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26. The Application of Operator Split Method to Large-Scale Reservoir Simulatios Part I. a Priori Estimates

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Introduction

High performance computing technology offers the petroleum industry the ability to solve previously prohibitive large-scale reservoir problems. In July 1999, our group, in cooperation with the Petroleum Exploration and Development Institute of Daqing Oil Field, China, ran a million-gridblock-scale reservoir simulation on DAWN 2000, which is a home-made supercomputer, and a loosely coupled PC cluster seperately. The parallel computing methods that we used derived from the domain decomposition methods with no overlap. The next goal of our group is to solve reservoir simulations with millions of gridblocks on parallel machines. Unfortunately, it seems that the original computing method is not scalable enough. We believe that the reason is rather geologic than mathematical. As the simulating area becomes larger and larger, the geologic faults will be more and more complicated. Therefore, the non-matching grids on the interfaces of the substructures will be increasing largely, and possessing entirely different properties. This will inevitably lead to the poor performance of the original computing methods.

The purpose of this paper is trying to find an effective way to remove as many of the geologic non-matching grids as possible from the interfaces. The operator split method, not a very new technique, proposed by Douglas and Dupont[JD71], can solve this problem. Because, for quite a few reservoir problems, the reservoir Ω can be taken to be unions of right prisms. Or, mathematically, $\Omega = \cup\Omega_i$, where $\Omega_i = \Omega_{xy}^i \times [0, l_i]$, $\Omega_{xy}^i \subset \mathbb{R}^2$. When only upright wells are available, the original reservoir problem can be divided into an xy-direction, two-dimensional problem and a z-direction, one-dimensional problem in some of the subdomains. So, the geologic non-matching grids on the interfaces can be greatly reduced.

For a detailed introduction of operator split method, see [JD71] and [Mar90]. Generalizations of this method to parabolic problems on nonrectangular regions were presented by Hayes [Hay81]. Special treatments for convection-diffusion problems, parabolic and hyperbolic equations were considered by Krishnamachari, Hayes and Russell[SHR89] (without theoretical analysis), Bramble, Ewing and Li[BEL89], Bialecki and Fernandes[BF93], and Fernandes and Fairweather [FF91]. Applications of these methods to problems in fluid flow, physics of semiconductors and elastoplastic dynamics were described by Hayes and Krishnamachari [HK84], Berezin and Yanenko[BY84], and Migual, Pinsky and Taylor[MPT83].

The main purpose of using operator split method here is to reduce the geologic non-matching grids on the interfaces, instead of saving the memory costs and the

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storage requirements as before. Based upon this method, we can further formulate the domain decomposition algorithms. We expect that this combining method can perform good scalability, and have nearly the same accuracy as the original methods, which will be proved in this paper.

In this paper, we will make some a priori estimates for the operator split method for reservoir problems. An optimal H^1 convergence rate will be proved. It is necessary to make a major, and probably unphysical assumption, as was done in [JR83], [Yua92], [Che94] and [JDEW83], that the sources and sinks are smoothly distributed and the resulting functions of interest are thus fairly smooth in space. The techniques involved to prove the error bounds are quite different from the standard ones presented by Douglas, Wheeler and Ewing, et.al.[JR83] [Yua92][JDEW83].

In this paper, we consider the single-phase, miscible displacement of one compressible fluid with another in a porous medium. A set of model equations is given as follows. For a more detailed description of the physical problem, see [Pea66]. Find the concentration $c = c(x,t)$ and $p = p(x,t)$ that satisfy the following equations:

$$d(c)\frac{\partial p}{\partial t} + \bigtriangledown \cdot u = d(c)\frac{\partial p}{\partial t} - \bigtriangledown \cdot (a(c) \bigtriangledown p) = q, \quad x \in \Omega, t \in \mathcal{J} \tag{1}$$

$$\phi\frac{\partial c}{\partial t} + b(c)\frac{\partial p}{\partial t} + u \cdot \bigtriangledown c - \bigtriangledown \cdot (D(u) \bigtriangledown c) = (\hat{c} - c)q, \quad x \in \Omega, t \in \mathcal{J} \tag{2}$$

where initial conditions and no flow boundary conditions are given by

$$p(x,0) = p_0(x), \qquad x \in \Omega \tag{3}$$
$$c(x,0) = c_0(x), \qquad x \in \Omega \tag{4}$$

and

$$u \cdot \nu = 0, \qquad x \in \partial\Omega \tag{5}$$
$$(D \bigtriangledown c - cu) \cdot \nu = 0, \qquad x \in \partial\Omega \tag{6}$$

