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21. Efficient and fast numerical methods to compute fluid flows in the geophysical β plane

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Introduction

We consider a fluid flow in a rotating sphere with an unit radius. The flow is incompressible and inviscid, and covers the sphere with a constant density. This kind of flow is called a geophysical flow, since it is one of the simplest models of atmospheric flows in the earth. In practical study of geophysical flows, we are sometimes interested in a local flow in the neighborhood of a certain point in the sphere. In that case, we consider flows in a plane which is tangent to the point as an approximation model. The plane is called the geophysical β plane. In the present article, we introduce an equation which describes a motion in the β plane. And a numerical procedure to compute the equation is formulated. Furthermore, we suggest an efficient technique to compute it fast and accurately by using a fast algorithm and a parallelization based on the idea of Domain Decomposition. As an example of its application, we compute a two-dimensional flow problem in the β plane and investigate the effectiveness of the fast method and the effect of rotation on the evolution numerically.

Numerical computations of the geophysical flows play an important role in the atmospheric research, such as the weather forecast and the investigation of the environmental issues. In spite of its importance, it is not easy to obtain useful and practical results since it costs too much to compute these problems for sufficiently fine resolutions. That is why a fast and accurate numerical method is required. The purpose of the study is to give an efficient numerical method to compute such flows and to show its effectiveness by applying it to some fluid problem.

In the next section, we suggest the fast numerical method: We consider the equation of the flows in the β plane, whose detailed definition and formulation is explained. Our numerical method called the point potential vortex method is introduced. Then, some techniques to compute it fast and accurately are appearing. In the third section, we show some results of the numerical computation of a fluid flow in the β plane: (1) effectiveness of the fast method and (2) investigation of the influence of rotation on the evolution of the flow. The last section is conclusions.

Numerical methods

The equation of motion of fluids in the β plane

Now, we introduce an equation of motion of the flows in the geophysical β plane. Let ϕ and λ be a latitude and longitude of a point in the sphere, respectively. When fixing a point (λ_0, ϕ_0) in the sphere, we consider the plane which is tangent to the point and

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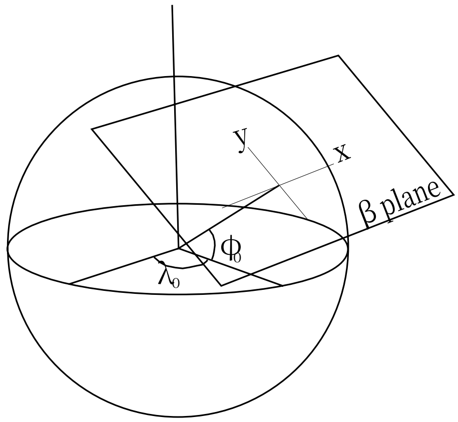

Figure 1: The β plane associated with the point (λ_0, ϕ_0) in the sphere.

introduce a new pair of variables (x, y) as follows (See Figure 1):

$$x = (\cos\phi_0)\lambda, \quad y = \phi - \phi_0, \quad (|x|, |y| << 1).$$

We define the stream function Ψ and the vorticity ω in the β plane as $\omega = \mathrm{rot}\mathbf{u}, \Delta\Psi = -\omega$. The velocity field $\mathbf{u}$ is recovered from the stream function by the formula $\mathbf{u} = (-\partial_y\Psi, \partial_x\Psi)$, and Δ is the two-dimensional Laplacian. Then, the equation of incompressible Euler flow in the β plane is given by

$$\frac{\partial}{\partial t}\Delta\Psi + \frac{\partial(\Psi, \Delta\Psi)}{\partial(x, y)} + \beta\frac{\partial\Psi}{\partial x} = 0, \tag{1}$$

where the second term is two-dimensional Jacobian:

$$\frac{\partial(a, b)}{\partial(x, y)} = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}.$$

