that R^n given by (14) can be equivalently put in the simpler form:

$$R^n(e,\mu)=\sum_{j<k}\left|\mu_{jk}^n+\beta e_j^n\right|_{0,\Gamma_{jk}}+\sum_{j>k}\left|\mu_{kj}^n+\beta e_k^n\right|_{0,\Gamma_{jk}}. \tag{15}$$

Theorem 1 *For a given $(u^0, \lambda^0) \in V \times \Lambda$, if iterative solutions $(u^n, \lambda^n) \in V \times \Lambda$ are computed by using (9), the pseudo-energy given by (15) satisfies*

$$R^n(e,\mu) = R^{n-1}(e,\mu) - 8Re\sum_{j,k}\langle \mu_{jk}^{n-1}, \beta e_j^{n-1}\rangle_{\Gamma_{jk}}.$$

Proof. Stating from (15), by suitable swapping of the indices j and k, we have

$$\begin{aligned}
R^n &= 2\sum_{j<k}\left|\mu_{jk}^n + \beta e_j^n\right|_{0,\Gamma_{jk}} \\
&= 2\sum_{j<k}\left|\mu_{kj}^{n-1} - \beta e_k^{n-1}\right|_{0,\Gamma_{jk}} && \text{by (13)} \\
&= 2\sum_{j>k}\left|\mu_{jk}^{n-1} - \beta e_j^{n-1}\right|_{0,\Gamma_{jk}} \\
&= 2\sum_{j>k}\left|-\mu_{kj}^{n-1} + \beta(e_k^{n-1} - 2e_j^{n-1})\right|_{0,\Gamma_{jk}} && \text{by (13)} \\
&= 2\sum_{j<k}\left|\mu_{jk}^{n-1} - \beta(e_j^{n-1} - 2e_k^{n-1})\right|_{0,\Gamma_{jk}} \\
&= 2\sum_{j<k}\left|\mu_{jk}^{n-1} + \beta e_j^{n-1} - 2(e_j^{n-1} - \beta e_k^{n-1})\right|_{0,\Gamma_{jk}} \\
&= R^{n-1} - 8\mathrm{Re}\sum_{j<k}\left\langle \mu_{jk}^{n-1} + \beta e_j^{n-1}, \beta(e_j^{n-1} - e_k^{n-1})\right\rangle_{\Gamma_{jk}} \\
&\qquad + 8\sum_{j<k}\left|\beta(e_j^{n-1} - e_k^{n-1})\right|_{0,\Gamma_{jk}} \\
&= R^{n-1} - 8\mathrm{Re}\sum_{j<k}\left\langle \mu_{jk}^{n-1} + \beta e_k^{n-1}, \beta(e_j^{n-1} - e_k^{n-1})\right\rangle_{\Gamma_{jk}} \\
&= R^{n-1} - 8\mathrm{Re}\sum_{j,k}\left\langle \mu_{jk}^{n-1}, \beta e_j^{n-1}\right\rangle_{\Gamma_{jk}}
\end{aligned}$$

since

$$\begin{aligned}
&\mathrm{Re}\left[\sum_{j<k}\left\langle -\beta e_k^{n-1}, \beta(e_j^{n-1} - e_k^{n-1})\right\rangle_{\Gamma_{jk}} + \sum_{j<k}\left\langle \mu_{jk}^{n-1}, \beta e_k^{n-1}\right\rangle_{\Gamma_{jk}}\right] \\
&= \mathrm{Re}\sum_{j<k}\left\langle \beta e_k^{n-1}, -\beta(e_j^{n-1} - e_k^{n-1}) + \mu_{jk}^{n-1}\right\rangle_{\Gamma_{jk}} \\
&= \mathrm{Re}\sum_{j>k}\left\langle \beta e_j^{n-1}, \beta(e_j^{n-1} - e_k^{n-1}) + \mu_{kj}^{n-1}\right\rangle_{\Gamma_{jk}} \\
&= -\mathrm{Re}\sum_{j>k}\left\langle \beta e_j^{n-1}, \mu_{jk}^{n-1}\right\rangle_{\Gamma_{jk}} && \text{by (13)} \\
&= -\mathrm{Re}\sum_{j>k}\left\langle \mu_{jk}^{n-1}, \beta e_j^{n-1}\right\rangle_{\Gamma_{jk}}.
\end{aligned}$$

