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44. FETI-DP: An Efficient, Scalable and Unified Dual-Primal FETI Method

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Introduction

The FETI algorithms are numerically scalable iterative domain decomposition methods. These methods are well documented for solving equations arising from the Finite Element discretization of second or fourth order elasticity problems. The one level FETI method equipped with the Dirichlet preconditioner was shown to be numerically scalable for second order elasticity problems while the two level FETI method was designed to be numerically scalable for fourth order elasticity problems (see [FR94, Far91b, Far91a, FR91, FR92, FM98, Rou95]).

The second level coarse grid is an enriched version of the original one level FETI method with coarse grid. The coarse problem is enriched by enforcing transverse displacements to be continuous at the corner points. This coarse problem grows linearly with the number of subdomains. Current implementations use a direct solution method to solve this coarse problem. However, the current implementation gives rise to a full matrix system. This full matrix can lead to increased storage requirements especially if working within a distributed memory environment. Also, the factorization and subsequent forward/backward substitutions of the second level coarse problem becomes the dominant factor in solving the global problem as the number of subdomains becomes large ($N_s > 1000$).

We introduce an alternative formulation of the two level coarse problem that leads to a sparse system better suited for a direct method. Then we show extensions to the alternate formulation that allow optional admissible constraints to be added to improve convergence. Lastly, we report on the numerical performance, parallel efficiency, memory requirements, and overall CPU time as compared to the classical two level FETI on some large scale fourth order elasticity problems.

The Dual-Primal FETI Method

Let Ω be partitioned into a set of N_s, non-overlapping subdomains (or substructures) Ω^s. Points where 3 or more subdomains intersect, are labeled as corner points which will remain primal variables. The mechanical interpretation of this particular method of mesh splitting can be viewed as making incisions into the mesh but leaving the corner points attached. This is analogous to the "tearing" stage of FETI. The "interconnecting" stage occurs only on the subdomain interfaces which now excludes the corner points. Typically, in fourth order elasticity problems, the corner points have 6 degrees of freedom (3 translations and 3 rotations). This method of mesh splitting

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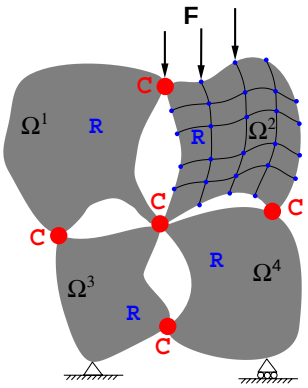

Figure 1: Dual-Primal Mesh Partitions

and corner point identification is illustrated in Figure 1: By splitting, u^s into two sub-vectors such that:

$$u = \begin{bmatrix} u_r \\ u_c \end{bmatrix} = \begin{bmatrix} u_r^1 \\ \vdots \\ u_r^{N_s} \\ u_c \end{bmatrix} \tag{1}$$

where u_r^s is the remaining subdomain solution vector and u_c is a global/primal solution vector over all defined corner degrees of freedom. The solution at the corner points is continuous by definition when the solution vector is constructed in this manner. Using this notation, we can split the subdomain stiffness matrix into:

$$K^s = \begin{bmatrix} K_{rr}^s & K_{rc}^s \\ K_{rc}^{s\,T} & K_{cc}^s \end{bmatrix} \tag{2}$$

Then the original FETI equilibrium equations can be modified using the following matrix partitioning where the subscripts c and r denote the corner and the remainder degrees of freedom.

$$\begin{bmatrix} K_{rr}^1 & \dots & 0 & K_{rc}^1 B_c^1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & K_{rr}^{N_s} & K_{rc}^{N_s} B_c^{N_s} \\ B_c^{1\,T} K_{rc}^{1\,T} & \dots & B_c^{N_s\,T} K_{rc}^{N_s\,T} & \sum_{s=1}^{N_s} B_c^{s\,T} K_{cc}^s B_c^s \end{bmatrix} \begin{bmatrix} u_r^1 \\ \vdots \\ u_r^{N_s} \\ u_c \end{bmatrix} = \begin{bmatrix} f_r^1 - B_r^{1\,T} \lambda \\ \vdots \\ f_r^{N_s} - B_r^{N_s\,T} \lambda \\ \sum_{s=1}^{N_s} B_c^{s\,T} f_c^s \end{bmatrix} \tag{3}$$

