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46. Best N-term capacitance approximation on sparse grids

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Introduction

In [GOS99], adaptive sparse grid spaces spanned by a finite number of tensor-product L_2-orthogonal Haar functions have been applied to capacitance calculations on a unit screen. In this note, we state asymptotically optimal approximation rates for this problem when choosing the best possible adaptive sparse grid space of a given dimension N. We also compare the results with other recent approaches to efficiently solve this problem and comment on some numerical tests. Details of the proofs and a discussion of the approximation-theoretical aspects have appeared in [Osw99].

For a flat square screen $I^2 \equiv [0,1]^2$, we consider the *single layer potential equation*

$$\frac{1}{4\pi}\int_{I^2}\frac{f(y)}{|x-y|_2}\,dy = g(x)\,, \qquad x \in I^2\,. \tag{1}$$

As this problem can be cast in variational form and leads to a symmetric $H^{-1/2}$-elliptic problem, Galerkin methods can be set up and allow for a straightforward analysis. E.g., convergence and error estimates in Sobolev norms (most naturally in the $H^{-1/2}$-related energy norm) can be obtained for many natural discretization spaces. There are two obstacles that trigger further investigations. First, one is interested in as small as possible computational subspaces since the discretization leads to *dense matrices* which is in contrast to the situation in finite element or finite difference methods for partial differential equations. Several approaches are under investigation (see [GOS99] for a brief discussion) to overcome this problem. We only mention adaptive wavelet compression schemes [Dah97, vPS97] and the hp-version of the boundary element method [Ste96] which will be used for comparison below. These methods also deal with the second obstacle: solutions of problems such as (1) exhibit very low global Sobolev smoothness due to dominant corner and edge singularities. For the important special case $g(x) \equiv 1$, the so-called *capacitance problem*

$$\frac{1}{4\pi}\int_{I^2}\frac{f(y)}{|x-y|_2}\,dy = 1\,, \quad x \in I^2\,, \tag{2}$$

the variational solution $f \in H^{-1/2}(I^2)$ does not even belong to $L_2(I^2)$. This leads to very slow convergence rates of any standard Galerkin method, both theoretically and practically. However, in analogy to elliptic problems in polyhedral domains, the 'bad' behavior of a solution f of (1) for smooth data g can be separated into a few singularity components associated with the edges and corners of I^2, i.e., one can write

$$f = f^{sing} + f^{reg}\,, \tag{3}$$

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where the *singular part* f^{sing} is a finite linear combination of specific, prescribed singularity functions (usually composed of terms of the form $\mathrm{dist}(x,F)^\alpha$ and $\log(\mathrm{dist}(x,F))$, where F is an edge or an vertex of I^2) while the *regular part* can be as smooth as wanted (limits are set by the smoothness class of g). See [vP89, vPa90] for details on the singularity decomposition (3) for (1) and similar screen problems. Thus, to obtain improved rates of convergence it would be enough to adapt the computational subspace such that it approximates the singularity functions in f^{sing} as well as the smooth part f^{reg}. In practical algorithms, this basic idea is implemented a priori (e.g., by using *graded meshes* in h- and hp-version boundary element methods [HMS97]) or by using *feedback adaptivity schemes*, e.g., based on *a posteriori error estimators*, as suggested by different authors [HMS97, Dah97].

