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5. On Schwarz Methods for Monotone Elliptic PDEs

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Introduction

The Schwarz Alternating Method was devised by H. A. Schwarz more than one hundred years ago to solve linear boundary value problems. It has garnered interest recently because of its potential as an efficient algorithm for parallel computers. See [Lio88], and [Lio89], the recent reviews [CM94], [LT94], and [XZ98], and the books [SBG96] and [QV99]. The literature for nonlinear problems is rather sparse. Besides Lions' works, see also [Bad91], [ZH92], [CD94], [Tai94], [TE98], [TX01], [Pao95], [Xu96], [DH97], [Lui00], [Lui01], and references therein. The effectiveness of Schwarz methods for nonlinear problems (especially those in fluid mechanics) has been demonstrated in many papers. See proceedings of the annual domain decomposition conferences.

This paper is a continuation of previous works by this author attempting to survey various classes of nonlinear elliptic PDEs for which Schwarz methods are applicable. We consider elliptic PDEs amenable to analysis by the monotone method (also known as the method of subsolutions and supersolutions).

The paper [KC67] was among the first to employ the monotone method to solve boundary value problems. Subsequent works by these two authors as well as by [Sat72], [Ama76], and many others have made this method into one of the important tools in nonlinear analysis. See [Pao92] for a very complete reference with many applications as well as a good bibliography. [Lio89] shows the convergence of a multiplicative Schwarz method for the Poisson's equation using the monotone method. Here, we prove convergence for an additive Schwarz method on finitely many subdomains for scalar as well as coupled systems of nonlinear elliptic PDEs. Our results on coupled systems can be applied to the three types of Lotka-Volterra models in population biology: competition, cooperation and predator-prey.

In the following section, we indicate convergence of two Schwarz methods for a class of scalar nonlinear elliptic PDEs. This is followed by a treatment of the so-called quasi-monotone non-increasing case of a coupled system of PDEs on finitely many subdomains. In the remaining part of this introduction, we set some notations.

Let Ω be a bounded, connected domain in $\mathbb{R}^N$ with a smooth boundary. Suppose Ω is composed of $m \geq 2$ subdomains, that is, $\Omega = \Omega_1 \cup \cdots \cup \Omega_m$. The boundary of each subdomain is also assumed to be smooth. Let $X = C^\alpha(\overline{\Omega}) \cap C^2(\Omega)$ for some $0 < \alpha < 1$. We shall look for solutions of PDEs lying in this space.

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Scalar Equations

Consider the PDE

$$-\triangle u = f(x,u) \text{ on } \Omega, \qquad u = h \text{ on } \partial\Omega. \tag{1}$$

A smooth function $\underline{u} \in X$ is a *subsolution* of the above PDE if

$$-\triangle \underline{u} - f(x,\underline{u}) \le 0 \text{ on } \Omega \quad \text{and} \quad \underline{u} \le h \text{ on } \partial\Omega.$$

Similarly, a *supersolution* is one which satisfies the above with both inequalities reversed.

Let us now record the assumptions for the above PDE. Suppose that it has a subsolution $\underline{u}$ and a supersolution $\overline{u}$ which satisfy $\underline{u} \le \overline{u}$ on Ω. Define the sector of smooth functions

$$\mathcal{A} \equiv \{u \in X,\ \underline{u} \le u \le \overline{u} \text{ on } \overline{\Omega}\}.$$

Assume f is a smooth (Holder continuous) function defined on $\overline{\Omega} \times \mathcal{A}$ and h is a smooth function defined on the boundary. In addition, suppose there exists some bounded non-negative function c defined on Ω so that

$$-c(x)(u-v) \le f(x,u) - f(x,v), \qquad x \in \Omega, \quad v \le u \in \mathcal{A}.$$

With these assumptions, it is known (section 3.2 in [Pao92]) that the PDE has a (not necessarily unique) solution in the sector $\mathcal{A}$.

We begin with a comparison lemma.

Lemma 1 *Suppose S is an open set. Let $w \in H^1(S) \cap C(\overline{S})$ satisfy*

$$\int_S (\nabla w \cdot \nabla \phi + cw\phi) \ge 0, \qquad \forall \text{ non-negative } \phi \in H_0^1(S) \tag{2}$$

and $w \ge 0$ on ∂S. Then $w \ge 0$ on $\overline{S}$.