The equation differs from the usual two-dimensional Euler equation in the effect of rotation (β-effect) of the third term. Therefore, the vorticity is no longer an invariant quantity along the trajectory of the fluid element just like in two-dimensional case. However, we can define a "potential vorticity" by

$$q = \omega + \beta y.$$

Then, substituting it to the equation (1), we obtain the following simple equations:

$$\frac{Dq}{Dt} = (\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})q = 0 \tag{2}$$

$$\Delta\Psi = -\omega, \quad (u, v) = (-\partial_y\Psi, \partial_x\Psi), \tag{3}$$

where $\frac{D}{Dt}$ represents the derivative along the trajectory of the fluid element which moves together with the fluid flows (material derivative).

What the equation (2) indicates is the potential vorticity is invariant along the trajectory of the fluid element. That means, when $(x(t), y(t))$ is a position of the fluid element at some time t, the potential vorticity at the position is given by the initial potential vorticity at $(x(0), y(0))$:

$$q(x(t), y(t), t) = q(x(0), y(0), 0) \equiv q_0.$$

Hence, the vorticity ω at the position $(x(t), y(t))$ is represented by

$$\omega(x(t), y(t), t) = q_0 - \beta y(t). \tag{4}$$

Based on the considerations in the section, we show a numerical method and some techniques to compute the evolution of the flows in the β plane fast and accurately in the following subsections.

The point potential vorticity method

At first, the velocity field is obtained by solving the Laplace equation (3). For the sake of simplicity, we impose the periodic boundary condition in the x direction. Then, the velocity field $(u, v) = (-\partial_y \Psi, \partial_x \Psi)$ are given by

$$\begin{aligned} u(x, y, t) &= -\frac{1}{2} \int \frac{\omega(\tilde{x}, \tilde{y}, t) \sinh 2\pi(y - \tilde{y})}{\cosh 2\pi(y - \tilde{y}) - \cos 2\pi(x - \tilde{x})} d\tilde{x} d\tilde{y}, \\ v(x, y, t) &= \frac{1}{2} \int \frac{\omega(\tilde{x}, \tilde{y}, t) \sin 2\pi(x - \tilde{x})}{\cosh 2\pi(y - \tilde{y}) - \cos 2\pi(x - \tilde{x})} d\tilde{x} d\tilde{y}. \end{aligned} \tag{5}$$

We must note that the integrals on the right hand side are singular integral. What follows is our numerical procedure.

1. We discretize the computational domain which includes the vorticity field. Then we obtain the discretization points $\{(x_n, y_n)\}, (n = 1, ..., N)$. We must discretize a sufficiently large region including no vorticity at the beginning because of generation of the "ghost vorticity", which we will explain later section.

2. We assume that the vorticity concentrates in the discretization points, (Point potential vortices).Then we approximate the vorticity field as follows:

$$\omega(x, y, t) = \sum_{n=1}^{N} (q_{n0} - \beta y_n) \delta(x - x_n, y - y_n), \tag{6}$$

where q_{n0} is initial potential vorticity and δ is Dirac's delta function.

3. Substituting (6) to (5), we compute the velocity field induced by the vorticity.

$$\begin{aligned} u_N(x, y, t) &= -\frac{1}{2N} \sum_{n=1}^{N} \frac{(q_{n0} - \beta y_n) \sinh 2\pi(y - y_n)}{\cosh 2\pi(y - y_n) - \cos 2\pi(x - x_n)}, \\ v_N(x, y, t) &= \frac{1}{2N} \sum_{n=1}^{N} \frac{(q_{n0} - \beta y_n) \sin 2\pi(x - x_n)}{\cosh 2\pi(y - y_n) - \cos 2\pi(x - x_n)}. \end{aligned} \tag{7}$$

4. The point potential vortices evolves by the induced velocity field:

$$\frac{dx_n}{dt} = u_N(x_n, y_n), \quad \frac{dy_n}{dt} = v_N(x_n, y_n), \quad (n = 1, ..., N).$$

We use the 4-th order Runge-Kutta method to compute the step 4. We refer the numerical scheme as the "point potential vortex method".

