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24. FEM-FSM Combined Method for 2D Exterior Laplace and Helmholtz Problems

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Introduction

Consider the Poisson equation $-\Delta u = f$ in a planar exterior domain of a bounded domain $\mathcal{O}$. Assume that $f = 0$ in the outside of a disc with sufficiently large diameter. The solution u is assumed to be bounded at infinity. Discretizing the problem, we employ the finite element method (FEM, in short) inside the disc, and the charge simulation method (CSM, in short) outside the disc. A result of mathematical analysis for this FEM-CSM combined method is reported in this paper.

CSM is a typical example of the fundamental solution method (FSM, in short), through which the solution of homogeneous partial differential equation is approximated as a linear combination of fundamental solutions of differential operator. Hence the combined method for 2D exterior Laplace problem is extendable to the planar exterior reduced wave equations. Our discretization procedure for the reduced wave equation is also described in the paper.

Boundary bilinear forms of Steklov type for exterior Laplace problems and its CSM-approximation form

Let D_a be the interior of the disc with radius a having the origin as its center, and let Γ_a be the boundary of D_a. Let $\Omega_e = (D_a \cup \Gamma_a)^C$, which is said to be the exterior domain. We use the notation $\mathbf{r} = \mathbf{r}(\theta)$ for the point in the plane corresponding to the complex number $re^{i\theta}$ with $r = |\mathbf{r}|$ where $|\mathbf{r}|$ is the Euclidean norm of $\mathbf{r} \in \boldsymbol{R}^2$. Similarly we use $\mathbf{a} = \mathbf{a}(\theta)$, and $\vec{\rho} = \vec{\rho}(\theta)$, corresponding to $ae^{i\theta}$ with $a = |\mathbf{a}|$, and $\rho e^{i\theta}$ with $\rho = |\vec{\rho}|$, respectively.

Fix a positive integer N. Set

$$\theta_1 = \frac{2\pi}{N}, \qquad \theta_j = j\theta_1 \quad \text{for} \quad j \in \boldsymbol{Z}.$$

Fix a positive number ρ so as to satisfy $0 < \rho < a$. Let

$$\vec{\rho}_j = \vec{\rho}(\theta_j), \quad \mathbf{a}_j = \mathbf{a}(\theta_j), \quad 0 \le j \le N-1.$$

The points $\vec{\rho}_j$, and $\mathbf{a}_j$, are said to be the source, and the collocation, points, respectively. The arrangement of the set of source points and collocation points introduced as above is called **the equi-distant equally phased arrangement of source points and collocation points**, in this paper.

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For functions $u(\mathbf{a}(\theta))$ and $v(\mathbf{a}(\theta))$ of $H^{1/2}(\Gamma_a)$, let us introduce the boundary bilinear form of Steklov type for exterior Laplace problem through the following formula:

$$b(u,v) = 2\pi \sum_{n=-\infty}^{\infty} |n| f_n \overline{g_n},$$

where f_n, and g_n, are continuous Fourier coefficients of $u(\mathbf{a}(\theta))$, and $v(\mathbf{a}(\theta))$, respectively. Namely f_n is defined through the following formula:

$$f_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\mathbf{a}(\theta)) e^{-in\theta} d\theta.$$

It is to be noted the following fact:

If $u(\mathbf{a}(\theta))$ is the boundary value on Γ_a of the function $u(\mathbf{r})$ satisfying the following boundary value problem (E) *of* (1) *with $\varphi = u(\mathbf{a}(\theta))$:*

$$(\mathrm{E}) \quad \begin{cases} -\Delta u = 0 & \text{in} \quad \Omega_e, \\ u = \varphi & \text{on} \quad \Gamma_a, \\ \sup\limits_{\Omega_e} |u| < \infty, \end{cases} \tag{1}$$

then

$$b(u,v) = -\int_{\Gamma_a} \frac{\partial u}{\partial r} v d\Gamma. \tag{2}$$

(In (2), $d\Gamma$ is the curve element of Γ_a. Namely $d\Gamma = a d\theta$ in the polar coordinate expression.)