We now show convergence of a (multiplicative) Schwarz sequence for the PDE (1) for the two subdomain case. For convenience, we suppress the dependence of f on $x \in \Omega$. Note that each subdomain problem is a linear one. Despite the possibility of multiple solutions, the Schwarz iteration always converges to a specific solution.

Theorem 1 *Let $u^{(0)} = u^{(-\frac{1}{2})} = \underline{u}$ on $\overline{\Omega}$ with $\underline{u} = h$ on $\partial\Omega$. Define the Schwarz sequence by ($n \ge 0$)*

$$-\triangle u^{(n+\frac{1}{2})} + cu^{(n+\frac{1}{2})} = f(u^{(n-\frac{1}{2})}) + cu^{(n-\frac{1}{2})} \text{ on } \Omega_1, \qquad u^{(n+\frac{1}{2})} = u^{(n)} \text{ on } \partial\Omega_1,$$

$$-\triangle u^{(n+1)} + cu^{(n+1)} = f(u^{(n)}) + cu^{(n)} \text{ on } \Omega_2, \qquad u^{(n+1)} = u^{(n+\frac{1}{2})} \text{ on } \partial\Omega_2.$$

Here, $u^{(n+\frac{1}{2})}$ is defined as $u^{(n)}$ on $\overline{\Omega} \setminus \overline{\Omega}_1$ and $u^{(n+1)}$ is defined as $u^{(n+\frac{1}{2})}$ on $\overline{\Omega} \setminus \overline{\Omega}_2$. Then $u^{(n+\frac{i}{2})} \to u$ in $C^2(\overline{\Omega}_i)$, $i = 1,2$, where u is a solution of (1) in $\mathcal{A}$. If v is any solution in $\mathcal{A}$, then $u \le v$ on $\overline{\Omega}$.

If $u^{(0)} = u^{(-\frac{1}{2})} = \overline{u}$ on $\overline{\Omega}$ with $\overline{u} = h$ on $\partial\Omega$ instead, then the same conclusion holds except that $u \ge v$ on $\overline{\Omega}$.

Sketch of Proof: We only consider the case $u^{(0)} = \underline{u}$ with $\underline{u} = h$ on $\partial\Omega$. The proof can be divided into four steps. First, we demonstrate that the sequence is monotone:

$$\underline{u} \leq u^{(n-\frac{1}{2})} \leq u^{(n)} \leq u^{(n+\frac{1}{2})} \leq \overline{u} \text{ on } \overline{\Omega}, \qquad n \geq 0. \tag{3}$$

Since the sequences are bounded above, the following limits are well defined on $\overline{\Omega}$

$$\lim_{n\to\infty} u^{(n+\frac{1}{2})} = u_1, \qquad \lim_{n\to\infty} u^{(n)} = u_2.$$

In the second step, we prove that the function u_i satisfies the same PDE on Ω_i using an elliptic regularity argument (see p. 102 in [Pao92]). We can also infer that the convergence to u_i is in the sense of $C^2(\overline{\Omega}_i)$. In the third step, we prove that $u_1 = u_2$ on $\overline{\Omega}$ which follows directly from (3). Define $u = u_1$. Then u is a solution of (1). Finally, if v is any other solution in $\mathcal{A}$, replace $\overline{u}$ by v in the above steps to obtain $u \leq v$ on $\overline{\Omega}$. This completes the sketch of the proof.

The above Schwarz iteration is an adaptation of the classical Schwarz iteration for the Poisson's equation. The next Schwarz method is called an additive Schwarz method. It generalizes the additive method for linear PDEs first introduced in [DW87]. It is sometimes preferable to the (multiplicative) Schwarz method above because the subdomain PDEs are independent and hence can be solved in parallel. We consider the general m-subdomain case.

Theorem 2 *Let $u^{(0)} = u_i^{(0)} = \underline{u}$ on $\overline{\Omega}$, $i = 1, \cdots, m$ with $\underline{u} = h$ on $\partial\Omega$. Define the additive Schwarz sequence by ($n \geq 1$)*

$$-\triangle u_i^{(n)} + cu_i^{(n)} = f(u_i^{(n-1)}) + cu_i^{(n-1)} \text{ on } \Omega_i, \qquad u_i^{(n)} = u^{(n-1)} \text{ on } \partial\Omega_i, \quad i = 1, \cdots, m.$$

Here, $u_i^{(n)}$ is defined as $u^{(n-1)}$ on $\overline{\Omega} \setminus \overline{\Omega}_i$ and

$$u^{(n)}(x) = \max_{1\leq i\leq m} u_i^{(n)}(x), \qquad x \in \overline{\Omega}.$$

Then $u_i^{(n)} \to u$ in $C^2(\Omega_i)$, $i = 1, \cdots, m$ where u is a solution of (1) in $\mathcal{A}$. If v is any solution in $\mathcal{A}$, then $u \leq v$ on $\overline{\Omega}$.