For simplicity, denote $\Omega = \Omega_{xy} \times [0,l]$, $\mathcal{J} = (0,T]$, and ν is the outward unit normal vector on $\partial\Omega$, the boundary of Ω. Here $a(c), b(c), d(c), \phi = \phi(x)$ are specific reservoir and fluid properties, u is the Darcy velocity of the fluid, $D(u)$ is the diffusion coefficient matrix which combines the effects of molecular diffusion and mechanical dispersion, $\hat{c}$ is the specific concentration at injection wells and the resident concentration at production wells, and $q = q(x,t)$ is the imposed external flow, positive for injection and negative for production.

In [JR83] the authors presented and analyzed certain numerical approximations for a two dimensional model. Extensions of these methods to more efficient time-stepping procedures and methods of characteristics[Yua92] have since been developed.

The paper is organized as follows: In §2, the variational form and the elliptic projections of the problem are introduced. In §3, the numerical procedures are described. In §4, some a priori estimates are presented, and in §5, the amount of calculations of the operator split method are estimated.

Variations and Projections

To obtain a variational form of (1) and (2), we multiply (1) and (2) by test functions $v, w \in H^1(\Omega)$, and integrate by parts, respectively. This yields

$$(\phi\frac{\partial c}{\partial t}, w) + (b(c)\frac{\partial p}{\partial t}, w) + (u \cdot \bigtriangledown c, w) + (D(u) \bigtriangledown c, \bigtriangledown w)$$

$$= ((\hat{c} - c)q, w), \qquad w \in H^1(\Omega), t \in \mathcal{J} \tag{7}$$

$$(d(c)\frac{\partial p}{\partial t}, v) + (a(c) \bigtriangledown p, \bigtriangledown v) = (q, v), \qquad v \in H^1(\Omega), t \in \mathcal{J} \tag{8}$$

Let $\mathcal{M}_h = \mathcal{M}_{h_c}, \mathcal{N}_h = \mathcal{N}_{h_p} \subset W^{1,\infty}$ denote the finite element spaces spanned by tensor product bases, where $\mathcal{M}_h = \text{span}[\psi_i^{(xy)}(x,y) \times \psi_j^{(z)}(z)], \mathcal{N}_h = \text{span}\,[\bar{\psi}_i^{(xy)}(x,y) \times \bar{\psi}_j^{(z)}(z)]$, and $\mathcal{M}_h, \mathcal{N}_h$ satisfy

$$\inf_{w_h \in \mathcal{M}_h} \|w - w_h\|_{1,q} \leq K\|w\|_{l+1,q} h_c^l, \quad w \in W^{l+1,q}, \quad 1 \leq q \leq \infty \tag{9}$$

and

$$\inf_{v_h \in \mathcal{N}_h} \|v - v_h\|_{1,q} \leq K\|v\|_{r+1,q} h_p^r, \quad v \in W^{r+1,q}, \quad 1 \leq q \leq \infty \tag{10}$$

respectively. We assume that all standard inverse relations hold on $\mathcal{M}_h$ and $\mathcal{N}_h$.

We project the solution of the differential problem (1) and (2) into the finite element spaces by means of coercive elliptic forms associated with the differential system. First, for $t \in \mathcal{J}$, let $\tilde{c} = \tilde{c}_h : \mathcal{J} \to \mathcal{M}_h$ be determined by the relations:

$$(D(u) \bigtriangledown (c - \tilde{c}), \bigtriangledown w) + (u \cdot \bigtriangledown (c - \tilde{c}), w) + \sigma_1 (c - \tilde{c}, w) = 0, \qquad w \in \mathcal{M}_h \tag{11}$$

where the constant σ_1 is chosen to be large enough to insure the coercivity of the bilinear form over $H^1(\Omega)$.

Similarly, let $\tilde{p} = \tilde{p}_h : calJ \to \mathcal{N}_h$ satisfy

$$(a(c) \bigtriangledown (p - \tilde{p}), \bigtriangledown v) + \sigma_2 (p - \tilde{p}, v) = 0, \qquad v \in \mathcal{N}_h \tag{12}$$

where σ_2 is assumed to be coercive over $H^1(\Omega)$.