A CSM approximate form for $b(u,v)$, which is denoted by $\overline{b}^{(N)}(u,v)$, is represented through the following formula (3):

$$\overline{b}^{(N)}(u,v) = -\frac{2\pi}{N} \sum_{j=0}^{N-1} \frac{\partial u^{(N)}(\mathbf{a}_j)}{\partial r} v^{(N)}(\mathbf{a}_j), \tag{3}$$

where $u^{(N)}(\mathbf{r})$, and $v^{(N)}(\mathbf{r})$, are CSM-approximate solutions for $u(\mathbf{r})$ satisfying (E) of (1) with $\varphi = u(\mathbf{a}(\theta))$, and $\varphi = v(\mathbf{a}(\theta))$, respectively. Namely $u^{(N)}(\mathbf{r})$ is determined through the following problem $(\mathrm{E}^{(\mathrm{N})})$ of (4) with $f(\mathbf{a}(\theta)) = u(\mathbf{a}(\theta))$:

$$(\mathrm{E}^{(\mathrm{N})}) \quad \begin{cases} u^{(N)}(\mathbf{r}) = \sum\limits_{j=0}^{N-1} q_j G_j(\mathbf{r}) + q_N, \\ u^{(N)}(\mathbf{a}_j) = f(\mathbf{a}_j), \quad 0 \le j \le N-1, \\ \sum\limits_{j=0}^{N-1} q_j = 0, \end{cases} \tag{4}$$

where

$$G_j(\mathbf{r}) = E(\mathbf{r} - \vec{\rho}_j) - E(\mathbf{r}), \qquad E(\mathbf{r}) = -\frac{1}{2\pi} \log r.$$

Problem ($\mathrm{E}^{(\mathrm{N})}$) of (4) is to find $N+1$ unknowns q_j, $0 \le j \le N$, and it is uniquely solvable for any fixed $\rho \in (0, a)$. See [KO88], [Ush98a], and [Ush98b].

Let us use the parameter γ as

$$\gamma = \frac{\rho}{a},$$

and let

$$N(\gamma) = \frac{\log 2}{-\log \gamma}\ .$$

Modifying the treatment in [KO88] appropriately, we have the following Theorem:

Theorem 1 *Fix a positive number b, $0 < b < a$. Let $u(\mathbf{r})$ be harmonic in a domain containing the exterior domain of the disc with radius b having the origin as its center. And let $u^{(N)}(\mathbf{r})$ be the solution of the problem ($\mathrm{E}^{(\mathrm{N})}$) of (4) with the data $f(\mathbf{a}(\theta)) = u(\mathbf{a}(\theta))$. Let $N \ge N(\gamma)$. Then there exist constants $B > 0$ and $\beta \in (0, 1)$, independent of u (with the property above) and N, such that the following two estimates are valid:*

$$\max_{\mathbf{r} \in \overline{\Omega}_e} \left| u(\mathbf{r}) - u^{(N)}(\mathbf{r}) \right| \le B \cdot \beta^N \cdot \max_{|\mathbf{r}|=b} |u(\mathbf{r})|\,,$$

$$\max_{\mathbf{r} \in \overline{\Omega}_e} \left| \mathrm{grad}\ u(\mathbf{r}) - \mathrm{grad}\ u^{(N)}(\mathbf{r}) \right|_{\mathbf{R}^2} \le B \cdot \beta^N \cdot \max_{|\mathbf{r}|=b} |u(\mathbf{r})|\,.$$

FEM-CSM combined method for exterior Laplace problems

Fix a simply connected bounded domain $\mathcal{O}$ in the plane. Assume that the boundary $\mathcal{C}$ of $\mathcal{O}$ is sufficiently smooth. The exterior domain of $\mathcal{C}$ is denoted by Ω. Fix a function $f \in L^2(\Omega)$ with the property that the support of f, $\mathrm{supp}(f)$, is bounded. Choose a so large that the open disc D_a may contain the union $\mathcal{O} \cup \mathrm{supp}(f)$ in its interior. The following Poisson equation (E) of (5) is employed as a model problem.

$$(\mathrm{E}) \quad \left\{ \begin{array}{rcll} -\Delta u &=& f & \text{in} \quad \Omega, \\ u &=& 0 & \text{on} \quad \mathcal{C}, \\ \sup\limits_{|\mathbf{r}|>a} |u| &<& \infty. & \end{array} \right. \tag{5}$$

The intersection of the domain Ω and the disc D_a is said to be the interior domain, denoted by Ω_i: $\Omega_i = \Omega \cap D_a$. Consider the Dirichlet inner product $a(u, v)$ for $u, v \in H^1(\Omega_i)$:

$$a(u, v) = \int_{\Omega_i} \mathrm{grad}\ u\ \mathrm{grad}\ v\ d\Omega.$$

Since the trace $\gamma_a v$ on Γ_a is an element of $H^{1/2}(\Gamma_a)$ for any $v \in H^1(\Omega_i)$, the boundary bilinear form of Steklov type $b(u, v)$ is well defined for $u, v \in H^1(\Omega_i)$. Therefore we can define a continuous symmetric bilinear form:

$$t(u, v) = a(u, v) + b(u, v)$$

for $u, v \in H^1(\Omega_i)$. Let $F(v)$ be a continuous linear functional on $H^1(\Omega_i)$ defined through the following formula:

$$F(v) = \int_{\Omega_i} fv \, d\Omega.$$

A function space V is defined as follows:

$$V = \left\{ v \in H^1(\Omega_i) : v = 0 \quad \text{on} \quad \mathcal{C} \right\}.$$