Without becoming too detailed, let us mention some theoretical approximation results for the h- and hp-version of the boundary element method, on the one hand, and the wavelet schemes, on the other. Throughout the paper, wavelets are *semi-orthogonal spline wavelets* of low order m, even though results for this class of ansatz spaces are valid under much more general assumptions [Dah97]. The boundary element spaces for the h-version are piecewise polynomials or splines of order m on certain sequences of partitions of I^2 (both quasi-uniform and adaptively refined ones) while in an hp-method, in addition, the polynomial degree may vary in each element of the underlying partition. Subsequently, we will specialize to the simplest case $m = 1$ of piecewise constant approximation. Our model problem will be (2). From [vPa90] it follows that the singular part f^{sing} of the solution f of (2) is representable as a sum of singularity functions, with the leading singularities of the form $\sim \mathrm{dist}(x,e)^{-1/2}$ near the interior part of any edge e of I^2, and $\sim \mathrm{dist}(x,P)^{\gamma-1}$, $\gamma \approx 0.2966...$, if x approaches a vertex P of I^2. This leads to $f \in H^{-\varepsilon}(I^2)$ for any $\varepsilon > 0$, a result which brakes down (due to the edge singularity) for $\varepsilon = 0$. Throughout the paper, the notation ε stands for an arbitrarily small positive parameter.

To make a fair comparison between different approximation methods, we will relate error quantities to the dimension $N = \dim V_N$ of the computational subspace V_N from which the Galerkin solution is determined, and not to a meshsize parameter of the underlying partition or to the level number of a space in a wavelet multiresolution analysis. We admit that this way of comparison is still disputable since computational work and storage limitations may be quite different for subspaces with the same N. All estimates are given for the best approximations in the $H^{-1/2}(I^2)$ norm,

$$e_N(f)_{-1/2} = \inf_{v_N \in V_N} \|f - v_N\|_{H^{-1/2}} \ ,$$

which is equivalent to estimating the error in energy norm between the Galerkin solution $f_N \in V_N$ and the solution f of (2). Moreover, capacitance errors $\delta_N \equiv |\mathcal{C} - \mathcal{C}_N|$, where

$$\mathcal{C} = \frac{1}{4\pi} \int_{I^2} f(y)\, dy \ , \qquad \mathcal{C}_N = \frac{1}{4\pi} \int_{I^2} f_N(y)\, dy \ , \tag{4}$$

are covered, too, since

$$\delta_N = \mathcal{C} - \mathcal{C}_N \asymp \|f - f_N\|^2_{H^{-1/2}} \asymp e_N(f)^2_{-1/2} \ . \tag{5}$$

- *h-version with quasi-uniform partitions and fixed polynomial degree resp. non-adaptive wavelet spaces.* Here, standard estimates

$$e_N(u)_{-1/2} \le C h_N^{t+1/2} \|u\|_{H^t} , \quad u \in H^t(I^2) , \ -1/2 < t \le m ,$$

hold with a mesh parameter $h_N \approx N^{-1/2}$, and lead in conjunction with the above regularity result for f to the estimate

$$e_N(f)_{-1/2} = \mathrm{O}(N^{-(1/4-\varepsilon)}) , \qquad N \to \infty , \tag{6}$$

The asymptotic behavior in (6) is independent of m, and much worse than the saturation order $\mathrm{O}(N^{-(m/2+1/4)})$ valid for approximating smooth functions from $H^m(I^2)$ with respect to the same spaces V_N.

- *h-version with graded meshes.* The estimate (6) can be improved if graded meshes are allowed for partitioning I^2, see [vPa90]. For I^2, these are based on tensor-product partitions where the univariate partitions have $n \asymp \sqrt{N}$ grid points $\xi_i \in (0,1)$ which behave like $\sim (i/n)^\beta$ near the left endpoint (analogous refinement is assumed at the right endpoint of $[0,1]$). For appropriate β, the above mentioned saturation order can be reached:

$$e_N(f)_{-1/2} = \mathrm{O}(N^{-(1/4+m/2)}) , \qquad N \to \infty , \tag{7}$$

for the associated spaces of piecewise polynomials or splines of order m on the above partitions, see [vP89, vPa90]. This improvement is achieved by allowing high aspect ratios of the rectangles (anisotropic refinement) near the edges.

