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19. A Mortar Finite Element Method for Plate Problems

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Introduction

In the paper we discuss two versions of mortar finite element methods applied to clamped plate problems. The problems are approximated by the nonconforming Morley and Adini element methods in each subregion into which the original region of the discussed problems have been partitioned. On the interfaces between subdomains and at crosspoints of subregions some continuity conditions are imposed.

The main results of the paper are the proof of the solvability of the discrete problems and their error bounds.

The mortar method is a domain decomposition method that allow us to use discretizations of different type with independent discretizations parameters in non-overlapping subdomains, see e.g. [BMP94], [BM97], [BB99] for a general presentation of the mortar method in the two and three dimensions for elliptic boundary value problems of second order.

In the paper mortar element methods for the locally nonconforming discretizations of the clamped plate problems are discussed. In [Lac98] there are formulated results for mortar method with nonconforming discrete Kirchoff triangle elements (DKT) for a similar problem while in [Bel97] the mortar method for the biharmonic problem is analyzed in the case of local spectral discretizations. The paper is based on the results which are obtained in the PhD thesis of the author, see [Mar99b], cf. also [Mar99a].

This paper is concerned with the mortar method where locally in the subdomains the nonconforming Adini and Morley plate finite elements are used. We restrict ourselves to the geometrically conforming version of the mortar method, i.e. the local substructures form a coarse triangulation. We first introduce independent local discretizations for the two discussed elements in each subdomain. The 2-D triangulations of two neighboring subregions do not necessarily match on their common interface, cf. Figure 1. The mortar technique for nonconforming plate elements which is discussed here requires the continuity of the solution at the vertices of subdomains and that the solution on two neighboring subdomains satisfies two mortar conditions of the L^2 type on their common interface. The form of these conditions depends on the local discretization methods and in some cases these conditions combine interpolants defined locally on interfaces. It follows from the fact that the respective traces of local functions also depend on the values of respective degrees of freedom at interior nodal points. We give error bounds for the both mortar methods. The results obtained in this paper can be generalized to analogous mortar discretizations of simply supported plate problems.

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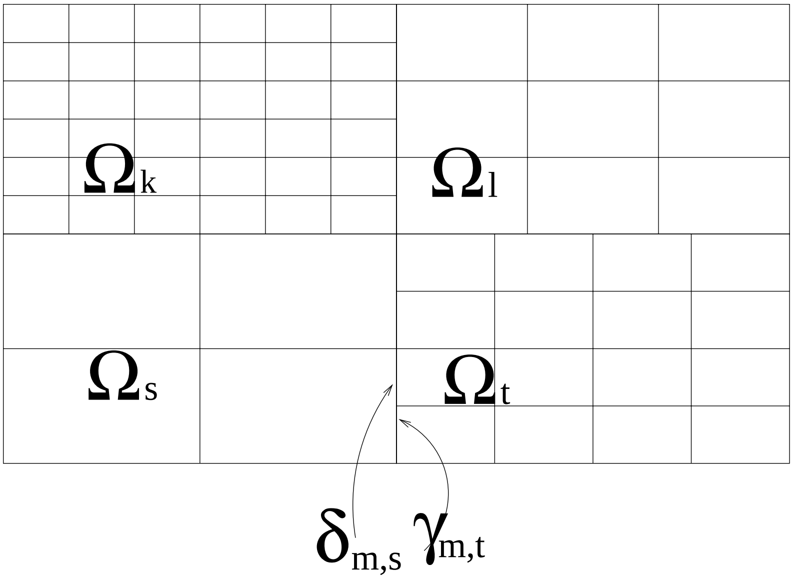

Figure 1: Nonmatching meshes.