Let:

$$\zeta^n = c^n - \tilde{c}^n, \ \mathcal{E}^n = \tilde{c}^n - C^n, \ \eta^n = p^n - \tilde{p}^n, \ \pi^n = \tilde{p}^n - P^n$$

If the following restrictions are valid:

(i)q is smoothly distributed, the coefficients are smooth, therefore the solution is smooth.

(ii)The coefficients a, d and ϕ are positively bounded below, as well as being smooth.

$$0 < a_* \leq a(c) \leq a^*, \quad 0 < d_* \leq d(c) \leq d^*, \quad 0 < \phi_* \leq \phi(x) \leq \phi^* \tag{13}$$

$D = (D_{ij}(u))_{3\times 3}$ is a positive definite matrix, and there exist constants D_*, D^*, $0 < D_* \le D^*$, such that for $\forall w \in R^2$,

$$D_*|w|^2 \le (D(u)w, w) \le D^*|w|^2 \tag{14}$$

It follows from [JR83],[Che94] that:

$$\|\zeta\|_{L^2} + h_c\|\zeta\|_{H^1} + \|\frac{\partial \zeta}{\partial t}\|_{L^2} + h_c\|\frac{\partial \zeta}{\partial t}\|_{H^1} \le K\{\|c\|_{H^{l+1}} + \|\frac{\partial c}{\partial t}\|_{H^{l+1}}\}h_c^{l+1} \tag{15}$$

$$\|\eta\|_{L^2} + h_p\|\eta\|_{H^1} + \|\frac{\partial \eta}{\partial t}\|_{L^2} + h_p\|\frac{\partial \eta}{\partial t}\|_{H^1} \le K\{\|p\|_{H^{r+1}} + \|\frac{\partial p}{\partial t}\|_{H^{r+1}}\}h_p^{r+1} \tag{16}$$

$$\|\tilde{c}\|_{W^1_\infty(\mathcal{J};W^1_\infty)} + \|\tilde{p}\|_{W^1_\infty(\mathcal{J};W^1_\infty)} \le K, \quad \|\frac{\partial^2 \eta}{\partial t^2}\|_{H^1} \le Kh_p^r \tag{17}$$

$$\|\frac{\partial^3 \eta}{\partial t^3}\|_{L^\infty} + \| \bigtriangledown \frac{\partial^2 \eta}{\partial t^2}\|_{L^\infty} + \| \bigtriangledown \frac{\partial^2 \zeta}{\partial t^2}\|_{L^\infty} \le K \tag{18}$$

where K is a positive constant that does not depend on h_c and h_p.

The Numerical Procedures

In this section, we present the numerical procedures of (1) and (2) by using operator split methods. The associated matrix problem, however, will not factor, since, in general, ϕ and $d(c)$ are not single tensor products. So, on the left-hand side of (7) and (8), ϕ and $d(c)$ are replaced with certain type of patch approximations, respectively. Using the approximate $\tilde{\phi}$ and d^n, perturbation terms can be added to the matrix problem, so that it does factor as desired. For $C = \sum_{i=1}^{m_c} \mu_i\psi_i$, $P = \sum_{i=1}^{m_p} \gamma_i\bar{\psi}_i$ and $w = \sum_{j=1}^{m_c} \nu_j\psi_j$, $v = \sum_{j=1}^{m_p} \kappa_j\bar{\psi}_j$, define

$$(\tilde{\phi}C, w) = \int_\Omega \{\sum_{i,j=1}^{m_c} \mu_i\psi_i\nu_j\psi_j\tilde{\phi}_{ij}\}dx \tag{19}$$

$$(d^nP, v) = \int_\Omega \{\sum_{i,j=1}^{m_p} \gamma_i\bar{\psi}_i\kappa_j\bar{\psi}_j d^n_{ij}\}dx \tag{20}$$

where

$$\tilde{\phi}_{ij} = \sqrt{\phi(x^i)\cdot\phi(x^j)} \quad , x^i \in supp(\psi_i) \tag{21}$$

$$d^n_{ij} = \sqrt{d(x^i, C^n)\cdot d(x^j, C^n)} \quad , x^i \in supp(\bar{\psi}_i) \tag{22}$$