- *hp-version on geometric meshes.* The best asymptotic estimates are known for the hp-method and a geometric tensor-product mesh (now the univariate meshes are given by $\xi_i \sim \sigma^{n/2-i}$, $\sigma < 1$, near the left endpoint of $[0,1]$). The result for the particular case under consideration (see [HMS97, Ste96]) is

$$e_N(f)_{-1/2} = \mathrm{O}(e^{-cN^{1/4}}) , \qquad N \to \infty . \tag{8}$$

- *Adaptive wavelet approximation.* The basic idea is to determine a wavelet space V_N as the linear span of N carefully selected wavelets ψ_λ from different levels of the underlying multiresolution analysis. Theoretically, assuming that $f = \sum_\lambda c_\lambda \psi_\lambda$ is decomposed into a wavelet series, and that $\Psi = \{\psi_\lambda\}$ forms a Riesz basis in $H^{-1/2}$, the best one can do is to select the terms with the N largest $|c_\lambda|$. An algorithm which uses this basic idea has been described in [Dah97]. The supporting approximation-theoretical result behind it has been known for some years [DJP92, Osw90]. It is now referred to under the name *nonlinear N-term approximation* and has found important applications to image compression and adaptive algorithms, see [DeV98]. The bad news, however, is that for our f this only leads to an estimate of

$$e_N(f)_{-1/2} = \mathrm{O}(N^{-(1/2-\varepsilon)}) , \qquad N \to \infty , \tag{9}$$

again independently of m. This is a slight improvement over (6) but even the h-version on optimally chosen graded meshes with piecewise constants does asymptotically better than any adaptive wavelet space. The main reason is that for

the wavelet bases considered in [Dah97, DeV98] (and in most of the literature on solving boundary integral equations by wavelet methods), the nonlinear N-term approximation models *optimal isotropic local h-refinement*. Thus, for resolving the dominating edge singularities in the solution of (2), too many wavelet functions are necessary to improve the resolution along edges. This effect does not occur for point singularities and is practically invisible for edge singularities that are weaker than those exhibited by the solutions of screen problems (compare [DD97]).

Clearly, from the above one would prefer graded resp. geometric meshes (combined with h- resp. hp-methods) over wavelet type methods for the application under consideration. The exponential convergence of the hp-method is hard to beat in the asymptotic range. However, since the implementation of an hp-method for integral equations is by no means trivial, simpler and less optimal methods may still have a chance. E.g., well-understood adaptivity and compression strategies, preconditioning, and canonical data structures are some advantages of wavelet methods that one might wish to explore.

Improving upon the relatively weak approximation potential for solutions of screen problems while still working in a wavelet multiresolution analysis is suggested by the results on *adaptive sparse grid spaces* in [GOS99]. In the present note we describe the approximation rates obtainable from these spaces in more quantitative terms. Roughly speaking, our general claim is that under the same assumptions on f, by changing from the traditional, isotropic wavelet constructions on I^2 to tensor-product, anisotropic wavelet systems Ψ^*, the unsatisfactory rates of (9) can be replaced by

$$e_N^*(f)_{-1/2} = \mathrm{O}(N^{-(1/4+m)}) , \qquad N \to \infty , \tag{10}$$

where $e_N^*(f)_{-1/2}$ describes now the best N-term approximation with respect to the new wavelet system Ψ^*. Our point is that, even without going to graded meshes, we can expect good results if standard wavelet systems are replaced by tensor-product wavelet systems. We give precise statements for the case $m = 1$ (piecewise constant approximation) in the next section. Numerical experiments are presented in the last part.

N-TERM APPROXIMATION BY HAAR FUNCTIONS

Let us give the definition of the Haar-wavelet systems ($m = 1$) under consideration. The characteristic function of a set Ω will be denoted by χ_Ω. Let $\mathcal{D}_j$ be the system of dyadic intervals Δ of length $|\Delta| = 2^{-j}$, $j \geq 0$, of $I \equiv [0, 1]$. Any $\Delta \in \mathcal{D}_j$ uniquely splits into left (Δ^+) and right (Δ^-) half-intervals from $\mathcal{D}_{j+1}$. Set