Discrete problems

Clamped plate problem

Let Ω be a polygonal domain in $\mathbb{R}^2$. The differential problem is to find $u^* \in H_0^2(\Omega)$ such that

$$a(u^*, v) = \int_\Omega f v \, dx \quad \forall v \in H_0^2(\Omega), \tag{1}$$

where u^* is the displacement, $f \in L^2(\Omega)$ is the body force,

$$a(u, v) = \int_\Omega \left[\triangle u \triangle v + (1-\nu)\left(2u_{x_1x_2}v_{x_1x_2} - u_{x_1x_1}v_{x_2x_2} - u_{x_2x_2}v_{x_1x_1}\right)\right] dx.$$

Here

$$H_0^2(\Omega) = \{v \in H^2(\Omega) : \; v = \partial_n v = 0 \;\text{ on }\; \partial\Omega\},$$

∂_n is the normal unit derivative outward to $\partial\Omega$, and $u_{x_ix_j} := D_iD_ju$ for $i, j = 1, 2$. The Poisson ratio ν satisfies $0 < \nu < 1/2$. It is well known that this problem has a unique solution, see e.g. [Cia91].

Let Ω be a union of non-overlapping polygonal subdomains that are arbitrary for the Morley element and are rectangles for the Adini element, i.e. $\overline{\Omega} = \bigcup_{k=1}^N \overline{\Omega}_k, \;\; \Omega_k \cap \Omega_l = \emptyset, \quad k \neq l$. We assume that the intersection of boundaries of two different subdomains $\partial\Omega_k \cap \partial\Omega_l, k \neq l$, is either the empty set, a vertex or a common edge. We assume the shape regularity of that decomposition, cf. [BS94].

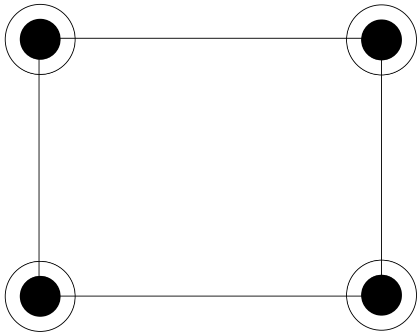

Figure 2: Adini element.

We triangulate each subdomain Ω_k into nonoverlapping rectangles for the Adini element and into triangles for the Morley one. The rectangles (or triangles) of this triangulation are denoted by τ_i and called elements. We assume that the arising fine triangulation $T_h(\Omega_k)$ is quasiuniform with parameter $h_k = \max(\text{ diam } \tau)$ for $\tau \in T_h(\Omega_k)$, cf. [BS94]. The triangulations for different Ω_k are independent and can be nonmatching across interfaces, i.e. on common edges of two subdomains, in general, cf. Figure 1.

Adini element

In this subsection, we introduce a mortar method that locally uses the Adini element, cf. Chapter 7, Section 49, p.298 in [Cia91]. The local finite element space $X_h^A(\Omega_k)$ of the Adini element is defined by

$$X_h^A(\Omega_k) = \{v \in L_2(\Omega_k): \quad v_{|\tau} \in P_3(\tau) \oplus \text{span}\{x_1^3 x_2, x_1 x_2^3\} \text{ for } \tau \in T_h(\Omega_k),$$

$$v, v_{x_1}, v_{x_2} \text{ continuous at the vertices of } \tau \text{ and}$$

$$v(a) = v_{x_1}(a) = v_{x_2}(a) = 0 \text{ for a vertex } a \in \partial\Omega_k \cap \partial\Omega\}$$

where $\tau \in T_h(\Omega_k)$ is a rectangular element, cf. Figure 2.

We also introduce the global space $X_h^A(\Omega) = \prod_k X_h^A(\Omega_k)$. For each interface $\overline{\Gamma}_{kl} = \partial\Omega_k \cap \partial\Omega_l$, we choose one side as a master denoted by $\gamma_{m,k} \subset \partial\Omega_k$ and the second one as a slave $\delta_{m,l} \subset \partial\Omega_l$ if $h_k \le h_l$, see Figure 1. This assumption is necessary for the proof of some technical results and is due to the fact that any local finite element function is not sufficiently regular.

We introduce additional auxiliary spaces on each slave (nonmortar) $\delta_{m,l} \subset \partial\Omega_l$. Let the first one denoted by $M_{1,3}^{h_l}(\delta_{m,l})$ be the space of C^1 smooth functions that are piecewise cubic except for two elements that touch the ends of $\delta_{m,l}$, where are piecewise linear, and let the second one $M_{0,1}^{h_l}(\delta_{m,l})$ be the space of continuous piecewise linear functions which are constant on the two elements which touch the ends of $\delta_{m,l}$.

