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6. Operator Theoretical Analysis to Domain Decomposition Methods

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Introduction

The purpose of the present paper is to give a brief summary of our recent study on the domain decomposition method from an operator theoretical point of view. There are a large number of works devoted to the mathematical analysis of the domain decomposition methods. Most of these works carry out the convergence analysis without any assumptions of general nature on the geometry of the decomposition. However, from the viewpoint of mathematical theory as well as from that of applications in science and engineering, we are seriously interested in the effect of relationships between the rate of convergence of iterations and the geometric shape of decomposed domains. Moreover, the choice of relaxation parameters is of importance. Our method enables us to get explicit convergence factors under some assumptions on geometric shapes of decomposed domains. Furthermore, our convergence theorems give information on the choice of relaxation parameters which guarantees a fast convergence.

The problem considered in this paper is well discussed in the monograph by A. Quarteroni and A. Valli (Domain Decomposition Methods for Partial Differential Equations, Oxford, 1999), and the results described here may be said to be particular cases of theorems presented in their monograph. However, the advantage of employing our method is already described above.

We shall present a rough sketch of the method of analysis and theorems without the proofs; for the complete proofs, we refer to [Fuj97], [FKKN96], [FFS98] and [FS97].

Model Problem

In order to fix the idea, let $\Omega \subset \mathbf{R}^2$ be a bounded domain with a Lipschitz boundary Γ, and consider the Poisson equation:

$$-\triangle u = f \text{ in } \Omega, \qquad u = \beta \text{ on } \Gamma. \tag{1}$$

We assume that $f \in L^2(\Omega)$ and $\beta \in H^{1/2}(\Gamma)$. The exact solution of (1) is denoted by $\tilde{u}$.

We divide the target domain Ω into two disjoint subdomains Ω_1 and Ω_2 by a smooth simple curve γ;

$$\overline{\Omega} = \overline{\Omega_1 \cup \Omega_2}, \quad \Omega_1 \cap \Omega_2 = \emptyset.$$

We assume that γ connects transversally two points on Γ. The outer unit normal vector to the boundary of a domain in consideration is denoted by n. If necessary, by

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ν we indicate the one to γ outgoing from Ω_1. Put $\Gamma_1 = \partial\Omega_1\backslash\gamma$ and $\Gamma_2 = \partial\Omega_2\backslash\gamma$. The curve γ is called the artificial boundary.

We consider the following domain decomposition algorithm which is well-known as the Dirichlet-Neumann (DN) method. Take a function $\mu^{(0)}$ defined on γ as the initial guess to $\tilde{u}|_\gamma$. Then, we successively generate $u_1^{(k)}$, $u_2^{(k)}$ and $\mu^{(k+1)}$, for $k = 0, 1, 2, \cdots$, by solving

$$\begin{cases} -\triangle u_1^{(k)} &= f & \text{in } \Omega_1, \\ u_1^{(k)} &= \beta & \text{on } \Gamma_1, \\ u_1^{(k)} &= \mu^{(k)} & \text{on } \gamma, \end{cases}$$

$$\begin{cases} -\triangle u_2^{(k)} &= f & \text{in } \Omega_2, \\ u_2^{(k)} &= \beta & \text{on } \Gamma_2, \\ \dfrac{\partial u_2^{(k)}}{\partial n} &= -\dfrac{\partial u_1^{(k)}}{\partial \nu} & \text{on } \gamma, \end{cases}$$

The value of $\mu^{(k)}$ is adapted by

$$\mu^{(k+1)} = (1-\theta)\mu^{(k)} + \theta u_2^{(k)}|_\gamma,$$

where θ is the relaxation parameter subject to $0 < \theta < 1$.