The three-level operator split method is defined by finding $\{C^n, P^n\} \in \mathcal{M}_h \times \mathcal{N}_h$ such that

$$(\tilde{\phi}\partial_t C^n, w) + (U^n \cdot \bigtriangledown C^n, w) + (D(U^n) \bigtriangledown C^n, \bigtriangledown w) + (b(C^n)\partial_t P^n, w)$$

$$+\lambda_1\Delta t(\tilde{\phi} \bigtriangledown \partial_t C^n, \bigtriangledown w) + \lambda_1^2(\Delta t)^2(\tilde{\phi}\frac{\partial^2}{\partial x\partial z}\partial_t C^n, \frac{\partial^2}{\partial x\partial z}w)$$

$$+\lambda_1^2(\Delta t)^2(\tilde{\phi}\frac{\partial^2}{\partial y\partial z}\partial_t C^n, \frac{\partial^2}{\partial y\partial z}w)$$

$$= ((\hat{C}^n - C^n)q^n, w) + ((\tilde{\phi} - \phi)\partial_t C^{n-1}, w), \quad w \in \mathcal{M}_h \tag{23}$$

suppose that U^n is given by

$$U^n = -a(C^n) \bigtriangledown P^n, \quad \text{for } \forall x \in \Omega \tag{24}$$

and

$$(d^n\partial_t P^n, v) + (a(C^n) \bigtriangledown P^n, \bigtriangledown v) + \lambda_2\Delta t(d^n \bigtriangledown \partial_t P^n, \bigtriangledown v)$$

$$+\lambda_2^2(\Delta t)^2(d^n\frac{\partial^2}{\partial x\partial z}\partial_t P^n, \frac{\partial^2}{\partial x\partial z}v) + \lambda_2^2(\Delta t)^2(d^n\frac{\partial^2}{\partial y\partial z}\partial_t P^n, \frac{\partial^2}{\partial y\partial z}v)$$

$$= (q^n, v) + ((d^n - d(C^n))\partial_t P^{n-1}, v), \quad v \in \mathcal{N}_h \tag{25}$$

where the computing order is $C^1, P^2, U^2, C^2, P^3, U^3, \cdots$. For stability, we require that $\lambda_1 > \frac{1}{2}D^*/\phi_*$ and $\lambda_2 > a^*/d_*$. We assume that the initial time steps are chosen small enough, so that $P^1 = P^0 = p_0$, and the initial values of C^1 are derived through some kind of iterative methods.

If we notice the fact that the concentration equation is normally convection-dominated, a scheme combining the operator split procedure with the method of characteristics can be defined by employing an approximation to the following characteristic vector. For each (x, t), we let $\tau(x, t)$ be the unit vector in the indicated characteristic direction such that

$$\frac{\partial}{\partial\tau(x,t)} = \frac{u(x, c, \bigtriangledown p)}{\sqrt{|u(x, c, \bigtriangledown p)|^2 + \phi^2(x)}}\frac{\partial}{\partial x} + \frac{\phi(x)}{\sqrt{|u(x, c, \bigtriangledown p)|^2 + \phi^2(x)}}\frac{\partial}{\partial t} \tag{26}$$

$$= (|u|^2 + \phi^2)^{-1/2}(u_1\frac{\partial}{\partial x} + u_2\frac{\partial}{\partial y} + u_3\frac{\partial}{\partial z} + \phi\frac{\partial}{\partial t}) \tag{27}$$

Let $\phi_c = (|u|^2 + \phi^2)^{1/2}$, we then see that (2) is equivalent to

$$\phi_c\frac{\partial c}{\partial\tau(x,t)} + b(c)\frac{\partial p}{\partial t} - \bigtriangledown \cdot (D(u) \bigtriangledown c) = (\hat{c} - c)q \tag{28}$$

and the variational form (7) becomes

$$(\phi_c\frac{\partial c}{\partial\tau}, w) + (b(c)\frac{\partial p}{\partial t}, w) + (D(u) \bigtriangledown c, \bigtriangledown w) = ((\hat{c} - c)q, w), \quad w \in H^1, t \in \mathcal{J} \tag{29}$$