We say that $u_k \in X_h^A(\Omega_k)$ and $u_l \in X_h^A(\Omega_l)$ for $\partial\Omega_l \cap \partial\Omega_k = \overline{\Gamma}_{kl}$ satisfy the mortar conditions if

$$\int_{\delta_m} (u_k - u_l)\psi\, ds = 0 \;\; \forall \psi \in M_{1,3}^{h_l}(\delta_{m,l}), \tag{2}$$

$$\int_{\delta_m} (I_{h_k}\partial_n u_k - I_{h_l}\partial_n u_l)\psi\, ds = 0 \;\; \forall \psi \in M_{0,1}^{h_l}(\delta_{m,l}), \tag{3}$$

where I_{h_l}, I_{h_k} are the standard piecewise linear interpolants onto the h_l and h_k meshes of $\delta_{m,l}$ and $\gamma_{m,k}$, respectively. Note that $I_{h_i}\partial_n u_i$, for $i = k, l$, equals the normal derivative of piecewise bilinear interpolant defined over Ω_i by the values of $\partial_n u_i$ at the vertices of rectangular elements of $T_h(\Omega_i)$.

We now define the discrete space V_h^A as the subspace of $X_h^A(\Omega)$ formed by functions which satisfy the mortar conditions (2) and (3) on all slave sides and are continuous at all crosspoints.

The discretization of (1) using V_h^A is of the form:
Find $u_h^A \in V_h^A$ such that

$$a_h(u_h^A, v) = \int_\Omega f v\, dx \quad \forall v \in V_h^A, \tag{4}$$

where $a_h(u,v) = \sum_{k=1}^N a_{h,k}(u,v)$ and

$$a_{h,k}(u,v) = \sum_{\tau \in T_h(\Omega_k)} \int_\tau \triangle u \triangle v + (1-\nu)(2u_{x_1x_2}v_{x_1x_2} - u_{x_1x_1}v_{x_2x_2} - u_{x_2x_2}v_{x_1x_1})\, dx. \tag{5}$$

The form $a_h(\cdot,\cdot)$ is positive definite over V_h^A what follows from the fact that $a_h(u,u) = 0$ implies that u is linear in all rectangles of $T_h(\Omega_k)$, then from the continuity of u, u_{x_1}, u_{x_2} at all vertices of the elements of $T_h(\Omega_k)$ follows that u is linear in Ω_k and from the mortar condition follows that u is linear in Ω. Then the boundary conditions yield $u = 0$.

Moreover, it has been proven in [Mar99b] that this form is uniformly elliptic on V_h^A what is stated in the following:

Theorem 1 *There exists a constant C independent of h_k and the number of subdomains such that for $u \in V_h^A$*

$$C\, \|u\|^2_{H_h^2(\Omega)} \le a_h(u,u),$$

where $\|u\|_{H_h^2(\Omega)} = (\sum_{k=1}^N \sum_{\tau \in T_h(\Omega_k)} \|u\|^2_{H^2(\tau)})^{1/2}$ is the so-called broken H^2-norm.

Hence

Proposition 1 *The problem (4) has a unique solution.*

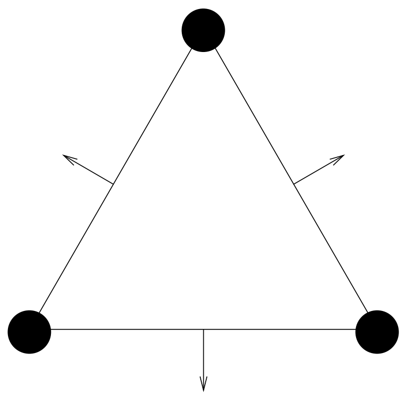

Figure 3: Morley element.

Morley element

In this subsection, we introduce a mortar method that locally uses the Morley element, e.g. cf. [LL75].

