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1 Additive Schwarz method for nonsymmetric problems : application to frictional multicontact problems

P. Alart [1], M. Barboteu [2], P. Le Tallec [3], M. Vidrascu [4]

Introduction

In this paper, we present a generalization of a Neumann-Neumann domain decomposition method for solving nonsymmetric elliptic systems in a scalable way. It uses the theoretical framework of Schwarz additive decomposition method and introduces a coarse space well adapted to nonsymmetric cases. The efficiency of this method is evaluated on nonsymmetric frictional contact problems.

In iterative substructuring, the parallel solution of a complex structural problem is achieved by splitting the original domain of computation in smaller nonoverlapping simpler subdomains, and by reducing the initial problem to an interface system to be solved by a parallel two-level preconditioned conjugate gradient method. Many variants of this approach have been proposed and investigated in the recent literature, all associated to different choices of preconditioners and of coarse spaces [BPS86], [Smi92], [LTDRV91].

Up to now, the main objectives when developing such preconditioners were to achieve efficiency and scalability even in presence of complex geometries, strongly heterogeneous coefficients, general elliptic operators (3D anisotropic elasticity, shells, etc ..) and arbitrary meshes (unstructured, nonmatching, etc ..). These objectives cannot be reached without an adequate coarse solver [DW92]. For FETI preconditioners, this coarse solver is introduced by strongly imposing a kinematic constraint at each iteration (rigid body modes in FETI1 [FR94], rigid and corner modes in FETI2, corner modes only in FETI DP [FLL$^+$01]). In balanced Neumann-Neumann techniques, this solver appears while imposing orthogonality to an adequate coarse space of singular modes. The recent applications have introduced two new key dimensions in the development of such a coarse solver, namely its ability to handle nonsymmetric operators, and its industrial feasibility (automatic construction and cost efficiency). In our case, this new perspective is motivated by multicontact frictional problems.

This evolution requires complete review of the construction process of such coarse solvers, which is done hereafter in the framework of the Neumann-Neumann Domain Decomposition Method. The key point is the construction of the local spaces $\bar{Z}_i$ of rigid motions. For symmetric problems, the space $\bar{Z}_i$ is the kernel $Ker\mathbf{S}^i$ of the local Schur operators, with the possible addition of corner modes for fourth order problems. For advection diffusion problems, the good choice is based on constants. In the general case, the choice of $\bar{Z}_i$ must both set the arbitrary constants to zero in the solutions of the local Neumann problems (thus ensuring a scale invariance of the related energy norm), and regularize these local problems. For this purpose, we will introduce dual rigid modes obtained by solving local adjoint regularized Neumann problems.

[1]Université Montpellier, alart@lmgc.univ-montp2.fr
[2]Université Perpignan, barboteu@univ-perp.fr
[3]Ecole Polytechnique, patrick.letallec@polytechnique.fr
[4]INRIA Rocquencourt, Marina.Vidrascu@inria.fr

The paper is organized as follows. The basic domain decomposition methodology is first reviewed (§2), with an application to frictional contact problems illustrating the difficulties arising in presence of nonsymmetric operators (§3). Such nonsymmetric problems are handled in (§4) by reformulating the two level Neumann-Neumann preconditioner to an additive Schwarz algorithm, and by defining an appropriate coarse space by duality. In the last section (§5), we test the efficiency of this updated general Neumann-Neumann preconditioner on the numerical solution of nonsymmetric structural problems with contact and friction.

Balancing method for symmetric systems

The basic idea in nonoverlapping domain decomposition methods is to split the domain Ω of study into N small nonoverlapping subdomains $\Omega^n (n = 1, N)$ and interfaces defined by : $\Omega = \bigcup_{n=1}^N \Omega^n \,\cup\, \bigcup_{n=1}^N \Gamma^n$ with $\Gamma^n = \partial\Omega^n \cap \left(\bigcup_{\substack{p=1\\ p\neq n}}^N \partial\Omega^p \right) - \partial\Omega$.
Substructuring techniques consist then in reducing the original global system to an interface problem by a block Gaussian elimination of the internal degrees of freedom and in iteratively solving the resulting variational interface problem :

$$\exists \bar{\mathbf{u}} \in \bar{V} \quad / \quad < \mathbf{S}\bar{\mathbf{u}}, \bar{\mathbf{v}} > = < \bar{\mathbf{f}}, \bar{\mathbf{v}} > \qquad \forall \bar{\mathbf{v}} \in \bar{V} = \mathrm{Tr}\, H(\Omega)|_\Gamma . \tag{1}$$

