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25. A Parallel Interface Preconditioner for the Mortar Element Method in Case of Jumping Coefficients

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Introduction

The paper is devoted to designing an interface preconditioner for the mortar element method. After brief overview of the problem in Introduction, we discuss the mortar element method with different types of the Lagrange multiplier spaces. Next, we consider the domain decomposition technique for the solution of mortar element systems and outline the general framework of the solution of saddle-point systems which result from the mortar element system. In the last two sections, we constuct the interface preconditioner for the saddle-point Schur complement, which is the goal of the paper, and we present numerical experiments illustrating the basic properties of the interface preconditioner.

Designing the interface preconditioner is one of the most difficult problems in the mortar element method. In this paper we continue development of the Dirichlet-Dirichlet preconditioner [DA99, KV99]. We extend the method to the case of arbitrary type of Lagrange multiplier space and large jumps of coefficients. The proposed algorithm possesses natural parallelism. It is illustrated on a set of numerical experiments.

The mortar element method with Lagrange multipliers

We consider a macro-hybrid P_1 finite element method with respect to a decomposition of the computational domain $\Omega \subset \mathbb{R}^3$ into m nonoverlapping regular shaped polyhedral subdomains Ω_i, $1 \le i \le m$, i.e., $\bar{\Omega} = \cup_{i=1}^m \bar{\Omega}_i$, $\Omega_i \cap \Omega_j = \emptyset, 1 \le i \ne j \le m$. We assume this decomposition to be geometrically conforming in the sense that if $\bar{\Theta}_{ij} = \bar{\Omega}_i \cap \bar{\Omega}_j \ne \emptyset, i \ne j$, then $\bar{\Theta}_{ij}$ is either a common vertex, a common edge, or a common face of Ω_i and Ω_j. We refer to $\mathcal{S} := \bigcup \{\bar{\Theta}_{ij} : \mid \Theta_{ij} \mid \ne 0, 1 \le i \ne j \le m\}$ as the skeleton of the decomposition. We further decompose the skeleton, according to

$$\mathcal{S} = \bigcup_{k=1}^{K} \bar{\gamma}_k = \bigcup_{k=1}^{K} \bar{\delta}_k, \tag{1}$$

into the so-called mortars γ_k and non-mortars δ_k, $1 \le k \le K$, where each mortar is the entire open face of two adjacent subdomains $\Omega_{M(k)}$ and $\Omega_{\bar{M}(k)}, 1 \le M(k) \ne \bar{M}(k) \le m$, i.e., $\gamma_k = \Theta_{M(k),\bar{M}(k)}$. The non-mortars δ_k denote the corresponding opposite side

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of the mortars γ_k. Choosing $H^{1/2}(\delta_k)$ as the trace space of $H^1(\Omega_{\bar{M}(k)})$ on δ_k, we introduce

$$V := \prod_{i=1}^{m} H^1(\Omega_i), \quad \Lambda := \prod_{k=1}^{K} H^{-1/2}(\delta_k).$$

We consider an elliptic problem in the macro-hybrid primal variational formulation [BF91]: Find $(u,\lambda) \in V \times \Lambda$ such that

$$\begin{array}{rcll} a(u,v) + b(\lambda,v) & = & l(v), & v \in V, \\ b(\mu,u) & = & 0, & \mu \in \Lambda. \end{array} \tag{2}$$

Here, the bilinear forms $a(\cdot,\cdot) : V \times V \to \mathbb{R}, b(\cdot,\cdot) : \Lambda \times V \to \mathbb{R}$ and the functional $l(\cdot) : V \to \mathbb{R}$ are given by

$$a(v,w) := \sum_{i=1}^{m} a_i(v,w), \quad a_i(v,w) := \int_{\Omega_i} [\rho \nabla v \cdot \nabla w + \varepsilon v w] dx,$$

$$b(\mu,v) := \sum_{k=1}^{K} b_k(\mu,v), \quad b_k(\mu,v) := < \mu, [v]_J >_{\delta_k}, \quad l(v) := \sum_{i=1}^{m} \int_{\Omega_i} f v dx,$$

where $[v]_J|_{\delta_k} := v|_{\Omega_{\bar{M}(k)}} - v|_{\Omega_{M(k)}}$, and $< \cdot,\cdot >_{\delta_k}$ refers to the dual pairing between $H^{-1/2}(\delta_k)$ and $H^{1/2}(\delta_k), f \in L_2(\Omega)$. For simplicity we assume that $\varepsilon(x) = \varepsilon_i \equiv const_i > 0$, $\rho(x) = \rho_i \equiv const_i > 0$ in Ω_i, $i = 1,\dots,m$.