If $u^{(0)} = u_i^{(0)} = \overline{u}$ on $\overline{\Omega}$ with $\overline{u} = h$ on $\partial\Omega$ instead, then the same conclusion holds except that $u \geq v$ on $\overline{\Omega}$.

Sketch of Proof: The details of this proof are quite similar to those of the last proof. Assume $u^{(0)} = \underline{u}$. The following monotone properties hold:

$$\underline{u} \leq u_i^{(n)} \leq u_i^{(n+1)} \leq \overline{u} \text{ on } \overline{\Omega}_i, \qquad \underline{u} \leq u^{(n)} \leq u^{(n+1)} \leq \overline{u} \text{ on } \overline{\Omega}, \tag{4}$$

$$u^{(n)} \leq u_i^{(n+1)} \text{ on } \overline{\Omega}, \qquad i = 1, \cdots, m. \tag{5}$$

The inequalities in (4) can be shown in a straightforward manner by induction using the maximum principle. To show the second set of inequalities in (4), take a fixed n and $x \in \Omega$. Then there is some integer i in between 1 and m inclusive so that $u^{(n)}(x) = u_i^{(n)}(x) \leq u_i^{(n+1)}(x) \leq u^{(n+1)}(x)$.

The inequality (5) can also be shown by induction. This can be done using the following (nontrivial) inequality

$$\int_{\Omega_i} (\nabla u^{(n)} \cdot \nabla \phi + c u^{(n)} \phi) \leq \int_{\Omega_i} \Big(f(u^{(n-1)}) + c u^{(n-1)} \Big) \phi, \qquad \forall \text{ non-negative } \phi \in H_0^1(\Omega_i).$$

which says that $u^{(n)}$ is a subsolution in some weak sense.

Next, we define on $\overline{\Omega}$, for $i = 1, \cdots, m$,

$$\lim_{n \to \infty} u_i^{(n)} = u_i, \qquad \lim_{n \to \infty} u^{(n)} = u_0$$

and show using elliptic regularity theory that the limit u_i satisfies the same PDE on $\Omega_i,\ i = 1, \cdots, m$ and that the convergence to u_i is in the sense of $C^2(\Omega_i)$. We have $u_i \leq u_0$ on $\overline{\Omega},\ i = 1, \cdots, m$. By (5), we have for any $j,\ u_0 \leq u_j \leq u_0 \leq u_i$. From these inequalities, we conclude that $u_i = u_j = u_0,\ 1 \leq i, j \leq m$. Define u to be this common function which must be a solution of (1) in $\mathcal{A}$. The proof of $u \leq v$ for any solution of (1) in $\mathcal{A}$ is the same as before.

Quasi-monotone Non-increasing Coupled Systems

Consider the system

$$-\triangle u = f(u, v), \qquad -\triangle v = g(u, v) \qquad \text{on } \Omega, \tag{6}$$

$$u = r, \qquad v = s \qquad \text{on } \partial\Omega.$$

The pairs of smooth functions $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ are called *subsolution and supersolution pairs* if they satisfy

$$-\triangle \underline{u} - f(\underline{u}, \overline{v}) \leq 0 \leq -\triangle \overline{u} - f(\overline{u}, \underline{v}) \text{ on } \Omega,$$

$$-\triangle \underline{v} - g(\overline{u}, \underline{v}) \leq 0 \leq -\triangle \overline{v} - g(\underline{u}, \overline{v}) \text{ on } \Omega, \text{ and}$$

$$\underline{u} \leq r \leq \overline{u}, \qquad \underline{v} \leq s \leq \overline{v} \qquad \text{on } \partial\Omega.$$

Furthermore, they are said to be *ordered* if

$$\underline{u} \leq \overline{u}, \qquad \underline{v} \leq \overline{v} \qquad \text{on } \overline{\Omega}.$$