The matrices $\mathbf{S} = \sum_{i=1}^N \mathbf{R}^i \mathbf{S}^i (\mathbf{R}^i)^t$ and $\mathbf{S}^i$ denote respectively the global Schur complement matrix (defined on Γ) and the local Schur complement matrices (defined on Γ^i by $\mathbf{S^i} = \bar{\mathbf{K}}^\mathbf{i} - (\mathbf{B^i})^\mathbf{t} (\mathring{\mathbf{K}}^\mathbf{i})^{-\mathbf{1}} \mathbf{B^i}$). Above, $(\mathbf{R}^i)^t$ is the restriction operator which goes from Γ to Γ^i, and $\mathbf{K^i} = \begin{pmatrix} \mathring{\mathbf{K}}^i & \mathbf{B^i} \\ (\mathbf{B^i})^\mathbf{t} & \bar{\mathbf{K}}^i \end{pmatrix}$ denotes the subdomain stiffness matrix, the first block corresponding to the internal degrees of freedom $\mathring{\mathbf{X}}^i$, the second one corresponding to the interface degrees $\bar{\mathbf{X}}^i$. The interface problem (1) can be solved by a preconditioned conjugate gradient method (symmetric cases) or the GMRES method (nonsymmetric cases). Hereafter, we use the multilevel Neumann-Neumann preconditioner. This iterative technique never requires the explicit calculation of the matrix $\mathbf{S}$. We have just to form the matrix vector products $\mathbf{S}\bar{\mathbf{p}}$ and $\mathbf{M^{-1}}\bar{\mathbf{r}}$ by solving independent auxiliary Dirichlet and Neumann problems on the local subdomains and a global coarse problem defined on a space of singular (rigid body) motions. Altogether, the product of the preconditioner $\mathbf{M}^{-1}$ and of the residual gradient $\bar{\mathbf{r}}$ has the following form,

$$\mathbf{M}^{-1}\bar{\mathbf{r}} = \sum_{i=1}^N \left\{ \mathbf{D}^i \, (\tilde{\mathbf{S}}^i)^{-1} \, (\mathbf{D}^i)^t \, \bar{\mathbf{r}} \right\} - \mathbf{G}\,\gamma,$$

where $\mathbf{D}^i$ is a weighting matrix, defining a local partition of unity on the interface and $(\tilde{\mathbf{S}})^{-1}$ denotes an regularized inverse of $\mathbf{S}^i$. Moreover, $\mathbf{G}\,\gamma$ is linear combination of subdomain rigid body motions over the interface obtained by projection of the residual onto this set of rigid body motions. In practice, the projection $\mathbf{G}\,\gamma$ is obtained by solving a global optimization problem over the interface Γ in order to minimize the residual [LT94] :

$$\min{}_{\mathbf{G}\gamma} \|\bar{\mathbf{r}}\|_\Gamma^2 := \min{}_{\mathbf{G}\gamma} \left\{ (\mathbf{S}\,(\mathbf{M}^{-1} - \mathbf{S}^{-1})\,\bar{\mathbf{r}})^t \,(\mathbf{M}^{-1} - \mathbf{S}^{-1})\,\bar{\mathbf{r}} \right\} . \tag{2}$$

This balanced preconditioner is very general and can be efficiently applied to linear or nonlinear three-dimensional elasticity problems using either matching or nonmatching grids [TSV94], to nonlinear plate or shell problems [TMV98].

A first "mechanical" nonsymmetric extension

As constructed above, the basic balanced Neumann Neumann preconditioner is not well adapted to nonsymmetric problems. Indeed the minimization problem (2) is not well defined for nonsymmetric Schur complement matrices. The numerical experiments [BAV01] also show that the behaviour of the iterative Schur complement solver (GMRES algorithm) is strongly perturbed when applied to structural problems with friction, i.e. when nonsymmetry is introduced in the tangent matrices [4]. The first idea is to replace the matrix $\mathbf{S}$ by the symmetrized matrix $\mathbf{S}^s$ ($\mathbf{S}^s = \mathbf{S} + \mathbf{S}^t$). Another choice is to use a symmetric matrix which has a mechanical meaning [BAV01] considering the interface reduced matrix $\mathbf{S}^*$ with a zero friction coefficient ($\mathbf{S}^* = \mathbf{S}_{\mu=0}$) to evaluate the norm of the difference between $\mathbf{M}^{-1}$ and $\mathbf{S}^{-1}$ and so to formulate the coarse problem. Then the minimization problem takes the following form :

$$\min{}_{\mathbf{G}\gamma} |||\bar{\mathbf{r}}|||_{\Gamma}^2 := \min{}_{\mathbf{G}\gamma} \left\{ (\mathbf{S}^* \; (\mathbf{M}^{-1} - \mathbf{S}^{-1}) \; \bar{\mathbf{r}})^t \; (\mathbf{M}^{-1} - \mathbf{S}^{-1}) \; \bar{\mathbf{r}} \right\}. \tag{3}$$