Using these notations, the following weak formulation problem (Π) of (6) is defined.

$$(\Pi) \quad \begin{cases} t(u,v) = F(v), & v \in V, \\ u \in V. \end{cases} \tag{6}$$

Admitting the equivalence between the equation (E) of (5) and the problem (Π) of (6), we consider the problem (Π) of (6) and its approximate ones hereafter.

Fix a positive number ρ so as to satisfy $0 < \rho < a$. For a fixed positive integer N, set the points $\vec{\rho}_j, \mathbf{a}_j, 0 \le j \le N-1$, as the equi-distant equally phased arrangement of source points and collocation points.

A family of finite dimensional subspaces of V :

$$\{V_N : N = N_0, N_0 + 1, \dots\}$$

is supposed to have the following properties:

$$(\mathrm{V_N} - 1) \quad V_N \subset C(\overline{\Omega_i}).$$

$$(\mathrm{V_N} - 2) \quad \begin{cases} \textit{For any } v \in V_N, \ v(\mathbf{a}(\theta)) \textit{ is an equi-distant piecewise linear} \\ \textit{continuous } 2\pi\textit{-periodic function with respect to } \theta. \end{cases}$$

$$(\mathrm{V_N} - 3) \quad \min_{v \in V_N} a(v - v_N) \le \frac{C}{N} ||v||_{H^2(\Omega_i)}, \qquad v \in V \cap H^2(\Omega_i).$$

In the property $(\mathrm{V_N} - 3)$, C is a constant independent of N and v, and

$$a(v) = a(v,v)^{1/2}, \quad v \in V.$$

To construct a family $\{V_N\}$ with the conditions $(\mathrm{V_N} - 1)$, $(\mathrm{V_N} - 2)$ and $(\mathrm{V_N} - 3)$, we employ the curved element technique due to [Zlá73] .

For $u, v \in H^1(\Omega_i) \cap C(\overline{\Omega_i})$, we define a bilinear form $\overline{t}^{(\mathrm{N})}(u,v)$ as follows.

$$\overline{t}^{(N)}(u,v) = a(u,v) + \overline{b}^{(N)}(u,v).$$

A family of approximate problems $(\overline{\Pi}^{(\mathrm{N})})$ of (7) is stated as follows.

$$(\overline{\Pi}^{(\mathrm{N})}) \quad \begin{cases} \overline{t}^{(\mathrm{N})}(\overline{u}_N, v) = F(v), & v \in V_N, \\ \overline{u}_N \in V_N. \end{cases} \tag{7}$$

We can show the following error estimate:

Theorem 2 *Suppose that* supp(f) *is contained in a disc* D_b *with the radius* $b(< a)$ *having the origin as its center. Let the function* $D(\xi)$ *of* $\xi \in (0,1)$ *be defined through*

$$D(\xi) = \frac{\xi}{(1-\xi)^3}.$$

Let $N \geq N(\gamma)$*. Then there is a constant* C *such that*

$$||u - \overline{u}_N||_{H^1(\Omega_i)} \leq C \left\{ B\beta^N + \frac{1 + D(\frac{b}{a})}{N} \right\} ||f||_{L^2(\Omega_i)},$$

where the constants B *and* $\beta \in (0,1)$ *are described in Theorem 1 for the set of parameters* $\{a, \rho, b\}$*. In the above, the constant* C *is independent of the inhomogeneous data* f *and* N*.*

Reduced wave problem in the outside of an open disc

Let k be the length of the wave number vector. Consider the following reduced wave problem ($\mathrm{E_f}$) of (8) in the exterior domain Ω_e of the circle Γ_a with radius a having the origin as its center.

$$(\mathrm{E_f}) \quad \left\{ \begin{array}{rcl} -\Delta u - k^2 u &=& 0 \ \text{ in } \Omega_e, \\ u &=& f \ \text{ on } \Gamma_a, \\ \lim\limits_{r\to\infty} \sqrt{r} \left\{ \frac{\partial u}{\partial r} - iku \right\} &=& 0. \end{array} \right. \tag{8}$$

In the above, f is a complex valued continuous function on Γ_a.