Define the sector

$$\mathcal{A} \equiv \left\{ \begin{bmatrix} u \\ v \end{bmatrix},\ u, v \in X,\ \underline{u} \leq u \leq \overline{u},\ \underline{v} \leq v \leq \overline{v} \text{ on } \overline{\Omega} \right\}.$$

Suppose $f, g \in C^1(\mathcal{A})$. Our system of PDEs is called *quasi-monotone non-increasing* if

$$\frac{\partial f}{\partial v}, \frac{\partial g}{\partial u} \leq 0 \text{ on } \mathcal{A}. \tag{7}$$

Suppose our system of PDEs is quasi-monotone non-increasing. Then it can be shown (section 8.4 in [Pao92]) that it has a solution (u, v) in $\mathcal{A}$. Without further assumptions, it may have more than one solution. Despite this, the following additive Schwarz sequence converges for an appropriately chosen initial guess. Note that the subdomain problems at each iteration are independent and are decoupled.

Theorem 3 *Suppose the system (6) is quasi-monotone non-increasing and let* $(\underline{u}, \underline{v})$ *and* $(\overline{u}, \overline{v})$ *be ordered subsolution and supersolution pairs. Consider any non-negative functions* $c, d \in C^\alpha(\overline{\Omega})$ *so that*

$$\frac{\partial f(u,v)}{\partial u} \geq -c, \qquad \frac{\partial g(u,v)}{\partial v} \geq -d, \qquad (u,v) \in \mathcal{A}. \tag{8}$$

For $i = 1, \cdots, m$, *let*

$$u^{(0)} = u_i^{(0)} = \underline{u} \text{ and } v^{(0)} = v_i^{(0)} = \overline{v} \text{ on } \overline{\Omega} \text{ with } \underline{u} = r \text{ and } \overline{v} = s \text{ on } \partial\Omega. \tag{9}$$

Define the Schwarz sequence for $i = 1, \cdots, m$ *and* $n \geq 1$

$$-\triangle u_i^{(n)} + c u_i^{(n)} = f(u_i^{(n-1)}, v_i^{(n-1)}) + c u_i^{(n-1)} \text{ on } \Omega_i, \qquad u_i^{(n)} = u^{(n-1)} \text{ on } \partial\Omega_i$$

$$-\triangle v_i^{(n)} + d v_i^{(n)} = g(u_i^{(n-1)}, v_i^{(n-1)}) + d v_i^{(n-1)} \text{ on } \Omega_i, \qquad v_i^{(n)} = v^{(n-1)} \text{ on } \partial\Omega_i.$$

Here, $u_i^{(n)}$ *and* $v_i^{(n)}$ *are defined as* $u^{(n-1)}$ *and* $v^{(n-1)}$, *respectively, on* $\overline{\Omega} \setminus \overline{\Omega}_i$ *while*

$$u^{(n)}(x) = \max_{1 \leq i \leq m} u_i^{(n)}(x), \quad v^{(n)}(x) = \min_{1 \leq i \leq m} v_i^{(n)}(x) \qquad \text{on } \overline{\Omega}.$$

Then $u_i^{(n)} \to \underline{u}_0$ *and* $v_i^{(n)} \to \overline{v}_0$ *in* $C^2(\Omega_i)$, $i = 1, \cdots, m$, *where* $(\underline{u}_0, \overline{v}_0)$ *is a solution of (6) in* $\mathcal{A}$. *If* (u, v) *is any solution in* $\mathcal{A}$, *then* $\underline{u}_0 \leq u$ *and* $v \leq \overline{v}_0$.

If $u^{(0)} = u_i^{(0)} = \overline{u}$ *and* $v^{(0)} = v_i^{(0)} = \underline{v}$ *on* $\overline{\Omega}$ *with* $\overline{u} = r$ *and* $\underline{v} = s$ *on* $\partial\Omega$ *replace the assumption (9), then the above Schwarz sequence satisfies* $u_i^{(n)} \to \overline{u}_0$ *and* $v_i^{(n)} \to \underline{v}_0$ *in* $C^2(\Omega_i)$, $i = 1, \cdots, m$, *where* $(\overline{u}_0, \underline{v}_0)$ *is also a solution of (6) in* $\mathcal{A}$. *If* (u, v) *is any solution in* $\mathcal{A}$, *then* $u \leq \overline{u}_0$ *and* $v \geq \underline{v}_0$.