This minimum is reached for the function $\mathbf{G}\,\gamma$ which cancels its gradient, which defines $\mathbf{G}\,\gamma$ as the solution of the following equality :

$$(\mathbf{G}^t \; \mathbf{S}^* \; \mathbf{G}) \; \gamma = -\mathbf{G}^t \; \mathbf{S}^* \; \sum_{i=1}^{N} \left(\mathbf{D}^i \; (\tilde{\mathbf{S}}^{\mathbf{i}})^{-1} \; (\mathbf{D}^i)^t \right) \; \bar{\mathbf{r}}, \tag{4}$$

which defines the coarse problem specially adapted to the nonsymmetry of the friction [BAV01]. As we will see later, the dependence due to nonsymmetry is reduced, but it is nonoptimal. So, to establish a general nonsymmetric preconditioner, we now introduce a generalisation of this preconditioner by viewing it as an additive Schwarz method.

Interpretation as additive Schwarz methods and general extension to nonsymmetric problems

The Neumann-Neumann preconditioner can in fact be viewed as an additive Schwarz technique [TV97] iteratively solving an interface problem with operator $A = \mathbf{S}$ on the interface space $\bar{V}$ using the preconditioner

$$\mathbf{M}^{-1} = \tilde{\mathbf{A}}_o^{-1} + \sum_i \mathbf{I}^i (\tilde{\mathbf{A}}^i)^{-1} (\mathbf{I}^i)^t.$$

Above, the operator $\tilde{\mathbf{A}}_0 = \tilde{\mathbf{S}}$ (resp. $\tilde{\mathbf{A}}^i = \tilde{\mathbf{S}}^{\mathbf{i}}$) denotes an approximate restriction of the original operator $\mathbf{S}$ onto the coarse space $\bar{V}_G = \sum_{i=1}^N D_i \bar{Z}_i \subset \bar{V}$ (resp. onto the local spaces $\bar{V}_i^{\perp}$), the local spaces $\bar{V}_i^{\perp} \subset \bar{V}_i = \mathrm{Tr}\; H(\Omega)|_{\Gamma^i}$ are locally defined by duality

$$\bar{V}_i^{\perp} = \{ \bar{\mathbf{v}}_f \in D_i \bar{V}_i, \quad < \mathbf{S}\bar{\mathbf{v}}_f, \bar{\mathbf{v}}_G >= 0, \qquad \forall \bar{\mathbf{v}}_G \in \bar{V}_G \}$$

and the extension from local to global space is given by $\mathbf{I}^i = (\mathbf{I} - \mathbf{P}_G)\mathbf{D}^i$, with $\mathbf{P}_G : \bar{V} \to \bar{V}_G$ the orthogonal $\tilde{\mathbf{S}}$ projection. This extension operator is in fact the key originality of the Neumann-Neumann preconditioner. With this notation, the additive Schwarz preconditioner reduces to the previous preconditioner

$$\mathbf{M}^{-1} = \tilde{\mathbf{S}}_0^{-1} + \sum_i (\mathbf{I} - \mathbf{P}_G)\mathbf{D}^i(\tilde{\mathbf{S}}^i)^{-1}(\mathbf{D}^i)^t(\mathbf{I} - \mathbf{P}_G)^t. \tag{5}$$

operating within the orthogonal of the coarse space, that is the image of the projection $(\mathbf{I} - \mathbf{P}_G)$.

The basic question is now to properly construct the local component $\bar{Z}_i$ of the coarse space $\bar{V}_G$. The objective is that its **orthogonal** complement (where the preconditioner lives) be nice. With a detailed examination, it can be observed that being nice means in fact that:
- the local Neumann solutions $\mathbf{w}^i$ must be scale invariant in energy norm, which requires to put all constants to zero in the local Neumann subproblems,
- the local Neumann subproblems must be regularized by adding a few boundary conditions.