Notation. The basic Hilbert space in our consideration is $X = L^2(\gamma)$. The usual $L^2(\gamma)$ inner product and norm are denoted by $(\cdot,\cdot)_X$ and $\|\cdot\|_X$, respectively. The space $V = H_{00}^{1/2}(\gamma)$ is familiar (See, for example, [LM72]), and the norm of V is denoted by $\|\cdot\|_V$. For any $\xi \in V$, a solution in $H^1(\Omega_1)$ of the harmonic problem

$$\triangle w = 0 \text{ in } \Omega_1, \quad w = 0 \text{ on } \Gamma_1, \quad w = \xi \text{ on } \gamma$$

is called the harmonic extension of ξ into Ω_1 and is denoted by $w = H_1\xi$. The harmonic extension $H_2\xi$ of ξ into Ω_2 is defined in the similar manner. As a consequence of the trace theorem (Theorem 1.5.2.3, [Gri85]) and the elliptic estimates, we have

$$C\|\xi\|_V \le \|\nabla H_i\xi\|_{L^2(\Omega_i)} \le C'\|\xi\|_V \quad (\forall \xi \in V) \tag{2}$$

with domain constants $C > 0$ and $C' > 0$, for $i = 1, 2$.

Amplification Operator for the Error

It is easy to derive that the error $\xi^{(k)} = \mu^{(k)} - \tilde{u}|_\gamma$ can be expressed as

$$\xi^{(k+1)} = (1-\theta)\xi^{(k)} - \theta S_2^{-1} S_1 \xi^{(k)}. \tag{3}$$

Here S_1 and S_2 stand for the Steklov-Poincaré (SP) operators corresponding to Ω_1 and Ω_2, respectively. The formal definition of S_1 is

$$S_1\xi = \frac{\partial(H_1\xi)}{\partial\nu}\Big|_\gamma$$

for $\xi \in V$. S_2 is also defined in the similar way. Actually, thought Kato's representation theorems concerning unbounded quadratic forms in a Hilbert space ([Kat76]), we have:

1. S_i is a positive and self-adjoint operator, and $S_i^{1/2}$ is so too.
2. The domain $D(S_i^{1/2})$ of $S_i^{1/2}$ coincides with V.
3. The identity $\|S_i^{1/2}\xi\|_X = \|\nabla H_i \xi\|_{L^2(\Omega_i)}$ holds for any $\xi \in V$.
4. $S_2^{-1}S_1$ with its domain $D(S_1)$ admits of a bounded extension H into V. In fact, H is given by
$$H = S_2^{-1/2}(S_1^{1/2}S_2^{-1/2})^* S_1^{1/2},$$
where * means the adjoint in X.

Therefore, the precise meaning of (3) is understood as:

$$\xi^{(k+1)} = A_\theta \xi^{(k)}, (k = 0, 1, 2, \cdots), \qquad \xi^{(0)} \in V,$$

where A_θ is the amplification operator for the error defined by

$$A_\theta = (1-\theta)I - \theta H, \quad (I \text{ is the identity}).$$

To treat A_θ as a self-adjoint operator, we employ the following device. Thus, we introduce a special inner product in V in terms of the SP operator:

$$((\xi, \eta)) = (S_2^{1/2}\xi, S_2^{1/2}\eta)_X, \qquad (\forall \xi, \eta \in V). \tag{4}$$

Then V again forms a Hilbert space with the new inner product (4). Moreover, by virtue of (2), we deduce that the corresponding norm $|||\cdot||| = ((\cdot,\cdot))^{1/2}$ is equivalent to $\|\cdot\|_V$ in V. Furthermore, under (4), H and therefore A_θ are self-adjoint in V.

Concerning the spectral radius $r_\sigma(A_\theta)$ of A_θ, as a direct consequence of the spectral mapping theorem (See, for example, [Yos80]), we have

$$r_\sigma(A_\theta) = \sup_{s \in \sigma(H)} |1 - \theta - \theta s|,$$

where $\sigma(H)$ denotes the spectrum of H.

Shape Conditions and Convergence Results

Throughout this section, we assume that γ is a line segment on the x_2-axis. In order to evaluate $r_\sigma(A_\theta)$, we introduce shape conditions of subdomains under the notation:

- R denotes reflection with respect to the x_2-axis defined by
$$R: \ (x_1, x_2) \mapsto (-x_1, x_2).$$
- T_m, m being a positive constant, denotes the contraction mapping along the x_1-axis defined by
$$T_m: \ (x_1, x_2) \mapsto \Big(\frac{x_1}{m}, x_2\Big).$$

Conditions ($\mathbf{I}_m$) and ($\mathbf{I}^l$). Let $1 \leq m < \infty$. We say that Condition (I_m) is satisfied if

$$RT_m \Omega_2 \subseteq \Omega_1.$$

On the other hand, for $1 \leq l \leq \infty$, we say that Condition (I^l) is satisfied if

$$RT_l \Omega_1 \subseteq \Omega_2.$$

In the above definition, we understand that Condition (I^l) is not satisfied if $l = \infty$. The following lemma is a consequence of Conditions (I_m) and (I^l).