$$\sum_{s=1}^{N_s} B_r^s u_r^s = 0 \tag{4}$$

Where the corner stiffness matrix, $K_{cc} = \sum_{s=1}^{N_s} {B_c^s}^T K_{cc}^s B_c^s$ is a global stiffness quantity, B_c^s maps the local corner equation numbering to global corner equation numbering, f_r^s is the external force applied on the r degrees of freedom, ${B_r^s}^T$ is a boolean matrix that extracts the interface of a subdomain, and λ are the Lagrange multipliers. Let K_{rr} denote the block diagonal subdomain stiffness matrix restricted to the remaining, r, points, K_{rc} the block column vector of subdomain coupling stiffness matrices, f_r the block column vector of subdomain force vectors, K_{cc} the global corner stiffness matrix and using the "rc" notation, we can rewrite the equilibrium equations in the more compact form:

$$\begin{bmatrix} K_{rr} & K_{rc} \\ K_{rc}^T & K_{cc} \end{bmatrix} \begin{bmatrix} u_r \\ u_c \end{bmatrix} = \begin{bmatrix} f_r - B_r^T \lambda \\ f_c \end{bmatrix} \tag{5}$$

Now we can invert the first equation for u_r noting that K_{rr} is a symmetric positive definite matrix due to the guarantee of enough corner points that remove all singularities. This is in contrast to all the previous FETI methods where the correct computation of the the null spaces was required to be accurately computed, leading to a natural coarse problem. This null space computation was seen as a liability when working with nonlinear structures where the size of the null space would vary from one tangent stiffness matrix to the next ([FPL00]). Then substitute the result into the compatibility equation (Eq. 4). With some algebraic manipulation we can derive the Dual-Primal FETI interface problem where the unknowns are λ, the Lagrange multipliers and u_c, the global corner degrees of freedom.

$$\begin{bmatrix} F_{rr} & F_{rc} \\ F_{rc}^T & -K_{cc}^* \end{bmatrix} \begin{bmatrix} \lambda \\ u_c \end{bmatrix} = \begin{bmatrix} d_r \\ -f_c^* \end{bmatrix} \tag{6}$$

where $F_{rr} = \sum_{s=1}^{N_s} B_r^s {K_{rr}^s}^{-1} {B_r^s}^T$, $F_{rc} = \sum_{s=1}^{N_s} B_r^s {K_{rr}^s}^{-1} K_{rc}^s B_c^s$, $d_r = \sum_{s=1}^{N_s} B_r^s {K_{rr}^s}^{-1} f_r^s$, and $f_c^* = \sum_{s=1}^{N_s} ({B_c^s}^T f_c^s - {B_c^s}^T {K_{rc}^s}^T {K_{rr}^s}^{-1} f_r^s)$. The corner degrees of freedom, u_c, are condensed out to form the following symmetric positive definite Dual-Primal FETI interface problem which we solve using a preconditioned conjugate gradient method. For a detailed derivation of this equation, please see [CFR00].

$$\left[F_{rr} + F_{rc} {K_{cc}^*}^{-1} F_{rc}^T\right] \lambda = d_r - F_{rc} {K_{cc}^*}^{-1} f_c^* \tag{7}$$

It can be seen that the new FETI operator has a coarse grid problem for which the stiffness matrix can be written as follows

$$K_{cc}^* = \sum_{s=1}^{N_s} \left[{B_c^s}^T K_{cc}^s B_c^s - (K_{rc}^s B_c^s)^T {K_{rr}^s}^{-1} (K_{rc}^s B_c^s)\right] \tag{8}$$

This new coarse problem has some highly beneficial properties over the existing two-level FETI coarse problem (see [FM98]). First, this new coarse problem is sparse

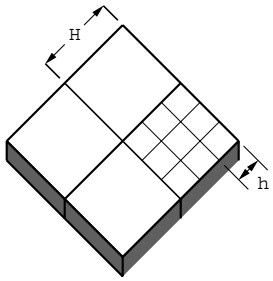

Figure 2: Model problem for numerical scalability studies

symmetric positive definite. Secondly, only one forward/backward substitution has to be performed per FETI iteration in comparison with two per iteration of the original FETI algorithms. We also note that the coarse problem is easily formed in parallel with subdomain operations. As with the original FETI coarse problem, it couples all of the subdomains and propagates the error at each FETI iteration.