Altogether, one only needs to impose implicitly that a few constants or boundary conditions $\mathbf{C}_\alpha^i$ be equal to zero for the solutions $\mathbf{w}^i$ of the local Neumann problems. We therefore need them to satisfy

$$\langle \mathbf{w}^i, \mathbf{C}_\alpha^i \rangle = 0, \forall \alpha,$$

that is

$$\langle \mathbf{K}^i \mathbf{w}^i, (\mathbf{K}^i)^{-t} \mathbf{C}_\alpha^i \rangle = 0, \forall \alpha,$$

or equivalently, since $\mathbf{w}^i$ is solution of a local Neumann problem with matrix $\mathbf{K}^i$

$$\langle \mathbf{D}^i \mathbf{S}^i \bar{\mathbf{v}}, (\mathbf{K}^i)^{-t} \mathbf{C}_\alpha^i \rangle = 0, \forall \alpha, \bar{\mathbf{v}} \in \bar{V}_G^\perp.$$

This is automatically guaranteed if $\bar{\mathbf{v}}$ is orthogonal to the function $\mathbf{D}^i(\mathbf{K}^i)^{-T}\mathbf{C}_\alpha^i$, that is if the local space is generated by the so called dual rigid modes as follows

$$\bar{Z}_i = \text{vect}((\mathbf{K}^i)^{-t}\mathbf{C}_\alpha^i).$$

Detailed algorithm

The adapted strategy which generalizes the approach of both the symmetric and the advection case, is thus given by the following steps [PAV00] :

1. Identify the local degrees of freedom $(P_{i\alpha})_{\alpha=1,N_i}$ which cancel all N_i rigid modes of subdomain i. In practice, this is done by identification of the small pivots in the factorization of the associated local stiffness matrix, with the possibility of choosing more degrees of freedom than necessary. For plate and shell problems, we can simply choose the degrees of freedom which lie on subdomain corners.

2. Introduce a regularization $\mathbf{K}_R^i$ of the local stiffness $\mathbf{K}^i$ on $V_i = H(\Omega^i)$

$$< \mathbf{K}_R^i \mathbf{v}^i, \hat{\mathbf{v}}^i > = < \mathbf{K}^i \mathbf{v}^i, \hat{\mathbf{v}}^i > + \sum_{\alpha,\beta} \mathbf{M}_{\alpha\beta}^i \mathbf{v}^i(P_{i\alpha})\hat{\mathbf{v}}^i(P_{i\beta}), \quad \forall \mathbf{v}^i, \hat{\mathbf{v}}^i \in V_i,$$

   the matrix $\mathbf{M}^i$ being a definite positive arbitrary matrix. For nonsymmetric problems, the matrices $\mathbf{K}^i$ and $\mathbf{K}_R^i$ are nonsymmetric.

3. Compute **dual rigid modes** $(\mathbf{v}^i_{G\alpha})_{\alpha=1,N_i}$ by solving local regularized Neumann problems set on the space V_i of subdomain displacement functions defined on subdomains i,

$$< (\mathbf{K}^i_R)^t \mathbf{v}^i_{G\alpha}, \hat{\mathbf{v}}^i >= \hat{\mathbf{v}}^i(P_{i\alpha}), \forall \hat{\mathbf{v}}^i \in V_i, \mathbf{v}^i_{G\alpha} \in V_i. \tag{6}$$

   For advection-diffusion problems or for unsteady problems, we must also introduce the dual constant mode defined by,

$$< (\mathbf{K}^i_R)^t \mathbf{v}^i_G, \hat{\mathbf{v}}^i >= \int_{\Omega^i} \hat{\mathbf{v}}^i, \quad \forall \hat{\mathbf{v}}^i \in V_i, \mathbf{v}^i_G \in V_i, \tag{7}$$

   in order to achieve scale invariance in the Neumann subproblems.

4. Introduce the local rigid space $Z_i = \text{vect}\left(\mathbf{v}^i_{G\alpha}, \alpha = 1, N_i\right)$.

The last construction leads to the local rigid spaces already introduced for symmetric cases [TV97] or for the advection-diffusion case [ATNV00]. The space Z_i does not depend on the choice of the regularized matrix $\mathbf{M}^i$ because all elements $\mathbf{v}^i$ of Z_i verify by construction,

$$< (\mathbf{K}^i)^t \mathbf{v}^i, \hat{\mathbf{v}}^i >= 0, \forall \hat{\mathbf{v}}^i \in V_i \text{ such that } \hat{\mathbf{v}}^i(P_{i\beta}) = 0, \forall \beta.$$