Lemma 1 *Let $1 \leq m < \infty$ and $1 \leq l \leq \infty$, and suppose that both Conditions (I_m) and (I^l) are satisfied. Then we have*

$$\frac{1}{l} \leq H \leq m. \tag{5}$$

That is, $1/l \leq s \leq m$ holds for any $s \in \sigma(H)$.

Therefore, concerning a convergence of the DN method, we obtain the following theorems:

Theorem 1 *Let $1 \leq m < \infty$ and $1 \leq l \leq \infty$, and suppose that both Conditions (I_m) and (I^l) are satisfied. For $0 < \theta < 1$, we define*

$$\tilde{r}(\theta) = \begin{cases} 1 - (1 + \frac{1}{l})\theta & \text{for } 0 < \theta \leq \frac{2}{m + l^{-1} + 2}, \\ (m+1)\theta - 1 & \text{for } \frac{2}{m + l^{-1} + 2} \leq \theta < 1. \end{cases}$$

Furthermore, assume that $0 < \theta < \dfrac{2}{m + l^{-1} + 1}$. Then $0 < \tilde{r} < 1$ and there exists a positive constant c_0 depending only on Ω_2 such that

$$\|\xi^{(k)}\|_V \leq c_0 \tilde{r}(\theta)^k \|\xi^{(0)}\|_V, \qquad (k = 1, 2, 3, \cdots).$$

Theorem 2 *Under the same assumptions of Theorem 1, we have*

$$\begin{aligned} \|u_1^{(k)} - \tilde{u}|_{\Omega_1}\|_{H^1(\Omega_1)} &\leq c_1 \tilde{r}(\theta)^k \|u_1^{(0)} - \tilde{u}|_{\Omega_1}\|_{H^1(\Omega_1)}, \\ \|u_2^{(k)} - \tilde{u}|_{\Omega_2}\|_{H^1(\Omega_2)} &\leq c_2 \tilde{r}(\theta)^k \|u_2^{(0)} - \tilde{u}|_{\Omega_2}\|_{H^1(\Omega_2)}, \end{aligned}$$

where c_1 and c_2 are domain constants.

Theorem 3 *Under the same assumptions of Theorem 1, by choosing $\theta = \dfrac{2}{m + l^{-1} + 2}$, we get $\tilde{r}_{opt} = \dfrac{m + l^{-1}}{m + l^{-1} + 2}$ as the optimal value of $\tilde{r}$.*

Optimality of (5)

The estimate (5) in Lemma 1 is really optimal in a certain sense. We below explain this fact with the aid of a simple example. We consider the case where Ω is a rectangle and γ is a line segment parallel to the lateral sides of the rectangle. Specifically, we assume that, for $0 < a_1 \le a_2$ and $b > 0$,

$$\begin{cases} \Omega_1 &= \{(x_1, x_2);\ -a_1 < x_1 < 0,\ 0 < x_2 < b\}, \\ \Omega_2 &= \{(x_1, x_2);\ 0 < x_1 < a_2,\ 0 < x_2 < b\}, \\ \gamma &= \{(0, x_2);\ 0 < x_2 < b\}. \end{cases}$$

Let $\xi \in V$ and write

$$\xi = \sum_{n=1}^{\infty} c_n \phi_n, \quad c_n = c_n(\xi) = (\xi, \phi_n)_X,$$

where ϕ_n are the eigenfunctions of (7) which will appear in Appendix. In this case, the harmonic extensions $w_1 = H_1\xi$ and $w_2 = H_2\xi$ can be expressed explicitly. In particular, we have

$$\frac{\partial w_1}{\partial x_1}\Big|_{x_1=0} = \sum_{n=1}^{\infty} \sqrt{\lambda_n} c_n \coth(\sqrt{\lambda_n} a_1)\phi_n(x_2),$$

where $\sqrt{\lambda_n} = n\pi/b$ and $\coth s = (e^s + e^{-s})(e^s - e^{-s})^{-1}$. Hence