Let Ω_i^h be a conformal simplicial triangulation of Ω_i, $i = 1,\dots,m$. We denote by V_i^h the space of P_1 conforming finite elements on Ω_i associated with triangulation Ω_i^h. It is obvious that the traces of $V_{\bar{M}(k)}^h$ and $V_{M(k)}^h$ on δ_k are, generally speaking, different.

We denote $\delta_k^h = \Omega_{\bar{M}(k)}^h \cap \delta_k$ and consider three choices of the discrete Lagrange multiplier space associated with continuous piecewise linear [BM94, Kuz95], piecewise constant [AT95] and the Dirac functions, respectively:

$$\Lambda^h(\delta_k) \;:=\; \Big\{ v = \sum_{i \in \{\mathcal{N}(\delta_k^h)\}} \beta_i \psi_i, \; \psi_i = \sum_{j \in \{\mathcal{B}(\delta_k^h)\}} \frac{(\varphi_i,\varphi_j)_{L_2(\delta_k)}}{\sum\limits_{l \in \{\mathcal{N}(\delta_k^h)\}} (\varphi_l,\varphi_j)_{L_2(\delta_k)}} \varphi_j + \varphi_i \Big\} \tag{3}$$

$$\Lambda^h(\delta_k) \;:=\; \Big\{ v|_\sigma \in P_0(\sigma), \; \sigma \in \mathcal{D}(\delta_k^h) \Big\} \tag{4}$$

$$\Lambda^h(\delta_k) \;:=\; \Big\{ v = \sum_{i \in \{\mathcal{N}(\delta_k^h)\}} \beta_i \boldsymbol{\delta}(x_i) \Big\} \tag{5}$$

Here $\mathcal{N}(\delta_k^h)$ is the set of inner nodes of δ_k^h, $\mathcal{B}(\delta_k^h)$ is the set of the nodes of δ_k^h lying on $\partial\delta_k$, and $\mathcal{D}(\delta_k^h)$ is the mesh dual to δ_k^h [Fei93]. The element σ of $\mathcal{D}(\delta_k^h)$ with a center node x_i is defined via the baricentric coordinates on elements e of δ_k^h surrounding x_i, $\lambda_j(e), j = 1,2,3 : \sigma = \{x | \lambda_j(x) \geq \max\limits_{l \neq j} \lambda_l, \lambda_j(x_i) = 1\}$. Notation $\boldsymbol{\delta}(x)$ stands for the Dirac function and φ_i stands for the standard Courant basis function, while $\{A\}$ denotes the set of indexes for nodes belonging to A.

Setting

$$V^h := \prod_{i=1}^{m} V_i^h \quad \text{and} \quad \Lambda^h := \prod_{k=1}^{K} \Lambda^h(\delta_k),$$

the mortar finite element approximation of (2) requires the computation of $(u, \lambda) \in V^h \times \Lambda^h$ such that

$$\begin{aligned} a(u,v) + b(\lambda, v) &= l(v),\ v \in V^h, \\ b(\mu, u) &= 0, \quad \mu \in \Lambda^h. \end{aligned} \tag{6}$$

We note that in contrast to (3),(4), in case (5) $\Lambda^h \not\subset \Lambda$ and the mortar finite elements are nonconforming ones. Since the paper is addressing a solution procedure for (6), we do not discuss approximation properties of (6) here.

In the sequel, we denote by $A \sim B$ the spectral equivalence between the matrices A and B or proportionality between values A and B, and by c or C, with or without subscripts, positive constants.

Domain decomposition solver

General framework

The finite element problem (6) results in the system of linear algebraic equations in the saddle-point form:

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} A_1 & & & 0 & B_1^T \\ & \cdot & & & \cdot \\ & & \cdot & & \cdot \\ & & & \cdot & \cdot \\ 0 & & & A_m & B_m^T \\ B_1 & \cdot & \cdot & B_m & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_m \\ \lambda \end{bmatrix} = \begin{bmatrix} f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_m \\ 0 \end{bmatrix}, \tag{7}$$

where the block representations of the matrices A and B are associated with the definition of the spaces V^h and Λ^h, while the matrix A and the vector f are specified by the bilinear form $a(u, v)$ and the functional $l(v)$, respectively. Under the assumptions made, matrices A_i are symmetric positive definite and the whole matrix of system (7) is nonsingular.