The fast algorithm and parallel implementation

Although the idea of the point potential vortex method is simple, it has not been easily applied to simulations of practical geophysical flows so far because of some difficulties. Here, we explain these difficulties and show some methods to overcome them.

Desingularization of the equation As we note in the previous section, the velocity field (5) is given as a singular integral. Due to the singularity, the round-off error has a seriously bad influence on numerical solutions. To get rid of the bad influence of the round-off error, we use the Krasny's desingularization technique[Kra86]. That is, taking a sufficiently small positive real number ϵ, we compute the following desingularized summation instead of (7):

$$\begin{aligned} u_N^\epsilon(x, y, t) &= -\frac{1}{2N} \sum_{n=1}^{N} \frac{(q_{n0} - \beta y_n) \sinh 2\pi(y - y_n)}{\cosh 2\pi(y - y_n) - \cos 2\pi(x - x_n) + \epsilon^2}, \\ v_N^\epsilon(x, y, t) &= \frac{1}{2N} \sum_{n=1}^{N} \frac{(q_{n0} - \beta y_n) \sin 2\pi(x - x_n)}{\cosh 2\pi(y - y_n) - \cos 2\pi(x - x_n) + \epsilon^2}. \end{aligned} \tag{8}$$

We can compute the velocity field (8) stably since they are bounded as long as $\epsilon \neq 0$.

Fast summation method Let N be the number of point potential vortices which is obtained by the discretizaion of the computational domain. The amount of computation which is required to compute the velocity field (8) for a point is $O(N)$. Therefore, it takes $O(N^2)$ operations to compute the Step 4 for all the points. Due to the rapid increase of the total amount of operations, it is difficult to use high resolution in the practical numerical computation. In order to overcome the difficulty, we apply the Draghicescu's fast algorithm. This algorithm reduces the total amount of operations to $O(N \log N)$, allowing approximation error to some extent. However, the method works well for a large number of N, since the approximation error reduces with $O(\frac{1}{N})$ and is negligible as N increases. As for the detailed algorithms and how to apply the algorithm to the periodic boundary condition, we would like the readers to refer to Draghicescu[Dra94] and Sakajo & Okamoto[SO98], respectively.

Parallel implementation We implement the fast numerical algorithm to a parallel computer. Now when the parallel computer has p CPUs, we assign $\frac{N}{p}$ point potential vortices to each processor and then compute the evolutions concurrently. Since the point potential vortices are obtained by discretizing the computational domain, the parallelization would be one of the Domain Decomposition techniques. In the present computation, we use a distributed parallel computer with four DEC Alpha 21264 processors.

N+M	8192	32768	131072
direct	143	2288	36789
fast	25	120	463
parallel+fast	12	51	156

Table 1: The elapsed time to compute the velocity field (8) in seconds

Results

We apply the point potential vortex method to the computation of a vortex sheet in the β plane. A vortex sheet is a surface across which the velocity of the fluid changes discontinuously. That means that initially the vorticity exists only in the vortex sheet, and outside of the vortex sheet there exists no vorticity. In the two-dimensional β plane, the vortex sheet is represented by a one-parameter curve: $(x(\Gamma, t), y(\Gamma, t))$, where Γ is circulation parameter along the sheet and t is time. We imposed the periodic boundary condition on the sheet as follows,

$$x(\Gamma + 1, t) = x(\Gamma, t) + 1, \quad y(\Gamma + 1, t) = y(\Gamma, t), \quad (0 \le \Gamma < 1).$$

A flat vortex sheet $(x, y) = (\Gamma, 0)$ is a steady state. We add a small disturbance to the steady state and take it as an initial condition of the numerical computation:

$$x(\Gamma, t) = \Gamma + 0.01 \sin 2\pi\Gamma, \quad y(\Gamma, t) = -0.01 \sin 2\pi\Gamma.$$

If we consider the ordinary two-dimensional vortex sheet, we have only to discretize the vortex sheet since the vorticity is invariant. However, since not the vorticity but the potential vorticity is invariant in the β plane approximation, the vorticity could be created as a result of the vertical movement of the point potential vortices even if it has no vorticity at the beginning. The created vorticity is called the ghost vorticity. Therefore, we have to discretize the sufficiently large regions to the y directions in this case by taking the creation of ghost vorticity into considerations.