$$\phi_\Delta = |\Delta|^{-1/2}\chi_\Delta , \qquad \psi_\Delta = |\Delta|^{-1/2}(\chi_{\Delta^+} - \chi_{\Delta^-}) , \quad \Delta \in \mathcal{D} = \cup_{j\geq 0}\mathcal{D}_j ,$$

for the univariate scaled box functions and Haar functions, respectively. The *standard bivariate Haar system* is given by

$$\Psi_H = \cup_{j\geq 0}\ \Psi_j ,$$

where Ψ_0 consists of the only function χ_{I^2}, and

$$\Psi_j = \{\psi_\Delta(x_1)\phi_{\Delta'}(x_2),\ \phi_\Delta(x_1)\psi_{\Delta'}(x_2),\ \psi_\Delta(x_1)\psi_{\Delta'}(x_2),\ \Delta,\Delta' \in \mathcal{D}_{j-1}\}$$

for $j \geq 1$.The supports of Haar functions from Ψ_j are dyadic squares of sidelength 2^{-j+1}, $j \geq 1$. In contrast, the Haar functions in the tensor-product bivariate Haar system

$$\Psi^*_H = \cup_{j_1,j_2\geq 0}\ \Psi^*_{j_1,j_2}\ ,$$

where

$$\Psi^*_{j_1,j_2} = \{\psi_\Delta(x_1)\psi_{\Delta'}(x_2)\ ,\ \Delta \in \mathcal{D}_{j_1-1}, \Delta' \in \mathcal{D}_{j_2-1}\}\ ,$$

possess rectangular support. For notational convenience, we defined $\mathcal{D}_{-1} = \{[0,2]\}$ and $\psi_{[0,2]} = \phi_I$. Obviously, both systems are complete orthonormal systems in $L_2(I^2)$.

We are interested in the behavior of best N-term approximations with respect to Ψ^*_H

$$e^*_N(f)_s = \inf_{\Psi^*_N \equiv \{\psi_1,\dots,\psi_N\}\subset\Psi^*_H}\ \inf_{v_N\in V^*_N \equiv \operatorname{span}\Psi^*_N} \|f - v_N\|_{H^s}\ , \quad N \geq 1\ , \tag{11}$$

in the $H^s(I^2)$-norm. Due to the approximation and smoothness properties of piecewise constant functions, only the range $-1 < s < 1/2$ is of interest. Two main theorems are established (for a detailed exposition and proofs, we refer to [Osw99]). The first theorem serves functions from spaces of functions with dominating mixed derivatives which can be defined as tensor products of univariate Sobolev spaces:

$$H^t_{\rm mix}(I^2) = H^t(I) \otimes H^t(I)\ , \qquad -\infty < t < \infty\ .$$

For $t = 0$, we have $H^0_{\rm mix}(I^2) \cong L_2(I^2)$ while $f \in H^1_{\rm mix}(I^2)$ if f belongs to $H^1(I^2)$ and additionally possesses a weak mixed derivative $\partial_{11} f \in L_2(I^2)$.

Theorem 1 *Let $f \in H^t_{\rm mix}(I^2)$ for some $-1/2 < t \leq 1$. Then its best N-term approximations with respect to Ψ^*_H in $H^s(I^2)$, $N \geq 1$, satisfy*

$$e^*_N(f)_s \leq C\|f\|_{H^t_{\rm mix}} \begin{cases} N^{-(t-s)}\ , & 0 < s < 1/2\ ,\ \ s < t \leq 1\ , \\ N^{-t}(1+\log N)^t\ , & s = 0 < t < 1\ , \\ N^{-1}(1+\log N)^{3/2}\ , & s = 0\ ,\ t = 1\ , \\ N^{-(t-s/2)}\ , & -1 < s < 0\ ,\ \ s/2 < t \leq 1\ . \end{cases}$$

In particular, if $f \in H^1_{\rm mix}(I^2)$ then

$$e^*_N(f)_{-1/2} \leq CN^{-5/4}\|f\|_{H^1_{\rm mix}}\ , \quad N \to \infty\ . \tag{12}$$