The linear problem (7) may be solved by several iterative techniques (the reader is referred to [HIK$^+$98, Kuz95] and references therein). The construction of a preconditioner R_λ for the matrix $BA^{-1}B^T$ is one of the most important issues. Usually R_λ is called to be an interface preconditioner, or a Lagrange multiplier preconditioner. One of possible constructions is the Dirichlet-Dirichlet preconditioner [DA99, KV99]. The goal of this paper is to develop the parallel version of the Dirichlet-Dirichlet preconditioner which is robust to both the types of Lagrange multipliers spaces and the jump of coefficients.

Interface preconditioner

Let $\Gamma_i := \partial\Omega_i \setminus \partial\Omega$, n_{Γ_i} be the number of nodes of $\Gamma_i^h := \partial\Omega_i^h \cap \Gamma_i$, $M_{\Gamma_i} \in \mathbb{R}^{n_{\Gamma_i} \times n_{\Gamma_i}}$ be the boundary mass matrix, d_i be the diameter of Ω_i, $i = 1, \ldots, m$.

We introduce the matrix $P_{\Gamma_i} = w_{1,\Gamma_i} w_{1,\Gamma_i}^T$, where $w_{1,\Gamma_i} = \frac{1}{\sqrt{|\Gamma_i|}} e_{\Gamma_i}$, $e_{\Gamma_i} = [1 \ldots 1]^T \in \mathbb{R}^{n_{\Gamma_i}}$. We note that $(M_{\Gamma_i} w_{1,\Gamma_i}, w_{1,\Gamma_i}) = 1$, and $P_{\Gamma_i} M_{\Gamma_i}$ are the M_{Γ_i}-orthogonal projectors, $i = 1, \ldots, m$. Let $\varepsilon_i \le c\rho_i/d_i^2$ and let $\bar{A}_i$ be a matrix generated on Ω_i^h by the bilinear form $a_i(u,v)$ with $\varepsilon = \rho_i/d_i^2$. The matrices A_i and $\bar{A}_i$ have the block forms

$$A_i = \begin{bmatrix} A_{\Gamma_i} & A_{\Gamma_i I_i} \\ A_{I_i \Gamma_i} & A_{I_i} \end{bmatrix} \text{ and } \bar{A}_i = \begin{bmatrix} \bar{A}_{\Gamma_i} & \bar{A}_{\Gamma_i I_i} \\ \bar{A}_{I_i \Gamma_i} & \bar{A}_{I_i} \end{bmatrix},$$

where $A_{\Gamma_i}, \bar{A}_{\Gamma_i} \in \mathbb{R}^{n_{\Gamma_i} \times n_{\Gamma_i}}$.

Lemma 1 [HIK+98, Kuz95] *Under the assumptions made*

$$\left(\bar{A}_{\Gamma_i} - \bar{A}_{\Gamma_i I_i} \bar{A}_{I_i}^{-1} \bar{A}_{I_i \Gamma_i}\right)^{-1} + \frac{1}{\varepsilon_i d_i} P_{\Gamma_i} \sim \left(A_{\Gamma_i} - A_{\Gamma_i I_i} A_{I_i}^{-1} A_{I_i \Gamma_i}\right)^{-1}. \tag{8}$$

The spectral equivalence takes place with constants independent of ρ_i, ε_i, d_i.

The above Lemma is used for the construction of a preconditioner to $BA^{-1}B^T$, since
$B_i A_i^{-1} B_i^T = B_{\Gamma_i} (A_{\Gamma_i} - A_{\Gamma_i I_i} A_{I_i}^{-1} A_{I_i \Gamma_i})^{-1} B_{\Gamma_i}^T$, where matrix B_{Γ_i} is the interface subblock of B_i, $B_i = (B_{\Gamma_i}, O)$. Using (8) we have

$$BA^{-1}B^T = \sum_{i=1}^m B_i A_i^{-1} B_i^T \sim \sum_{i=1}^m \frac{1}{\varepsilon_i d_i} B_{\Gamma_i} P_{\Gamma_i} B_{\Gamma_i}^T + \bar{G}, \tag{9}$$

$$\bar{G} = \sum_{i=1}^m B_{\Gamma_i} \left(\bar{A}_{\Gamma_i} - \bar{A}_{\Gamma_i I_i} \bar{A}_{I_i}^{-1} \bar{A}_{I_i \Gamma_i}\right)^{-1} B_{\Gamma_i}^T. \tag{10}$$