Numerical Scalability

Now we would like to test the numerical scalability of the Dual-Primal FETI method for fourth order elasticity problems. The chosen tests show the numerical scalability with respect to the number of subdomains, size of the subdomains, and the size of the elements. The model problem for these tests is a 1×1 square mesh discretized into 3 node shell elements. Let h denote the size of an individual element and H denote the size of one subdomain. The first numerical test keeps the number of subdomains at 64, while varying the size of h and the effect on the number of iterations to converge to $1.0E-6$ is observed.

H	h	N_{dof}	FETI-2	FETI-DP
1/8	1/40	5,166	23 itr.	17 itr.
1/8	1/80	19,926	30 itr.	22 itr.
1/8	1/160	78,246	36 itr.	28 itr.
1/8	1/320	310,086	44 itr.	34 itr.
1/8	1/640	1,234,566	51 itr.	41 itr.

One can see that the number of iterations remains roughly constant for both the two level FETI method and for the Dual-Primal method as the size of the problem is increased from $5,166$ dof to over 1 million dof.

The second numerical test fixes the size of the problem and varies the number of subdomains used to solve the problem. Again the number of iterations remains approximately constant over a large range of N_s for both the two level FETI method and the Dual-Primal FETI method..

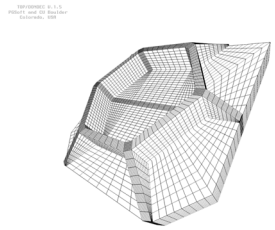

Figure 3: Finite element model of a diffraction grating

H	h	N_s	FETI-2	FETI-DP
1/8	1/640	64	51 itr.	17 itr.
1/10	1/640	100	47 itr.	22 itr.
1/16	1/640	256	47 itr.	28 itr.
1/20	1/640	400	47 itr.	34 itr.
1/40	1/640	1,600	40 itr.	41 itr.
1/64	1/640	4,096	36 itr.	28 itr.

The last numerical test holds the size of the subdomains constant while increasing the size of the overall problem. In this test, the condition number of the two level FETI method should remain roughly the same (see [FM98]). We see that both methods exhibit this trend for a large range of N_s.

H	h	N_s	FETI-2	FETI-DP
1/2	1/20	4	12 itr.	12 itr.
1/4	1/40	16	24 itr.	19 itr.
1/8	1/80	64	30 itr.	22 itr.
1/16	1/160	256	32 itr.	24 itr.
1/32	1/320	1,024	34 itr.	25 itr.
1/64	1/640	4,096	36 itr.	28 itr.

The Augmented Dual-Primal FETI Method

After testing FETI-DP on a range of fourth order problems, we decided to test a second order elasticity problem. The motivation was to see if we could improve the existing one level FETI technology. As the reader can see, the following results were not encouraging for this diffraction grating problem with 120,987 degrees of freedom.

N_s	FETI-1	FETI-DP	Augmented FETI-DP
56	81 itr. (281 sec.)	190 itr. (534 sec.)	63 itr. (284 sec.)
128	51 itr. (115 sec.)	115 itr. (273 sec.)	38 itr. (129 sec.)

The initial thought was to investigate how the new Dual-Primal coarse problem could be extended to improve convergence. This can be accomplished by forcing the residual to be orthogonal to a chosen set of vectors at each iteration of the FETI algorithm.

Let Q be a matrix of arbitrarily chosen vectors, r the residual, then we can enforce the following equation to enhance convergence:

$$Q^T r = Q^T \sum_{s=1}^{N_s} B_r^s u_r^s = Q_r^T u_r = 0 \tag{9}$$

We insert these equations within the formulation by introducing new Lagrange multipliers, μ, to enforce the constraints associated with Eq. 9.

$$\begin{bmatrix} K_{rr} & K_{rc} & Q_r \\ K_{rc}^T & K_{cc} & 0 \\ Q_r^T & 0 & 0 \end{bmatrix} \begin{bmatrix} u_r \\ u_c \\ \mu \end{bmatrix} = \begin{bmatrix} f_r - B_r^T \lambda \\ f_c \\ 0 \end{bmatrix} \tag{10}$$