The solution $u = u(\mathbf{r})$ of the problem ($\mathrm{E_f}$) of (8) is represented as

$$u = \sum_{n=-\infty}^{\infty} f_n \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} e^{in\theta},$$

where f_n is the continuous Fourier coefficient of the function $f(\mathbf{a}(\theta))$, and $H_n^{(1)}(z)$ is the n-th Hankel function of the first kind.

The boundary bilinear form $b(u,v)$ of Steklov type corresponding to the problem ($\mathrm{E_f}$) of (8) is given by the following formula:

$$b(u,v) = 2\pi \sum_{n=-\infty}^{\infty} \mu_{|n|} f_n \overline{g}_n,$$

where

$$\mu_n = k \frac{\dot{H}_n^{(1)}(ka)}{H_n^{(1)}(ka)} \quad \text{with} \quad \dot{H}_n^{(1)}(z) = \frac{d}{dz} H_n^{(1)}(z) \quad \text{for} \quad n = 0, 1, 2, \ldots .$$

FSM approximate problem for the reduced wave problem in the outside of an open disc

Fix a positive number ρ so as to satisfy $0 < \rho < a$. For a fixed positive integer N, set the points $\vec{\rho}_j, \mathbf{a}_j, 0 \le j \le N-1$, as the equi-distant equally phased arrangement of source points and collocation points.

The FSM approximate problem $(\mathrm{E}_\mathrm{f}^{(\mathrm{N})})$ of (9) for the problem (E_f) of (8) in the case of equi-distant equally phased arrangement of source points and collocation points is defined through the following:

$$(\mathrm{E}_\mathrm{f}^{(\mathrm{N})}) \quad \begin{cases} u^{(N)}(\mathbf{r}) &= \sum_{j=0}^{N-1} q_j G_j(\mathbf{r}), \\ u^{(N)}(\mathbf{a}_j) &= f(\mathbf{a}_j),\ 0 \le j \le N-1. \end{cases} \tag{9}$$

We use basis functions $G_j(\mathbf{r})$ in this problem represented as follows, with the use of the constant multiple of the fundamental solution of Helmholtz equation, $H_0^{(1)}(kr)$,

$$G_j(\mathbf{r}) = H_0^{(1)}(k|re^{i\theta} - \rho e^{i\theta_j}|), \quad 0 \le j \le N-1.$$

FSM approximate form for the boundary bilinear form of Steklov type

Setting

$$g(\theta) = H_0^{(1)}(k|ae^{i\theta} - \rho|),$$

we define for $l \in \boldsymbol{Z}$,

$$g_l = g(\theta_l).$$

The two-sided infinite sequence $\{g_l : l = 0, \pm 1, \pm 2, \ldots\}$ has the period N. Further it is symmetric with respect to $N/2$. A **wave propagation matrix** G is defined through

$$G = (g_{jk})_{0 \le j,k \le N-1}, \qquad g_{jk} = g_{k-j}, \quad 0 \le j,k \le N-1.$$

It is to be noted that the matrix G is a complex valued symmetric cyclic square matrix of order N. The problem $(\mathrm{E}_\mathrm{f}^{(\mathrm{N})})$ of (9) is represented as

$$(\mathbf{E}) \quad G\mathbf{q} = \mathbf{f}, \qquad \text{with} \quad \mathbf{q} = (q_j)_{0 \le j \le N-1}, \quad \mathbf{f} = (f(\mathbf{a}_j))_{0 \le j \le N-1}.$$

Denote eigenvalues of the matrix G by $\lambda_j,\ 0 \le j \le N-1$. Then we have the following representation:

$$\lambda_j = \sum_{l=0}^{N-1} g_l \omega^{jl}, \quad 0 \le j \le N-1, \quad \text{with} \quad \omega = e^{i\theta_1}.$$

All the eigenvalues of G differ from zero if and only if the matrix G is regular. Therefore the problem $(\mathrm{E}_\mathrm{f}^{(\mathrm{N})})$ of (9) is uniquely solvable if and only if the following condition holds good:

$$\lambda_j \ne 0, \quad 0 \le j \le N-1.$$

Assuming the above condition, define an FSM approximate boundary bilinear form $\overline{b}^{(N)}(u,v)$ of the boundary bilinear form $b(u,v)$ through the same formula (3) as in the case of exterior Laplace problem, in which $u^{(N)}(\mathbf{r})$, and $v^{(N)}(\mathbf{r})$, are solutions of the FSM approximate problem $(\mathrm{E_f^{(N)}})$ of (9) with the boundary data $f = u(\mathbf{a}(\theta))$, and $f = v(\mathbf{a}(\theta))$, respectively.