Theorem 1 [KV99] *Let the symmetric positive definite matrix D be such that the spectrum of $D\bar{G}$ belongs to the interval $[c_1, c_2]$, $0 < c_1 < c_2$ and let*

$$R_\lambda := \sum_{i=1}^m \frac{1}{\varepsilon_i d_i} B_{\Gamma_i} P_{\Gamma_i} B_{\Gamma_i}^T + D^{-1}. \tag{11}$$

Then

$$R_\lambda \sim BA^{-1}B^T. \tag{12}$$

The spectral equivalence takes place with constants independent of ρ_i, ε_i, d_i, m and dependent on c_1, c_2.

Matrix R_λ is a modification of D^{-1} by a low rank matrix $XX^T = \sum\limits_{i=1}^m \frac{1}{\varepsilon_i d_i} B_{\Gamma_i} P_{\Gamma_i} B_{\Gamma_i}^T$ with $X = \left(\ldots, \frac{1}{\sqrt{\varepsilon_i d_i |\Gamma_i|}} B_{\Gamma_i} e_{\Gamma_i}, \ldots\right)$. The solution of a system with matrix R_λ may be found by evaluations of matrix D:

$$R_\lambda^{-1} = D - DX(I_m^{-1} + X^T D X)^{-1} X^T D,$$

where $I_m \in \mathbb{R}^{m\times m}$ is the identity matrix. Thus, in order to construct a good preconditioner for $BA^{-1}B^T$ we have to find a preconditioner D to $\bar{G}$ such that $D\bar{G} \sim I$ and D is easily multiplied by a vector.

In order to motivate our further constructions, we briefly review already developed ones. Let us suppose for a moment that $\rho_i = 1$, $i = 1,\dots,m$. In [KV99] and in [DA99] the following constructions were investigated, respectively:

$$\widetilde{D} = \sum_{i=1}^{m} B_{\Gamma_i}\left(\bar{A}_{\Gamma_i} - \bar{A}_{\Gamma_i I_i}\bar{A}_{I_i}^{-1}\bar{A}_{I_i\Gamma_i}\right)B_{\Gamma_i}^T, \tag{13}$$

$$\widetilde{D} = (BB^T)^{-1}B\bar{A}B^T(BB^T)^{-1}. \tag{14}$$

The choice (13) provides an easy parallel implementation, while (14) is not well parallelized, since the global matrix BB^T is to be factorized. A parallel iterative inversion of BB^T seems to be too expensive in view of large condition number of BB^T (of order of 100 in cases (3),(4)). On the other hand, the choice (13) yields the small ratio c_2/c_1 only in the case (5), in contrast to (14) providing satisfactory results in the case (3). The main reason for that is a mutual annihilation of the jump matrices in the product $\widetilde{D}\bar{G} = (BB^T)^{-1}B\bar{A}B^T(BB^T)^{-1}B\bar{A}B^T$.

A natural compromise between (13) and (14) is an approximation of $(BB^T)^{-1}$ by a block diagonal matrix whose blocks are associated with interfaces. The construction of this matrix will be considered later.

Another important modification stems from the properties of the Neumann-Neumann preconditioner [DRLT91, MB96]. Preconditioning the interface Schur complement by assembling Neumann problems requires certain weights for the Neumann problems [KMV93]. By analogy with the Neumann-Neumann preconditioner we weight the Dirichlet problems in (13) by diagonal matrices w_{Γ_i}. The entries of w_{Γ_i} are reciprocal to the number of host subdomains Ω_i, for any interface node.