Sketch of Proof: We only consider the case where $u^{(0)} = \underline{u}$ and $v^{(0)} = \overline{v}$. The proof can be divided into four steps. We first show that the following monotone properties hold on $\overline{\Omega}$ for $i = 1, \cdots, m$,

$$\underline{u} \leq u_i^{(n)} \leq u_i^{(n+1)} \leq \overline{u}, \qquad u^{(n)} \leq u^{(n+1)}, \qquad u^{(n)} \leq u_i^{(n+1)} \tag{10}$$

and

$$\underline{v} \leq v_i^{(n+1)} \leq v_i^{(n)} \leq \overline{v}, \qquad v^{(n+1)} \leq v^{(n)}, \qquad v_i^{(n+1)} \leq v^{(n)}. \tag{11}$$

Since the sequences are bounded, the following limits on $\overline{\Omega}$ are well defined

$$\lim_{n\to\infty} u_i^{(n)} = \underline{u}_i, \qquad \lim_{n\to\infty} v_i^{(n)} = \overline{v}_i \qquad i = 1, \cdots, m,$$

and

$$\lim_{n\to\infty} u^{(n)} = \underline{u}_0, \qquad \lim_{n\to\infty} v^{(n)} = \overline{v}_0.$$

In the second step, we prove, using a similar elliptic regularity argument as before, that the limit functions satisfy the following PDEs on Ω_i:

$$-\triangle \underline{u}_i = f(\underline{u}_i, \overline{v}_i), \qquad -\triangle \overline{v}_i = g(\underline{u}_i, \overline{v}_i), \qquad i = 1, \cdots, m, \tag{12}$$

and that convergence to $\underline{u}_i$ and to $\overline{v}_i$ is in the sense of $C^2(\Omega_i)$. Third, we demonstrate that the functions $\underline{u}_i$ are identical. This follows because from (10) and the definition of $u^{(n)}$,

$$u_i^{(n)} \le u^{(n)} \le u_j^{(n+1)} \le u^{(n+1)} \le u_i^{(n+2)}, \qquad 1 \le i, j \le m.$$

Take the limit to obtain $\underline{u}_i = \underline{u}_j = \underline{u}_0$ on $\overline{\Omega}$. Similarly, we use (11) to show $\overline{v}_i = \overline{v}_j = \overline{v}_0$ on $\overline{\Omega}$ for $1 \le i, j \le m$. From (12), it follows that $(\underline{u}_0, \overline{v}_0)$ is a solution of (6).

Fourth, we prove that any solution (u, v) of (6) in $\mathcal{A}$ must satisfy

$$\underline{u}_0 \le u \text{ and } v \le \overline{v}_0 \text{ on } \Omega_i. \tag{13}$$

This follows from the observation that $(\underline{u}, v)$ and $(u, \overline{v})$ form subsolution and supersolution pairs. Apply the above result to establish (13).

One example where a quasi-monotone non-increasing system occurs is the Lotka-Volterra competition model

$$-\triangle u = u(a_1 - b_1 u - c_1 v), \qquad -\triangle v = v(a_2 - b_2 u - c_2 v).$$

Here u, v stand for the population of two species competing for the same food sources and/or territories and all other variables are positive constants.

Similarly, it can be shown that the additive Schwarz method converges for other types of coupled systems. For instance, a quasi-monotone non-decreasing system is one where $f_v, g_u \ge 0$ on $\mathcal{A}$ in place of (7). (The definition of subsolution and supersolution pairs is slightly different though.) One example where a quasi-monotone non-decreasing system occurs is the Lotka-Volterra cooperating model

$$-\triangle u = u(a_1 - b_1 u + c_1 v), \qquad -\triangle v = v(a_2 + b_2 u - c_2 v).$$

Here u, v stand for the population of two species which have a symbiotic relationship and all other variables are positive constants.

A third class of coupled systems, known as mixed quasi-monotone is one where $f_v, -g_u \le 0$ on $\mathcal{A}$ in place of (7). Using essentially the same technique, one can show that the additive Schwarz method also works for this class of problems as well. One example where a mixed quasi-monotone system occurs is the Lotka-Volterra predator-prey model

$$-\triangle u = u(a_1 - b_1 u - c_1 v), \qquad -\triangle v = v(a_2 + b_2 u - c_2 v).$$

Here u stands for the population of a prey while v denotes the population of a predator and all other variables are positive constants.