When solving for C^{n+1}, we define for each $x \in \Omega$,

$$\bar{x} = x - \frac{U^n(x)}{\phi(x)}\Delta t, \quad \bar{C}^n(x) = C^n(\bar{x}) \tag{30}$$

It is assumed that no flow occurs across the boundary. If $\bar{x}$ crosses over the boundary $\partial\Omega$, we can replace it with its mirror image point along the normal direction of $\partial\Omega$. We represented this point by $\bar{\bar{x}}$. Therefore, $\bar{C}^n$ is well defined. To approximate (29), we use a backward difference quotient for $\partial c/\partial\tau$ along the characteristic. Specifically, we take

$$(\frac{\partial c}{\partial \tau})^{n+1}(x) \approx \phi \frac{c^{n+1}(x) - c^n(\bar{x})}{\Delta t \phi_c} \tag{31}$$

so that

$$\phi_c \frac{\partial c^{n+1}}{\partial \tau} \approx \phi \frac{c^{n+1} - \bar{c}^n}{\Delta t} \tag{32}$$

The numerical scheme based on combining the operator split procedure with the method of characteristics for the concentration equation can be defined as

$$(\tilde{\phi}\partial_t C^n, w) + (U^n \cdot \bigtriangledown C^n, w) + (D(U^n) \bigtriangledown C^n, \bigtriangledown w) + (b(C^n)\partial_t P^n, w)$$

$$+\lambda_1 \Delta t(\tilde{\phi} \bigtriangledown \partial_t C^n, \bigtriangledown w) + \lambda_1^2 (\Delta t)^2 (\tilde{\phi}\frac{\partial^2}{\partial x \partial z}\partial_t C^n, \frac{\partial^2}{\partial x \partial z} w)$$

$$+\lambda_1^2 (\Delta t)^2 (\tilde{\phi}\frac{\partial^2}{\partial y \partial z}\partial_t C^n, \frac{\partial^2}{\partial y \partial z} w) = ((\hat{C^n} - C^n)q^n, w)$$

$$+((\tilde{\phi} - \phi)\partial_t C^{n-1}, w) - (\phi \frac{C^n - \bar{C}^n}{\Delta t}, w), \qquad w \in \mathcal{M}_h \tag{33}$$

The matrix problem associated with (23)-(25) , similarly for (33),(24),(25), is given by

$$\mathbb{K}_c^n(\mu^{n+1} - \mu^n) = \Phi^n \tag{34}$$
$$\mathbb{K}_p^n(\gamma^{n+1} - \gamma^n) = \Psi^n \tag{35}$$

where

$$\mathbb{K}_c^n = (Diag_c)^{1/2}\mathbb{K}_c(Diag_c)^{1/2} \quad , \mathbb{K}_p^n = (Diag_p^n)^{1/2}\mathbb{K}_p(Diag_p^n)^{1/2}$$

$$Diag_c = \begin{bmatrix} \tilde{\phi}(x^1) & & \\ & \ddots & \\ & & \tilde{\phi}(x^{m_c}) \end{bmatrix}, Diag_p^n = \begin{bmatrix} d(x^1, C^n) & & \\ & \ddots & \\ & & d(x^{m_p}, C^n) \end{bmatrix}$$

$$\mathbb{K}_c^{ij} = ((\psi_j, \psi_i) + \lambda_1 \Delta t(\bigtriangledown\psi_j, \bigtriangledown\psi_i) + \lambda_1^2(\Delta t)^2[(\frac{\partial^2\psi_j}{\partial x\partial z}, \frac{\partial^2\psi_i}{\partial x\partial z}) + (\frac{\partial^2\psi_j}{\partial y\partial z}, \frac{\partial^2\psi_i}{\partial y\partial z})])$$

$$\mathbb{K}_p^{ij} = ((\bar{\psi}_j, \bar{\psi}_i) + \lambda_2 \Delta t(\bigtriangledown\bar{\psi}_j, \bigtriangledown\bar{\psi}_i) + \lambda_2^2(\Delta t)^2[(\frac{\partial^2\bar{\psi}_j}{\partial x\partial z}, \frac{\partial^2\bar{\psi}_i}{\partial x\partial z})(\frac{\partial^2\bar{\psi}_j}{\partial y\partial z}, \frac{\partial^2\bar{\psi}_i}{\partial y\partial z})])$$