We return to construction of block diagonal approximation of $(BB^T)^{-1}$. Let $B_{\Gamma_i,j}$ be the j-face block of matrix B_{Γ_i}, then

$$BB^T = \sum_{i=1}^{m} B_{\Gamma_i}B_{\Gamma_i}^T = \sum_{i=1}^{m}\begin{pmatrix} B_{\Gamma_i,1}B_{\Gamma_i,1}^T & B_{\Gamma_i,1}B_{\Gamma_i,2}^T & \cdots \\ B_{\Gamma_i,2}B_{\Gamma_i,1}^T & B_{\Gamma_i,2}B_{\Gamma_i,2}^T & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \tag{15}$$

$$\sum_{i=1}^{m}\begin{pmatrix} B_{\Gamma_i,1}w_{\Gamma_i}B_{\Gamma_i,1}^T & & \\ & B_{\Gamma_i,2}w_{\Gamma_i}B_{\Gamma_i,2}^T & \\ & & \ddots \end{pmatrix} + \sum_{i=1}^{m}\begin{pmatrix} B_{\Gamma_i,1}(1-w_{\Gamma_i})B_{\Gamma_i,1}^T & B_{\Gamma_i,1}B_{\Gamma_i,2}^T & \cdots \\ B_{\Gamma_i,2}B_{\Gamma_i,1}^T & B_{\Gamma_i,2}(1-w_{\Gamma_i})B_{\Gamma_i,2}^T & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Such a decomposition of matrix BB^T turns out to be numerically reasonable in the sense that the inverse of the first term in (15) is a suitable substitution for $(BB^T)^{-1}$, according to numerical evidence.

Taking into account the above observations we present the parallel version of Dirichlet-Dirichlet preconditioner, in the case $\rho_i = 1$, $i = 1,\dots,m$:

$$D = \sum_{i=1}^{m} F_{\Gamma_i}^{-1}B_{\Gamma_i}\omega_{\Gamma_i}\left(\bar{A}_{\Gamma_i} - \bar{A}_{\Gamma_i I_i}\bar{A}_{I_i}^{-1}\bar{A}_{I_i\Gamma_i}\right)\omega_{\Gamma_i}B_{\Gamma_i}^T F_{\Gamma_i}^{-1}, \tag{16}$$

$$F_{\Gamma_i} = blockdiag\{F_{\Gamma_i,j}\}, \quad F_{\Gamma_i,j} = B_{\Gamma_i,j}\,\omega_{\Gamma_i}\,B^T_{\Gamma_i,j} + B_{\Gamma_{i^\star},j^\star}\,\omega_{\Gamma_{i^\star}}\,B^T_{\Gamma_{i^\star},j^\star}.$$

Here, $\Omega_{i^\star}$ is the neighbor-subdomain to Ω_i with shared faces j and $j^\star$. We note the factorization of F_{Γ_i} is feasible since F_{Γ_i} is a sparse matrix.

Now let $\rho_i > 0$ be arbitrary. Then

$$\bar{G} = \sum_{i=1}^{m} B_{\Gamma_i}\left(\bar{A}_{\Gamma_i} - \bar{A}_{\Gamma_i I_i}\bar{A}_{I_i}^{-1}\bar{A}_{I_i\Gamma_i}\right)^{-1} B^T_{\Gamma_i} \equiv \tag{17}$$

$$\equiv \sum_{i=1}^{m} \frac{1}{\sqrt{\rho_i}} B_{\Gamma_i}\left(\rho_i^{-1}\bar{A}_{\Gamma_i} - \rho_i^{-1}\bar{A}_{\Gamma_i I_i}\bar{A}_{I_i}^{-1}\bar{A}_{I_i\Gamma_i}\right)^{-1} B^T_{\Gamma_i}\frac{1}{\sqrt{\rho_i}},$$

and the problem of construction D is reduced to the case $\rho_i = 1$ by substitutions $B_{\Gamma_i} \to B_{\Gamma_i}/\sqrt{\rho_i}$, $\bar{A}_i \to \bar{A}_i/\rho_i$. Thus, the general form of the Dirichlet-Dirichlet preconditioner is

$$D = \sum_{i=1}^{m} F_{\Gamma_i}^{-1}\frac{1}{\sqrt{\rho_i}} B_{\Gamma_i}\omega_{\Gamma_i}\left(\rho_i^{-1}\bar{A}_{\Gamma_i} - \rho_i^{-1}\bar{A}_{\Gamma_i I_i}\bar{A}_{I_i}^{-1}\bar{A}_{I_i\Gamma_i}\right)\omega_{\Gamma_i} B^T_{\Gamma_i}\frac{1}{\sqrt{\rho_i}} F_{\Gamma_i}^{-1},$$