$$S_1\phi_n = \sqrt{\lambda_n} \coth(\sqrt{\lambda_n} a_1)\phi_n,$$

since $c_j = (\phi_n, \phi_j)_X = \delta_{n,j}$ (Kronecker's delta). This means that $\zeta_n^{(1)} = \sqrt{\lambda_n} \coth(\sqrt{\lambda_n} a_1)$ are the eigenvalues of S_1. In the similar way, $\zeta_n^{(2)} = \sqrt{\lambda_n} \coth(\sqrt{\lambda_n} a_2)$ are the eigenvalues of S_2. ϕ_n is the eigenfunction of S_1 and S_2 corresponding to $\zeta_n^{(1)}$ and $\zeta_n^{(2)}$, respectively. The operator H is a compact operator in X, since, from Lemma 2 in Appendix, the imbedding operator from V into X is compact. Therefore, the spectrum of H consists of only the set of the eigenvalues $\{\zeta_n^{(1)}/\zeta_n^{(2)}\}_{n=1}^{\infty}$. As a result, by Rayleigh's principle,

$$\sup_{\xi \in V} \frac{((H\xi, \xi))}{|||\xi|||^2} = \frac{\zeta_1^{(1)}}{\zeta_1^{(2)}} = \frac{\tanh(\pi a_2/b)}{\tanh(\pi a_1/b)} \equiv \tau(a_1, a_2, b).$$

In addition, we have

$$\inf_{\xi \in V} \frac{((H\xi, \xi))}{|||\xi|||^2} = \inf_{n \ge 1} \frac{\zeta_n^{(1)}}{\zeta_n^{(2)}} = 1,$$

since $\zeta_n^{(1)}/\zeta_n^{(2)}$ is a non-increasing sequence in n and is greater than 1. Therefore we obtain

$$1 \le H \le \tau(a_1, a_2, b).$$

On the other hand, both Conditions (I_m) and (I^l) are satisfied with $l = 1$ and $m = a_2/a_1$;

$$1 \le H \le \frac{a_2}{a_1}. \tag{6}$$

We now note that

$$1 < \tau(a_1, a_2, b) < \frac{a_2}{a_1},\ (b > 0) \quad \text{and} \quad \tau(a_1, a_2, b) \to \frac{a_2}{a_1},\ (b \to \infty).$$

This means that the estimate (6) by Conditions (I_m) and (I^l) is really optimal when b is sufficiently large for fixed a_1 and a_2.

Remarks

1. Numerical results to exemplify our theoretical results are presented in [Fuj97], [FKKN96], [FFS98] and [FS97].
2. Our method of analysis works for some other domain decomposition algorithms, for instance, the Neumann-Neumann method proposed in [BGLTV89].
3. The similar problem for the Stokes equations is discussed in [Sai00]. There a new important role of the inf-sup constant is revealed.

Appendix. An Equivalent Norm to $\|\cdot\|_V$

We assume that γ is a line segment on the x_2-axis. Let $\{\lambda_n\}_{n=1}^{\infty}$ be the set of the eigenvalues of the eigenvalue problem:

$$-\frac{d^2}{dx_2^2}\phi = \lambda\phi \text{ in } \gamma, \quad \phi = 0 \text{ on } \partial\gamma \tag{7}$$

and, let $\phi_n = \phi_n(x_2)$ be the eigenfunction corresponding to λ_n which is normalized as $\|\phi_n\|_X = 1$. Then each element $\xi \in X$ can be expanded by the Fourier series as follows:

$$\xi = \sum_{n=1}^{\infty} c_n\phi_n \quad \text{with } c_n = (\xi, \phi_n)_X.$$

We introduce

$$U^{1/4} = \Big\{\xi = \sum_{n=1}^{\infty} c_n\phi_n \in X;\ \ \sum_{n=1}^{\infty} c_n^2\lambda_n^{1/2} < \infty\Big\}$$

with the norm

$$\|\xi\|_{U^{1/4}} = \Big(\sum_{n=1}^{\infty} c_n^2 + \sum_{n=1}^{\infty} \lambda_n^{1/2}c_n^2\Big)^{1/2} \quad \text{for } \xi = \sum_{n=1}^{\infty} c_n\phi_n \in U^{1/4}. \tag{8}$$