Remark 1 *The reader should observe that the form of pseudo-energy defined in (14) or (15) and both Theorem 1 and its proof are entirely independent of the sesquilinear form $a(\cdot,\cdot)$, and hence Theorem 1 is independent of governing model problem. (Our result depends only on the interface condition (8).) An implication of this observation is that Theorem 1 is valid for a wide range of problems, such as Maxwell and elasticity problems, obviously extending our model problem introduced in the previous section.*

Theorem 2 *The energy R^n can be expressed as*

$$R^n(e,\mu) = R^0(e,\mu) - 8\beta \sum_{k=1}^{n-1}\sum_{j=1}^{J} Re a_j(e_j^k, \beta e_j^k)_j.$$

Now, take the real part in (12) to obtain

$$\mathrm{Re}\sum_k \langle \mu_{jk}^n, e_j\rangle_{\Gamma_{jk}} = \mathrm{Re}a_j(\nabla e_j^n, \nabla e_j^n).$$

and choose $\beta = \beta_R + i\beta_I$ with positive real and nonnegative imaginary parts. Then, under additional assumptions on B_I and α_I such that $B_I \geq 0$ and $\alpha_I \geq 0$, which are indeed physically valid, we have

$$\begin{aligned}\mathrm{Re}a_j(e_j^n, \beta e_j^n)_j &= \beta_R\Big[(A\nabla e_j^n, \nabla e_j^n)_j + (B_R e_j^n, e_j^n)_j + \langle \alpha_R e_j^n, e_j^n\rangle_\Gamma\Big] \\ &\quad + \beta_I\Big[(B_I e_j^n, e_j^n)_j + \langle \alpha_I e_j^n, e_j^n\rangle_{\Gamma_j}\Big] > 0.\end{aligned}$$

In this case, we can conclude from (2) that e_j^n tends to zero as $n \to \infty$.

Remark 2 *For the Jacobi case with the same form of energy as in (15), it is well-known after Després [Des91] that the corresponding decay relations to Theorems 1 and 2 have the form*

$$R^n(e,\mu) = R^{n-1}(e,\mu) - 4Re\sum_{j,k}\langle \mu_{jk}^{n-1}, \beta e_j^{n-1}\rangle_{\Gamma_{jk}},$$

and

$$R^n(e,\mu) = R^0(e,\mu) - 4\beta\sum_{k=1}^{n-1}\sum_{j=1}^{J} Re a_j(e_j^k, \beta e_j^k)_j.$$

Therefore we conclude that the Seidel scheme is exactly as twice fast as the Jacobi scheme.

Red-black Gauss-Seidel procedure

Jacobi-type iterative algorithms are easily parallelizable, but Seidel-type are not easily parallelizable. In order to parallelize the introduced Seidel scheme, we propose a *red-black Seidel scheme* with efficiency equivalent to the Seidel-type one. For this, divide the subdomain indices into the two parts J_R and J_B, so that

$$\bar\Omega = \left[\cup_{j\in J_R}\bar\Omega_j\right] \bigcup \left[\cup_{j\in J_B}\bar\Omega_j\right], \quad \Omega_j \cap_{j\neq k} \Omega_k = \emptyset,$$

and every element $\Omega_j, j \in J_R$, is not adjacent to any element $\Omega_k, k \in J_B$.

With an initialization, the red-black iteration scheme is then the altenations of the following steps

1. $\forall j \in J_R$, solve (7) for $u^n \in V$ with

$$\lambda_{jk}^n = -\lambda_{kj}^{n-1} + \beta\big(u_j^n(\xi_{jk}) - u_k^{n-1}(\xi_{jk})\big)$$

2. $\forall j \in J_B$, solve (7) for $u^n \in V$ with

$$\lambda_{jk}^n = -\lambda_{kj}^n + \beta\big(u_j^n(\xi_{jk}) - u_k^n(\xi_{jk})\big).$$

The pseudo-energy for the red-black Seidel-type iterative procedure takes the similar form as (14) or (15) for errors

$$\begin{aligned} R^n &:= R(e^n, \mu^n) = \sum_{j\in J_R} \big|\mu_{jk}^n + \beta e_j^n\big|_{0,\Gamma_{jk}} + \sum_{j\in J_B} \big|\mu_{jk}^n + \beta(e_j^n - 2\beta e_k^n)\big|_{0,\Gamma_{jk}} \\ &= \sum_{j\in J_R} \big|\mu_{jk}^n + \beta e_j^n\big|_{0,\Gamma_{jk}} + \sum_{k\in J_B} \big|\mu_{kj}^n + \beta e_k^n\big|_{0,\Gamma_{jk}} \end{aligned}$$

The same arguments as the *Gauss-Seidel case* lead the analogous results as Theorems 1 and 2, and Remark 1.

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