$$F_{\Gamma_i} = blockdiag\{F_{\Gamma_i,j}\},$$

$$F_{\Gamma_i,j} = \frac{1}{\sqrt{\rho_i}} B_{\Gamma_i,j}\,\omega_{\Gamma_i}\,B^T_{\Gamma_i,j}\frac{1}{\sqrt{\rho_i}} + \frac{1}{\sqrt{\rho_{i^\star}}} B_{\Gamma_{i^\star},j^\star}\,\omega_{\Gamma_{i^\star}}\,B^T_{\Gamma_{i^\star},j^\star}\frac{1}{\sqrt{\rho_{i^\star}}},$$

which may be rewritten as:

$$D = \sum_{i=1}^{m} F_{\Gamma_i}^{-1} B_{\Gamma_i}\omega^\rho_{\Gamma_i}\left(\bar{A}_{\Gamma_i} - \bar{A}_{\Gamma_i I_i}\bar{A}_{I_i}^{-1}\bar{A}_{I_i\Gamma_i}\right)\omega^\rho_{\Gamma_i} B^T_{\Gamma_i} F_{\Gamma_i}^{-1}, \tag{18}$$

$$F_{\Gamma_i,j} = B_{\Gamma_i,j}\,\omega^\rho_{\Gamma_i}\,B^T_{\Gamma_i,j} + B_{\Gamma_{i^\star},j^\star}\,\omega^\rho_{\Gamma_{i^\star}}\,B^T_{\Gamma_{i^\star},j^\star}, \quad \omega^\rho_{\Gamma_i} = \omega_{\Gamma_i}/\rho_i.$$

It is clear that (18) differs from (16) only in the scaled count matrices $w^\rho_{\Gamma_i}$.

Remark. The presence of Dirichlet boundary conditions for the original problem reduces the rank of XX^T since subdomains with a Dirichlet part of the boundary do not contribute to XX^T [Kuz95].

Numerical experiments

We present the effect of the Dirichlet-Dirichlet preconditioner for the model operator $-\nabla\cdot\rho\nabla + \varepsilon$ with Neumann boundary conditions. The domain Ω is a union of four similar tetrahedra Ω_i sharing one common edge:

$$\Omega = \left\{x \mid \sum_{i=1}^{3}|x_i| < \frac{1}{2},\, x_1 > 0\right\},\ m = 4,$$

$$\Omega_1 = \{x \in \Omega, x_2 < 0, x_3 < 0\}\,,\ \Omega_2 = \{x \in \Omega, x_2 < 0, x_3 > 0\}\,,$$

$$\Omega_3 = \{x \in \Omega, x_2 > 0, x_3 < 0\}\,,\ \Omega_4 = \{x \in \Omega, x_2 > 0, x_3 > 0\}\,.$$

We compare the Dirichlet-Dirichlet preconditioner R_λ for $BA^{-1}B^T$ in three cases of Lagrange multiplier spaces, (3), (4), (5). The comparison will be done for different types of tetrahedral meshes: quasi-uniform, shape-regular, and anisotropic. We distinguish the above types of meshes by the metric $H = diag\{H_1, H_2, H_3\}$ in which the meshes Ω_i^h become quasi-uniform, i.e. consist of the given number N_T of shape-regular (in metric H) tetrahedra of the same size (in metric H). In the tables below we show the estimated condition number of preconditioned Schur complement $BA^{-1}B^T$ and the number of PCG iterations applied to a system with $BA^{-1}B^T$ in order to reduce the Euclidean norm of residual by a factor of 10^6.