The local finite element space $X_h^M(\Omega_k)$ is defined by, see Figure 3,

$$X_h^M(\Omega_k) = \{v \in L_2(\Omega_k) : \ v_{|\tau} \in P_2(\tau), \ v \text{ continuous at vertices of}$$

$$\tau \in T_h(\Omega_k) \text{ and } \partial_n v \text{ continuous at midpoints of edges of } \tau \text{ and}$$

$$v(p) = \partial_n v(m) = 0 \ \text{ for a vertex } p \in \partial\Omega \text{ and a midpoint } m \in \partial\Omega\}.$$

We also introduce a global space $X_h^M(\Omega) = \prod_{k=1}^N X_h^M(\Omega_k)$ as in the previous subsection.

We now select an open disjoint side Γ_{kl} of $\partial\Omega_k$, $\overline{\Gamma}_{kl} = \partial\Omega_k \cap \partial\Omega_l$, denote it by $\gamma_{m,k}$ and name as master (mortar) if $h_k \leq h_l$, cf. Figure 1. This assumption like for the Adini element is necessary for the proof of some technical results and is due to the fact that any local finite element function is not sufficiently regular. The side of $\Gamma_{kl} \subset \partial\Omega_l$ is called slave (nonmortar) and is denoted by $\delta_{m,l}$. As $h_k \leq h_l$ and the both triangulations are quasiuniform, we can assume that the two end elements of the h_l-triangulation of the slave $\delta_{m,l}$, i.e. the ones that touch the ends of $\delta_{m,l}$, are longer than the respective elements of the h_k-triangulation of the master $\gamma_{m,k}$.

We introduce additionally two auxiliary spaces on each slave (nonmortar) $\delta_{m,l}$. Let the first one denoted by $M_{-1,0}^{h_l}(\delta_{m,l})$, be the space of functions which are piecewise constant on the h_l triangulation of $\delta_{m,l}$.

For the simplicity of presentation, we also assume that the both 1-D triangulations of the interface Γ_{kl}, the h_k one of its master $\gamma_{m,k}$ and the h_l one of its slave $\delta_{m,l}$, have even numbers of the elements. Let consider $\delta_{m,l}$ and let $\overline{\delta}_{m,l,h} = \{p_0, p_1, \ldots, p_{N_{m,l}}\}$ be a set of vertices of the h_l triangulation of this slave, ($N_{m,l}$ is even). Then we introduce an operator $I_{2h_l,2} : C(\overline{\delta}_{m,l}) \to C(\overline{\delta}_{m,l})$ defined by the values of u at all points of $\overline{\delta}_{m,l,h}$ as follows:

- $I_{2h_l,2}u \in P_2$ on each $[p_i, p_{i+2}]$ for even $i < N_{m,l}$,
- $I_{2h_l,2}u(p_i) = u(p_i)\ \ p_i \in \overline{\delta}_{m,l,h}$.

The operator $I_{2h_k,2}$ that corresponds to the h_k mesh of master $\gamma_{m,k}$ is defined in the same way.

We next define an auxiliary space $M_{0,2}^{2h_l}(\delta_{m,l})$ as follows

$$M_{0,2}^{2h_l}(\delta_{m,l}) = \{v \in C(\overline{\delta}_{m,l}) : \ v \in P_2([p_i, p_{i+2}]) \text{ for even } i \neq 0, N_{m,l} - 2, \tag{6}$$

$$\text{and } v \in \ P_1([p_i, p_{i+2}]) \text{ for } i = 0, N_{m,l} - 2\}.$$

We now introduce the two mortar conditions on the interface $\Gamma_{kl} = \gamma_{m,k} = \delta_{m,l}$:

$$\int_{\delta_m} (I_{2h_k,2}u_k - I_{2h_l,2}u_l)\psi \, ds = 0 \ \ \forall \psi \in M_{0,2}^{2h_l}(\delta_{m,l}) \tag{7}$$

and

$$\int_{\delta_m} (\partial_n u_k - \partial_n u_l)\phi \, ds = 0 \ \ \forall \phi \in M_{-1,0}^{h_l}(\delta_{m,l}). \tag{8}$$

We next define the discrete space V_h^M as the subspace of $X_h^M(\Omega)$ formed by functions which satisfy the mortar conditions (7) and (8) on all slave sides and are continuous at all crosspoints.