We discretize the vortex sheet along the sheet and obtain N point potential vortices $(x_n, y_n), (n = 1, ..., N)$, whose initial position is

$$x_n(0) = \frac{n}{N} + 0.01 \sin 2\pi \frac{n}{N}, \quad y_n(0) = -0.01 \sin 2\pi \frac{n}{N}, \quad (n = 1, ..., N).$$

and initial potential vorticity is $q_{n0} = \frac{1}{N} + \beta y_n(0)$. We also discretize the outer regions by grids and obtain M point potential vortices, $(\tilde{x}_n(t), \tilde{y}_n(t))$, whose initial potential vorticity is $\tilde{q}_{n0} = 0 + \beta \tilde{y}_n(0)$, $(1 \le n \le M)$.

Effectiveness of the fast computation

We show the effectiveness of the fast algorithm and parallelization. The desingularization parameter is fixed to 0.1. Table 1 shows the elapsed time to compute the velocity field (8) in second when we change the number of discretization $N + M$. When we use the direct summation of $O(N^2)$, the computational time increases rapidly. On the other hand, the time is very small when we use the fast summation method. It takes

N+M	8192	32768	131072
error	2.26e-07	5.35e-08	9.46e-09

Table 2: Maximum approximation error of the fast algorithm

about 80 times faster than the direct summation when $N = 131072$ point potential vortices are used. Moreover, as a result of the implementation of the fast algorithm to the parallel computer, we achieve more than 230 times faster computation for $N = 131072$ points are used. Table 2 shows the maximum approximation error of the fast algorithm. As $N + M$ increase, the error gets smaller. The fast algorithm yields very accurate computation for a large number of point potential vortices.

These two results indicates that the more we use point potential vortices the more accurate and faster we can compute the velocity field.

An application - a vortex sheet in the β plane

We discretize the vortex sheet by 65536 points and the other no vorticity region $[0, 1] \times [-2.0, 2.0]$ by 128×512 grid points. The desingularization parameter ϵ is 0.1. We change only the parameter β to see the effect of rotation to the evolution.

Figure 2 shows the time evolution of the vortex sheet: (a) $\beta = 0$ (no rotation), (b) $\beta = 5$ (mild rotation) and (c) $\beta = 10$ (fast rotation). At first, when there exists no β-effect (column (a)), it evolves in the same way as the two-dimensional vortex sheet, which Krasny computed[Kra86]. Nearly flat vortex sheet becomes unstable and roll-up and then generates the spiral structure in the middle of the region. Next, when $\beta = 5$ (column (b)), it forms the spiral structure as well. However, the center of the spiral moves toward the northwest. This movement is due to the effect of rotation, which is well-known as the Rosby effect. Note that the number of winding of the spiral becomes few. At last, when $\beta = 10$ (column (c)), it begins forming the spiral structure at $t = 1.0$ but it hardly grows. Instead, an another spiral structure is generated at $t = 2.0$. The result indicates that faster rotation results in the appearance of the new spiral structure, which would be a new feature of the β-effect.

Conclusions

We suggest the point potential vortex method to compute the geophysical fluids in the β plane numerically. The fast summation method and the implementation to the parallel computer based on the Domain Decomposition approach makes us possible to execute the numerical computation accurately and fast. The hybrid combination of these two fast numerical method brings us a possibility to try the numerical computations of various practical geophysical flows.