The space $U^{1/4}$ forms a Hilbert space equipped with the norm $\|\cdot\|_{U^{1/4}}$. Moreover, $U^{1/4}$ coincides with the domain $D(L^{1/4})$ of the fractional power of L, where L means a minus Laplacian on γ with the zero boundary condition. Then we have, by virtue of [Fuj67], $V = U^{1/4}$ with the equivalent norm (8). This implies, in view of the closed graph theorem, that

$$C\|\xi\|_V \le \|\xi\|_{U^{1/4}} \le C'\|\xi\|_V, \quad (\forall\xi \in V).$$

Namely, we can employ $\|\cdot\|_{U^{1/4}}$ as the norm of V. This sometimes gives a better viewpoint of our discussion. For instance, the following proposition is an easy consequence of (8).

Lemma 2 *$U^{1/4} = D(L^{1/4})$ is compactly imbedded in X, if γ is a line segment.*

Proof We set

$$i_N\xi = \sum_{n=1}^{N} c_n\phi_n \quad \text{for } \xi = \sum_{n=1}^{\infty} c_n\phi_n \in X.$$

The operator i_N is a degenerate operator from $U^{1/4}$ into X. Let i be the imbedding operator from $U^{1/4}$ into X. Then we can calculate as

$$\|(i - i_N)\xi\|_X^2 = \sum_{n=N+1}^{\infty} c_n^2 \le \lambda_{N+1}^{-1/2} \sum_{n=1}^{\infty} c_n^2 \lambda_n^{1/2} \le \lambda_{N+1}^{-1/2} \|\xi\|_{U^{1/4}}^2.$$

Thus we have $\|i - i_N\|_{U^{1/4},X} \le \lambda_{N+1}^{-1/4} \to 0$ as $N \to \infty$. Since degenerate operators i_N are compact, i is also compact. ■

References

[BGLTV89]Jean-François Bourgat, Roland Glowinski, Patrick Le Tallec, and Marina Vidrascu. Variational formulation and algorithm for trace operator in domain decomposition calculations. In Tony Chan, Roland Glowinski, Jacques Périaux, and Olof Widlund, editors, *Domain Decomposition Methods*, pages 3–16, Philadelphia, PA, 1989. SIAM.

[FFS98]Hiroshi Fujita, Makoto Fukuhara, and Norikazu Saito. On the rate of convergence of iterations in the domain decomposition methods. In Zhong-Ci Shi and Masatake Mori, editors, *Proceedings of the third China-Japan seminar on numerical mathematics*, pages 30–43, Beijing and New-York, 1998. Science Press.

[FKKN96]Hiroshi Fujita, Masashi Katsurada, Asako Kobari, and Yoshiaki Nagasaka. Analytical and numerical study of convergence of the domain decomposition method, I. *Mem. Inst. Sci. Tech. Meiji Univ.*, 35(8):103–135, 1996. in Japanese.

[FS97]Hiroshi Fujita and Norikazu Saito. An analytical study of optimal speed of convergence of iterations in ddm under certain shape assumptions of domains. In *Computational Science for the 21st Century*, pages 139–148, New-York, 1997. John Wiley & Sons.

[Fuj67]Daisuke Fujiwara. Concrete characterization of the domains of fractional powers of some elliptic differential operators of the second order. *Proc. Japan Acad.*, 43:82–86, 1967.

[Fuj97]Hiroshi Fujita. Remarks on the domain-dependence of convergence rate of iterations in a certain domain decomposition method. In *Collection of Papers on Geometry, Analysis and Mathematical Physics*, pages 71–84, River Edge, NJ, 1997. World Science Publishing.

[Gri85]P. Grisvard. *Elliptic problems in nonsmooth domains.* Pitman Publishing, Boston, 1985.

[Kat76]Tosio Kato. *Perturbation Theory for Linear Operators*. Springer-Verlag, second edition, 1976.

[LM72]Jacques-Louis Lions and Enrico Magenes. *Nonhomogeneous Boundary Value Problems and Applications*, volume I. Springer, New York, Heidelberg, Berlin, 1972.

[Sai00]N. Saito. On the domain-dependence of convergence rate in a domain decomposition method for the Stokes equations. *East-West J. of Numer. Math.*, 8(1):37–55, 2000.

[Yos80]Kôsaku Yosida. *Functional Analysis*. Springer-Verlag, sixth edition, 1980.