The discretization of (1) using V_h^M is of the form:
Find $u_h^M \in V_h^M$ such that

$$a_h(u_h^M, v) = \int_\Omega fv \, dx \quad \forall v \in V_h^M, \tag{9}$$

where $a_h(u,v) = \sum_{k=1}^N a_{h,k}(u,v)$ and $a_{h,k}(u,v)$ are defined as in (5). The form $a_h(\cdot,\cdot)$ is positive definite over V_h^M. It follows from the fact that $a_h(u,u) = 0$ yields that u is piecewise linear in the triangles of $T_h(\Omega_k)$, then the continuity of u at all vertices and $\partial_n u$ at all midpoints of elements of $T_h(\Omega_k)$ yields that u is linear in Ω_k and finally from the mortar conditions follows that u is linear in Ω. The boundary conditions yield $u = 0$.

As in the case of Adini mortar method, we have the uniform ellipticity of the form $a_h(\cdot,\cdot)$ over V_h^M, cf. [Mar99a] and [Mar99b], i.e.

Theorem 2 *There exists a constant C independent of h_k and the number of subdomains such that for $u \in V_h^M$*

$$C \, \|u\|_{H_h^2(\Omega)}^2 \leq a_h(u,u),$$

where $\|u\|_{H_h^2(\Omega)}$ is the broken H^2-norm.

Thus we obtain

Proposition 2 *The problem (9) has a unique solution.*

Error estimates

We have the following error estimates for the both elements:

Theorem 3 *Assume that u^*, the solution of (1), is in the space $H_0^2(\Omega) \cap H^4(\Omega)$. Then for the Adini element*

$$\|u^* - u_h^A\|_{H_h^2(\Omega)}^2 \leq C_A \sum_{k=1}^{N} \left(h_k^2 |u^*|_{H^3(\Omega_k)}^2 + h_k^4 |u^*|_{H^4(\Omega_k)}^2 \right),$$

and for the Morley element

$$\|u^* - u_h^M\|_{H_h^2(\Omega)}^2 \leq C_M \sum_{k=1}^{N} \left(h_k^2 |u^*|_{H^3(\Omega_k)}^2 + h_k^4 |u^*|_{H^4(\Omega_k)}^2 \right),$$

where u_h^A and u_h^M are the solutions of (4) and (9), respectively, $\|v\|_{H_h^2(\Omega)}$ is the broken H^2-norm, and C_A, C_M are positive constants independent of u^*, any h_k, and the number of subdomains.

Remark on Additive Schwarz Methods

In this section we make a brief remark on the parallel methods of Schwarz type for solving the discrete problems (4) and (9). The detailed discussion will be published elsewhere.

In [Mar99b] a parallel algorithm for solving (4) was constructed and analyzed. This is a iterative substructuring method, i.e. it is applied to the Schur complement of the discrete problem, i.e. interior variables are first eliminated using some direct methods. The method is described in terms of an Additive Schwarz Method (ASM), cf. [SBG96]. We decompose a discrete space into a sum of subspaces which consists of a coarse space, local one dimensional spaces associated with degrees of freedom of order one at vertices of subdomains, and certain local spaces associated with interfaces. The coarse space is not standard and can be named an exotic one.

A Neumann-Neumann method for solving systems of linear equations arising from conforming mortar discretizations of a plate problem which is constructed and analyzed in [Mar99b], can be adapted to the nonconforming cases of the Adini and Morley discretizations considered in this paper. The analysis of the Neumann-Neumann methods for the Adini case can be done in a similar way to that in [Mar99b] utilizing some technical results which can also be found in [Mar99b], while the case of the Morley element requires some new technical results which have been obtained and which will be published elsewhere.

The described methods are almost optimal, i.e. the number of iterations required to decrease the energy norm of the error by a conjugate gradient method is proportional to $(1 + \log(\frac{H}{\underline{h}}))$, where $H = \max_i(\mathrm{diam}\,\Omega_i)$ and $\underline{h} = \inf_i h_i$.

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