$$\Phi_i^n = ((\hat{C^n} - C^n)q^n, \psi_i) - (U^n \cdot \bigtriangledown C^n, \psi_i) - (D(U^n) \bigtriangledown C^n, \bigtriangledown\psi_i)$$
$$-(b(C^n)\partial_t P^n, \psi_i) + ((\tilde{\phi} - \phi)\partial_t C^{n-1}, \psi_i)$$
$$\Psi_i^n = (q^n, \bar{\psi}_i) - (a(C^n) \bigtriangledown P^n, \bigtriangledown\bar{\psi}_i) + ((d^n - d(C^n))\partial_t P^{n-1}, \bar{\psi}_i)$$

Notice that $\mathcal{M}_h$ and $\mathcal{N}_h$ are spanned by tensor product bases, so K_c and K_p can be rewritten in the following manner:

$$[\mathbb{I} \otimes (\mathbb{C}_{xy} + \lambda \Delta t \mathbb{A}_{xy})][(\mathbb{C}_z + \lambda \Delta t \mathbb{A}_z) \otimes \mathbb{I}] \tag{36}$$

where $\mathbb{C}_{xy}, \mathbb{A}_{xy}$ correspond to a two-dimensional problem in horizontal planes of Ω, while $\mathbb{C}_z, \mathbb{A}_z$ to a one-dimensional problem along the vertical lines in Ω.

"A Priori" Error Estimates

In order to derive the optimal H^1 error estimates for the procedures (23)-(25), and (33),(24),(25), We need to let ∂_t act on the both sides of the error equation of the pressure equation. Quite a few of the technical treatments were involved. After a careful calculation, we obtain

Theorem 1 *Suppose the restrictions of §2 be satisfied, and there is no flow at the initial time, i.e.* $p_0 \equiv const$. *The parameters* h_p *and* h_c *are chosen such that* $h_p^r = o(h_c), h_c^l = o(h_p)$, $r, l \geq 2$, $\Delta t = O(h_c^2) = O(h_p^2)$. *If* $\lambda_1 > \frac{1}{2} D^*/\phi_*$, $\lambda_2 > a^*/d_*$, *and the initial values of* C^0 *and* C^1 *satisfy*

$$\|C^1 - c^1\|_{H^1}^2 + \Delta t \|\partial_t (C - c)^0\|_{L^2}^2 \leq K(h_p^{2r} + h_c^{2l} + (\Delta t)^2)$$

Then for h_c *and* h_p *sufficiently small, we have*

$$\max_{1 \leq n \leq M} \{\|C^n - c^n\|_{H^1}^2 + \|U^n - u^n\|_{L^2}^2\}$$
$$+ \Delta t \sum_{n=1}^{M-1} \{\|\partial_t (C - c)^n\|_{L^2}^2 + \|\partial_t (P - p)^n\|_{L^2}^2\} \leq K(h_p^{2r} + h_c^{2l} + (\Delta t)^2)$$

From the estimates, we know that the operator split method can maintain the optimal H^1 accuracy. Therefore, the new parallel computing method, the DDM combining with the operator split method, can have nearly the same numerical accuracy as the origianal method we used before.

Work Estimates

Suppose there are $m_p = m(h_p), m_c = m(h_c)$ unknowns for the pressure equation and the concentration equation respectively. The factorization of the matrices $\mathbb{K}_c$ and $\mathbb{K}_p$ requires $O(m_c^{3/2} + m_p^{3/2})$ operations, but this is done only once and used at all successive time steps. The evaluation of $(Diag_c)^{-1/2}$ requires $O(m_c)$ operations, while $(Diag_p^n)^{-1/2}$ requires $O(m_p)$ operations for each time level. The solution, given the factorization of $\mathbb{K}_c$ and $\mathbb{K}_p$, requires $O(m_c \log m_c + m_p \log m_p)$ operations. If $\Delta t = O(h_p^r) = O(h_c^l)$, i.e. $\Delta t = O(m_p^{-r/3}) = O(m_c^{-l/3})$, and $r, l \geq 2$, then the total number of operations needed is $O(m_p^{r/3+1} \log m_p + m_c^{l/3+1} \log m_c)$, which is nearly optimal since the solution is defined by $O(m_p^{r/3+1} + m_c^{l/3+1})$ parameters for a first-order correct-in-time method.

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