This estimate is applicable to the smooth part f^{reg} of solutions to (1), to achieve the $\mathrm{O}(N^{-5/4})$ error bound in practice, one can, e.g., take subspaces spanned by the following subset of $\asymp 2^J$ Haar functions:

$$\Psi^*_{H,J} = \cup_{j_1,j_2\geq 0\,:\,j_1+j_2+\frac{1}{4}\max(j_1,j_2)\leq\frac{9}{8}J}\ \Psi^*_{j_1,j_2}\ . \tag{13}$$

Note that $\Psi^*_{H,J}$ spans a subspace of the standard sparse grid space of level J.

The second result covers certain types of *singularity functions*. We call $f \in L_1(I^2)$ an *edge singularity function with exponent* $\alpha \in [0,1)$ if it has a continuous derivative $\partial_{11} f$ in the open square $(0,1)^2$ and satisfies

$$|\partial_{kl} f(x_1,x_2)| \le C(\min(x_1,1-x_1))^{-\alpha-k}(\min(x_2,1-x_2))^{-\alpha-l}$$

for all $(x_1,x_2) \in (0,1)^2$ and $0 \le k,l \le 1$. E.g., the singular part f^{sing} in (3) of the solution f of (2) possesses this property with $\alpha = 0.7034...$ (a more detailed analysis shows that for the capacitance problem better representations of f^{sing} can be found which would lead to edge singularity functions with $\alpha = 1/2$ as the appropriate value).

Theorem 2 *Let $-1 < s < 1/2$, and f be an edge singularity function with exponent α, where $0 \le \alpha < \min(1/2-s, 1/2-s/2)$. Then $f \in H^s(I^2)$ and*

$$e^*_N(f)_s \le C \begin{cases} N^{-(1-s)}\,, & 0 < s < 1/2\,, \\ N^{-1}(\log N)^{3/2}\,, & s = 0\,, \\ N^{-(1-s/2)}\,, & -1 < s < 0\,, \end{cases} \qquad N \to \infty\,. \tag{14}$$

Roughly speaking, by optimally choosing N Haar functions from $\Psi^*_H(I^2)$, an edge singularity function with exponent α satisfying the above condition possesses the same asymptotic N-term approximation rate as smooth functions from $H^1_{\rm mix}(I^2)$. For the case $s = -1/2$, we can have $0 \le \alpha < 3/4$ which leads according to our above remarks to

$$e^*_N(f^{sing})_{-1/2} \le CN^{-5/4}\,, \quad N \to \infty\,, \tag{15}$$

for the singular part of the solution f of the capacitance problem (2). Since, at the same time, we can assume that $f^{reg} \in H^1_{\rm mix}(I^2)$ in (3), the two estimates (12) and (15) yield an analogous estimate for f itself. Finally, from (4) we see that the capacitance $\mathcal{C}$ of the unit square screen can be approximated at a rate of $\mathrm{O}(N^{-5/2})$ if optimal selections of N Haar functions from $\Psi^*_H(I^2)$ are used to build discretization spaces.

NUMERICAL TESTS

In Section 3.3-4 of [GOS99], capacitance approximations have been computed for full grid (fg-), sparse grid (sg-) and adaptive sparse grid (asg-) spaces. To reach a relative capacitance error δ^{rel}_N of approximately 10^{-3}, subspaces V_N of dimension $N = 65536$, $N = 1280$, and $N = 68$, respectively, were needed. The proofs of Theorem 1 and 2 suggest the use of new asg-spaces with slightly improved convergence properties (see Table 1). In order to achieve the above-mentioned asymptotical error estimate $\mathrm{O}(N^{-5/2})$, it should be sufficient to take the union of the set $\Psi^*_{J_0,H}$ defined in (13) which serves the regular part f^{reg}, and a set Ψ^*_N consisting of $N \asymp 2^{J_0}$ functions from $\Psi^*_H(I^2)$ producing the N largest contributions to the upper bound

$$\|f^{sing}\|^2_{H^{-1/2}} \le C \sum_{j_1,j_2} \sum_{\psi \in \Psi^*_{j_1,j_2}} 2^{-\max(j_1,j_2)} |c_\psi(f^{sing})|^2 \tag{16}$$