References

[Ama76]H. Amann. Fixed point equations and nonlinear eigenvalue problems in ordered banach spaces. *SIAM Rev.*, 18:620–709, 1976.

[Bad91]L. Badea. On the schwarz alternating method with more than two subdomains for nonlinear monotone problems. *SIAM J. N. A.*, 28:179–204, 1991.

[CD94]X. C. Cai and M. Dryja. Domain decomposition methods for monotone nonlinear elliptic problems. In D. Keyes and J. Xu, editors, *Domain decomposition methods in scientific and engineering computing*, pages 335–360. AMS, 1994.

[CM94]Tony F. Chan and Tarek P. Mathew. Domain decomposition algorithms. In *Acta Numerica 1994*, pages 61–143. Cambridge University Press, 1994.

[DH97]M. Dryja and W. Hackbusch. On the nonlinear domain decomposition method. *BIT*, pages 296–311, 1997.

[DW87]Maksymilian Dryja and Olof B. Widlund. An additive variant of the Schwarz alternating method for the case of many subregions. Technical Report 339, also Ultracomputer Note 131, Department of Computer Science, Courant Institute, 1987.

[KC67]H. B. Keller and D. S. Cohen. Some positone problems suggested by nonlinear heat generation. *J. Math. Mech.*, 16:1361–1376, 1967.

[Lio88]Pierre-Louis Lions. On the Schwarz alternating method. I. In Roland Glowinski, Gene H. Golub, Gérard A. Meurant, and Jacques Périaux, editors, *First International Symposium on Domain Decomposition Methods for Partial Differential Equations*, pages 1–42, Philadelphia, PA, 1988. SIAM.

[Lio89]Pierre Louis Lions. On the Schwarz alternating method. II. In Tony Chan, Roland Glowinski, Jacques Périaux, and Olof Widlund, editors, *Domain Decomposition Methods*, pages 47–70, Philadelphia, PA, 1989. SIAM.

[LT94]Patrick Le Tallec. Domain decomposition methods in computational mechanics. In J. Tinsley Oden, editor, *Computational Mechanics Advances*, volume 1 (2), pages 121–220. North-Holland, 1994.

[Lui00]S. H. Lui. On schwarz alternating methods for nonlinear elliptic pdes. *SISC*, 2000.

[Lui01]S. H. Lui. On schwarz alternating methods for the incompressible Navier-Stokes equations. *SISC*, 2001.

[Pao92]C. V. Pao. *Nonlinear parabolic and elliptic equations.* Plenum Press, 1992.

[Pao95]C. V. Pao. Block monotone iterative methods for numerical solutions of nonlinear elliptic equations. *Numer. Math.*, 72:239–262, 1995.

[QV99]A. Quarteroni and A. Valli. *Domain Decomposition Methods for Partial Differential Equations.* Oxford Science Publications, 1999.

[Sat72]D. H. Sattinger. Monotone methods in nonlinear elliptic and parabolic boundary balue problems. *Indianna Univ. Math. J.*, 21:979–1000, 1972.

[SBG96]Barry F. Smith, Petter E. Bjørstad, and William Gropp. *Domain Decomposition: Parallel Multilevel Methods for Elliptic Partial Differential Equations.* Cambridge University Press, 1996.

[Tai94]X. C. Tai. Domain decomposition for linear and nonlinear elliptic problems via function or space decomposition. In D. Keyes and J. Xu, editors, *Domain decomposition methods in scientific and engineering computing*, pages 335–360. AMS, 1994.

[TE98]X.-C. Tai and M. S. Espedal. Rate of convergence of some space decomposition method for linear and non-linear elliptic problems. *SIAM J. Numer. Anal.*, 35:1558–1570, 1998.

[TX01]X. C. Tai and J. Xu. Global convergence of subspace correction methods for convex optimization problems. *Math. Comput.*, 2001.

[Xu96]J. Xu. Two-grid discretization techniques for linear and nonlinear pdes. *SIAM J. N. A.*, 33:1759–1777, 1996.

[XZ98]J. Xu and J. Zou. Some nonoverlapping domain decomposition methods. *SIAM Rev*, 40:857–914, 1998.

[ZH92]J. Zou and H. C. Huang. Algebraic subproblem decomposition methods and parallel algorithms with monotone convergence. *J. Comput. Math.*, 10:47–59, 1992.