The resulting FETI operator has the same form as given in 7. Following the same procedure used to derive Eq. 6, we arrive at the following expression for the augmented Dual-Primal FETI coarse grid which is non-singular for a well-posed non-floating structure but because of the μ Lagrange multiplier, we have negative eigen values.

$$\tilde{K}_{cc}^* = \sum_{s=1}^{N_s} \begin{bmatrix} B_c^{s^T} K_{cc}^s B_c^s - B_c^{s^T} K_{rc}^{s^T} K_{rr}^{s^{-1}} K_{rc}^s B_c^s & -B_c^{s^T} K_{rc}^{s^T} K_{rr}^{s^{-1}} Q_r^s \\ -Q_r^{s^T} K_{rr}^{s^{-1}} K_{rc}^s B_c^s & -Q_r^{s^T} K_{rr}^{s^{-1}} Q_r^s \end{bmatrix} \tag{11}$$

These Q matrices can be chosen to be the average x,y, or z jump along a subdomain edge resulting in an edge by edge sparsity pattern for the augmented set of equations.. There has been a clear advantage to writing the equations on a per edge basis as it has improved convergence dramatically, improved CPU times, and restored numerical scalability with respect to second order elasticity problems.

For higher order elements, such as 10 node tetrahedron, FETI-DP has shown to be much more efficient than the one level FETI method. The following results were obtained from a large-scale structural solid model discretized using 10 node tetrahedrons of a BMW engine. The entire engine model has over 1 million degrees of freedom which was decomposed into 823 subdomains and computed on an Origin 2000 machine. It took the one level FETI method 243 iterations to converge while it took 90 iterations for FETI-DP.

N_p	FETI-1	Augmented FETI-DP
3	1,476 sec.	604 sec.
6	773 sec.	334 sec.
12	461 sec.	247 sec.
24	207 sec.	140 sec.

Parallel Scalability

We conclude this paper with a large-scale example problem that highlights the advantages of FETI-DP. The following problem is a shell model of a wheel rim composed of over 313856 elements, 156017 nodes, and containing 936, 102 degrees of freedom. Three points were fixed along the inner rim, effectively constraining the model. Then a gravity load was applied to the model which was decomposed into 500 subdomains.

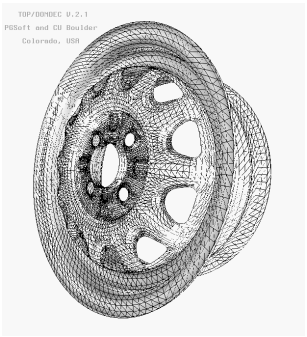

Figure 4: Finite element model of wheel rim

As one can see, the reduction in CPU time is dramatic for the FETI-DP method. The PSLDLT parallel sparse solver shows a large improvement over the two level FETI method for low numbers of processors while the FETI-DP method is faster for $Np = 1$ all the way to $Np = 24$. The speed-up numbers for the two level FETI method and the FETI-DP method are nearly identical for these runs on an Origin 2000.

N_p	FETI-2	PSLDLT	FETI-DP
1	2,995 s (1.0)	1,631 s (1.0)	1,594 s (1.0)
4	789 s (3.8)	502 s (3.2)	370 s (4.3)
8	371 s (8.1)	301 s (5.4)	196 s (8.1)
16	214 s (13.9)	218 s (7.5)	116 s (13.7)
20	179 s (16.7)	200 s (8.2)	99 s (16.1)
24	157 s (19.0)	200 s (8.2)	86 s (18.5)

Conclusion

We have shown a modification to the classical FETI method where the local operators are symmetric positive definite. This eliminates the necessity for computing the local null spaces. which also removes the original FETI coarse problem. The new Dual-Primal FETI method has a global coarse problem associated with the global corner displacements. This coarse grid was shown to have as good as or better than convergence for fourth order plates and shells problems with respect to the two level FETI method. For second order problems, the new Dual-Primal FETI coarse grid has to be augmented with optional constraints to remain numerically scalable. The Dual-Primal FETI method is more robust, more efficient and typically faster than the classical FETI methods for large numbers of subdomains.

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