Table 1: Relative capacitance errors for various V_N

fg-spaces		sg-spaces		asg-spaces [GOS99]		new asg-spaces	
N	δ_N^{rel}	N	δ_N^{rel}	N	δ_N^{rel}	N	δ_N^{rel}
4	0.08302	3	0.08302	20	0.00921	9	0.02516
16	0.04584	8	0.04589	32	0.00495	17	0.00738
64	0.02490	20	0.02511	44	0.00268	25	0.00254
256	0.01310	48	0.01340	56	0.00150	33	0.00130
1024	0.00677	112	0.00708	68	0.00090	41	0.00098
4096	0.00346	256	0.00373	80	0.00060	61	0.00069
16384	0.00175	576	0.00197	92	0.00044	81	0.00051
65536	0.00089	1280	0.00105	104	0.00037	101	0.00036

for the singular part f^{sing} from (3). The proof of Theorem 2 also shows that the unknown Haar-Fourier coefficients $c_\psi(f^{sing})$ can be replaced by computable upper bounds. These, in turn, can be obtained from using appropriate majorants for f^{sing} (such as appearing in the definition of edge singularity functions with exponent α or obtained directly from the available singularity decompositions, see [HMS97, GOS99]). Tuning N, J_0, and choosing different majorants may lead to further improvement.

In our experiments, the sets Ψ_N^* have been obtained from thresholding the sequence $\{2^{-\max(j_1,j_2)}|c_\psi(f^\alpha)|^2\}$ for the function $f^\alpha(x_1,x_2) = x_2^{-\alpha}$ (which mimics a singularity along the edge $x_2 = 0$ of the unit square) and a straightforward symmetrization step (note that for the solution of (2) satisfies $f(x_1,x_2) = f(x_1,1-x_2) = f(1-x_1,x_2)$). Using the values $\alpha = 1/2$, $J_0 = 3$, we found that the above-mentioned relative error of 10^{-3} can be reached by using 41 ansatz functions (the constant function from $\Psi_{0,0}^*$ and four functions with support along the edges from each of the sets $\Psi_{0,j}^* \cup \Psi_{j,0}^*$, $j = 2,\ldots,11$). This hints at the importance of dealing with the edge singularities adequately, and in the first place.

We also performed some a posteriori analysis by first computing the numerical solution on a sufficiently large adaptive sparse grid space (dimensions $N_{\max} = 277$ and $N_{\max} = 409$ have been tried), and then applying the above thresholding procedure to the obtained set of approximate Fourier coefficients. For small $N << N_{\max}$, this procedure leads to essentially the same spaces as used to produce the results of the last column of Table 1. Lack of space prevents us from giving more details (see the extended version of this note at `http://cm.bell-labs.com/who/poswald`).

Conclusion

It is demonstrated that properly selected, small subsystems of the tensor-product Haar system can be used as ansatz functions in a Galerkin scheme for the single layer potential equation to obtain the capacitance of a square screen with a relative accuracy of up to 10^{-4} in a highly efficient way. Theoretical support is given by providing sharp asymptotic estimates for the best N-term approximation with regard to this Haar system in Sobolev norms and various classes of functions (including those typical for the solutions of the single layer potential equation), and comparing them with analogous results for other popular approximation schemes for this problem.

The results also highlight, under model assumptions, the importance of anisotropic refinement along the edges of the screen and represent an interesting improvement over the use of graded meshes. The advantage is that only mesh-structures based on coordinate-wise dyadic refinement need to be implemented and that in an adaptive scheme that selects the right subset of the Haar system on this mesh-structure, the overall approximation rate measured in terms of dimensions of the resulting computational subspaces is even better.

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