Coef.	Mesh	quasi-uniform			isotropic		anisotropic	
ρ_i	N_T	800	6000	39000	800	6000	800	6000
		$\Lambda^h(\delta_k)$ from (3)						
$\rho_{1,2,3,4} = 1$	cond(#it)	29(19)	45(25)	29(26)	37(17)	44(20)	26(17)	32(22)
$\rho_{1,2} = 1, \rho_{3,4} = 10^4$	cond(#it)	36(13)	18(11)	17(12)	81(23)	170(24)	59(18)	18(13)
$\rho_{1,3} = 1, \rho_{2,4} = 10^4$	cond(#it)	52(19)	83(23)	41(21)	91(21)	81(22)	44(17)	69(22)
		$\Lambda^h(\delta_k)$ from (4)						
$\rho_{1,2,3,4} = 1$	cond(#it)	29(20)	36(25)	28(24)	34(16)	36(17)	23(16)	29(19)
$\rho_{1,2} = 1, \rho_{3,4} = 10^4$	cond(#it)	32(13)	19(12)	17(12)	74(20)	170(22)	51(19)	18(12)
$\rho_{1,3} = 1, \rho_{2,4} = 10^4$	cond(#it)	49(19)	83(24)	35(21)	83(20)	81(22)	35(15)	65(23)
		$\Lambda^h(\delta_k)$ from (5)						
$\rho_{1,2,3,4} = 1$	cond(#it)	17(16)	18(20)	20(26)	17(14)	18(18)	16(13)	18(18)
$\rho_{1,2} = 1, \rho_{3,4} = 10^4$	cond(#it)	20(12)	41(19)	22(20)	20(11)	36(17)	23(12)	20(17)
$\rho_{1,3} = 1, \rho_{2,4} = 10^4$	cond(#it)	18(13)	19(15)	21(19)	18(10)	19(12)	18(10)	19(13)

Table 1: Condition number of $R_\lambda^{-1}BA^{-1}B^T$ and #PCG iteration, $\varepsilon = 1$.

In Table 1 the quasi-uniform mesh is obtained on the basis of the metric $H_1 = H_2 = H_3 = 1$, and the isotropic and anisotropic refinements to the common edge are defined by $H_1 = H_2 = H_3 = 0.5/(\sqrt{y^2+z^2} + 0.01)$ and $H_1 = 1, H_2 = H_3 = 0.5/(\sqrt{y^2+z^2}+0.025)$, respectively. The meshes are generated in such a way that they do not match on the interfaces. It implies that the number of tetrahedra in Ω_i^h is equal to N_T only approximately. We consider three different distributions of coefficients ρ_i in Ω: no jump, two simply connected subdomains with constant coefficient, and the chess pattern.

In the next example we consider the effects of small value of coefficient ε_i and large number of subdomains m. The domain $\Omega = (0,1)^3$ is split into $m = 6$ (resp. 48 or 384) tetrahedron subdomains Ω_i of the same diameter $d_i = \sqrt{3}$ (resp. $\sqrt{3}/2$ or $\sqrt{3}/4$), $i = 1, \ldots, m$. We consider the Helmholtz operator $-\Delta + \varepsilon$ with homogeneous Neumann boundary condition on $\partial\Omega$ and restrict the set of possible triangulations by quasi-uniform ones and take the Lagrange multiplier space (3). The number of tetrahedra N_T in Ω_i^h is chosen to be equal to 800. Slightly worse performance in the cases $m = 48, 384$ is due to presence of the crosspoints.

In Table 3 we present the parallel properties of the method in terms of the execution time of PCG iterations measured on different sets of processors. The measurement was obtained using a DEC TruCluster with Dec alpha processors running at 400 MHz.

$\varepsilon \setminus m$		$m = 6$	$m = 48$	$m = 384$
$\varepsilon = 1$	cond (# it)	26(26)	64(44)	84(45)
$\varepsilon = 10^{-2}$	cond (# it)	35(20)	61(33)	62(28)
$\varepsilon = 10^{-4}$	cond (# it)	29(15)	40(18)	19(7)

Table 2: Condition number of $R_\lambda^{-1}BA^{-1}B^T$ and #PCG iteration, quasi-uniform meshes, $\rho_{1,2,3,4} = 1$, $\Lambda^h(\delta_k)$ from (3).

The Fortran code uses MPI library for interprocessor communications.

N_T	#Processors	2	4	8
800	time of PCG it.	3.0	1.5	0.9
2000	time of PCG it.	7.7	3.8	1.9

Table 3: Execution time of PCG iterations (sec), $m = 48$, quasi-uniform meshes, $\varepsilon = 1$, $\rho_{1,2,3,4} = 1$.

Conclusions

The paper is addressing the construction of parallel interface preconditioner for the mortar element method. The new version of the Dirichlet-Dirichlet method is discussed. It is easy to parallel and it is robust to such "bad" parameters of an elliptic boundary value problem as the number of subdomains, the mesh refinement, the jump of the diffusion coefficient, the small value of perturbation parameter. Numerical experiments exhibited the basic properties of the method.

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