FEM-FSM combined method for the reduced wave problem in the exterior of a general scattering body

Fix a simply connected bounded domain $\mathcal{O}$ in the plane. Assume that the boundary $\mathcal{C}$ of $\mathcal{O}$ is sufficiently smooth. The exterior domain of $\mathcal{C}$ is denoted by Ω. Let g be a function representing the plane wave with the wave number vector (l,m). More precisely, set

$$g(x,y) = e^{i(lx+my)}, \qquad l^2 + m^2 = k^2.$$

Consider the following reduced wave problem (E) of (10).

$$(\mathrm{E}) \quad \left\{ \begin{array}{rcll} -\Delta u - k^2 u & = & 0 & \text{in} \quad \Omega, \\ u + g & = & 0 & \text{on} \quad \mathcal{C}, \\ \lim\limits_{r\to\infty} \sqrt{r}\left\{\frac{\partial u}{\partial r} - iku\right\} & = & 0. & \end{array} \right. \qquad (10)$$

As in the case of Poisson equation in the second section, the intersection of the domain Ω and the disc D_a is said to be the interior domain, denoted by Ω_i.

For complex valued functions $u, v \in H^1(\Omega_i)$, consider the Dirichret inner product $a(u,v)$:

$$a(u,v) = \int_{\Omega_i} \text{grad } u \text{ grad } \overline{v} \, d\Omega,$$

where $\overline{v}$ represents the complex conjugate of v. Further the L^2 inner product for $u, v \in L^2(\Omega_i)$ is denoted by $m(u,v)$:

$$m(u,v) = \int_{\Omega_i} u\overline{v} \, d\Omega.$$

Since the trace $\gamma_a v$ on Γ_a is an element of $H^{1/2}(\Gamma_a)$ for any $v \in H^1(\Omega_i)$, we can see the boundary bilinear form of Steklov type $b(u,v)$ is well defined for $u, v \in H^1(\Omega_i)$ (See, for example, [Zeb92].). Therefore we can define a continuous bilinear form:

$$t(u,v) = a(u,v) - k^2 m(u,v) + b(u,v)$$

for $u, v \in H^1(\Omega_i)$. Hereafter, denoting the function space $H^1(\Omega_i)$ by W, let

$$V = \{v \in W : v = 0 \quad \text{on} \quad \mathcal{C}\}.$$

With these notations, the following weak formulation problem (Π) of (11) is defined.

$$(\Pi) \quad \left\{ \begin{array}{ll} t(u,v) = 0, & v \in V, \\ u + g = 0 & \text{on} \quad \mathcal{C}, \\ u \in W. & \end{array} \right. \qquad (11)$$

Admitting the equivalence between the equation (E) of (10) and the problem (Π) of (11), we consider the problem (Π) of (11) and its approximate ones hereafter.

A family of finite dimensional subspaces of W,

$$\{W_N : N = N_0, N_0 + 1, \dots\},$$

is supposed to have the following properties:

$$(\mathrm{W_N} - 1) \quad W_N \subset C(\overline{\Omega_i}).$$

$$(\mathrm{W_N} - 2) \quad \begin{cases} \textit{For any } v \in W_N,\ v(\mathbf{a}(\theta)) \textit{ is an equi-distant piecewise linear} \\ \textit{continuous } 2\pi\textit{-periodic function with respect to } \theta. \end{cases}$$

Define an approximate space V_N of V through

$$V_N = W_N \cap V.$$

For $u, v \in H^1(\Omega_i) \cap C(\overline{\Omega_i})$, set

$$\bar{t}^{(N)}(u,v) = a(u,v) - k^2 m(u,v) + \bar{b}^{(N)}(u,v).$$

Fix an element g_N of W_N which coincides with g at the nodal points on the interior boundary $\mathcal{C}$.

Now we can set the following approximate problem $(\overline{\Pi}^{(\mathrm{N})})$ of (12).

$$(\overline{\Pi}^{(\mathrm{N})}) \quad \begin{cases} \bar{t}^{(N)}(\bar{u}_N, v) = 0, & v \in V_N, \\ \bar{u}_N + g_N = 0 & \text{on} \quad \mathcal{C}, \\ \bar{u}_N \in W_N. \end{cases} \tag{12}$$

Thus we have formulated an FEM-FSM combined method for the reduced wave problem in the exterior of a general scattering body.

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