As one of the examples, we apply the numerical scheme to the computation of the vortex sheet in the β plane. We find the northwestward movement of the spiral structure and the appearance of the new spiral structure due to the effect of rotation. The analytic investigation of these phenomena remains in the future.

The point potential vortex method could be extended to the case of the flows in the rotating sphere. The formulation is the same as the present method. That is, the potential vorticity is also preserved along the trajectory of the fluid element. However, since there is no fast summation method to compute the velocity field fast, the extension wouldn't be completed. The development of the fast algorithm for the velocity field in the sphere is challenging.

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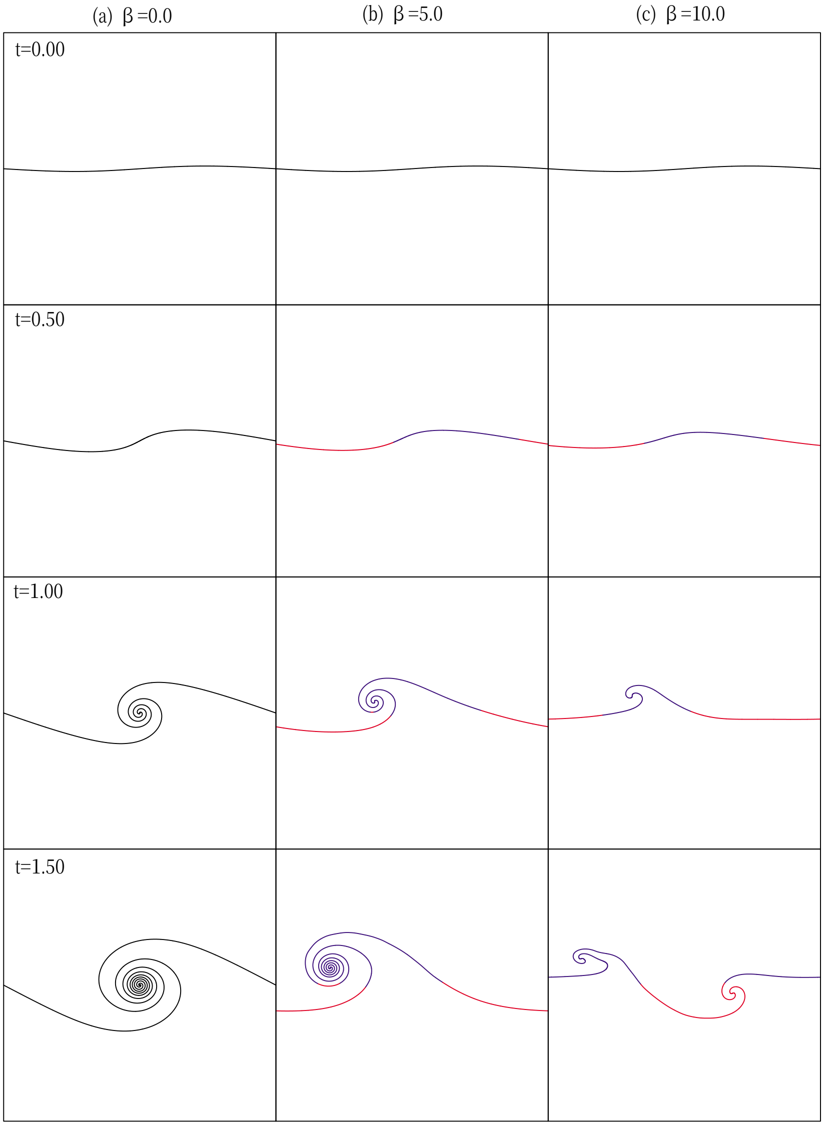

Figure 2: The time evolution of the vortex sheet in the β plane. (a) $\beta = 0.0$, (b) $\beta = 5.0$ and (c) $\beta = 10.0$