With this choice, the 2-level Neumann-Neumann preconditioner takes the form defined in (5)

$$\mathbf{M}^{-1}\mathbf{S} = \mathbf{P}_G + \sum_{i=1}^{N} (\mathbf{I} - \mathbf{P}_G)\mathbf{D}^i(\tilde{\mathbf{S}}^i)^{-1}(\mathbf{D}^i)^t(\mathbf{I} - \mathbf{P}_G)^t\mathbf{S}. \tag{8}$$

Above, the regularized Schur inverse $(\tilde{\mathbf{S}}^i)^{-1}$ acting on a given linear form L_i defined on the local interface space $\bar{V}_i'$ yields the interface vector $(\tilde{\mathbf{S}}^i)^{-1}L_i = Tr(\mathbf{w}^i)_{\Gamma^i}$ obtained by solution of the local regularized Neumann problem :

$$< \mathbf{K}^i_R \mathbf{w}^i, \hat{\mathbf{v}}^i >= L_i(Tr(\hat{\mathbf{v}}^i)_{|\Gamma^i}), \quad \forall \hat{\mathbf{v}}^i \in V_i, \mathbf{w}^i \in V_i. \tag{9}$$

Our construction ensures that the solutions $\mathbf{w}^i = (\tilde{\mathbf{S}}^i)^{-1}(\mathbf{D}^i)^t(\mathbf{I} - \mathbf{P}_G)^t\bar{\mathbf{r}}$ of the local Neumann problems have rigid constants $\mathbf{w}^i(P_{i\alpha})$ fixed to zero. Indeed, by definition of the dual rigid modes $\mathbf{v}^i_{G\alpha}$ and by the construction of $\mathbf{w}^i$ and by the projection $\mathbf{P}_G$, we have :

$$\mathbf{w}^i(P_{i\alpha}) =< (\mathbf{K}^i_R)^t \mathbf{v}^i_{G\alpha}, \mathbf{w}^i >=< \bar{\mathbf{r}}, (\mathbf{I} - \mathbf{P}_G)\mathbf{D}^i \mathbf{v}^i_{G\alpha} >= 0. \tag{10}$$

This value of the rigid constant on $\mathbf{w}^i$ cancels the effect of the regularization. We have indeed: $< \mathbf{K}^i_R \mathbf{w}^i, \mathbf{w}^i >=< \mathbf{K}^i \mathbf{w}^i, \mathbf{w}^i >$, which guarantees in some way the optimality of our algorithm.

Application to frictional contact problems

Nonsymmetric frictional contact problems

The behaviour of multicontact structures is characterized by a multiplicity of contact interfaces between deformable structure bodies. These large nonlinear problems constitute a class of problems well suited to the use of the above numerical substructuring techniques. The modelling of the frictional contact problem is first based on a hybrid formulation presented in Alart and Curnier [PC91]. Following this augmented Lagrangian approach [PC91], the equilibrium of a discretized contact bodies system is governed by the system of nonlinear equations

$$\left\{ \begin{array}{l} F_{int} - F_{ext} + \mathbf{N}^t \mathcal{F}(\mathbf{u}, \lambda) = 0, \\ -\frac{1}{r}(\lambda - \mathcal{F}(\mathbf{u}, \lambda)) = 0, \end{array} \right. \tag{11}$$

where $\mathbf{N}$ is a restriction operator from Ω to Γ_c (Γ_c is the contact boundary). The notation $\mathbf{u}$ stands for kinematic variables (displacements or rotations) and λ for the static variables (contact forces or torques). Moreover, $\mathcal{F}(\mathbf{u}, \lambda)$ defines the discretized contact operator, with r the corresponding penalty coefficient, F_{int} and F_{ext} denote respectively the internal and the external discretized forces,

$$< F_{int}(\mathbf{u}), \hat{\mathbf{v}} >= \int_{\Omega} E\sigma(\nabla_{sym}\mathbf{u}) : \nabla_{sym}\hat{\mathbf{v}} \quad \text{and} \quad < F_{ext}, \hat{\mathbf{v}} >= \int_{\Omega} f \cdot \hat{\mathbf{v}},$$

and $\mathcal{F}(\mathbf{u}, \lambda)$ is the assembly of elementary contributions according to the notion of contact element [PC91]. For sake of simplicity, the local contact operator is presented for a contact between a deformable body and a rigid obstacle in a bidimensional modelling. Consequently the displacement $\mathbf{u}$ concerns only the node of the body on Γ_c and λ the contact force exerted by Γ_c on the obstacle. It is convenient to split it into normal and tangential components $\lambda = \lambda_n \mathbf{n} + \lambda_t$ and to express $\mathcal{F}^e(\mathbf{u}, \lambda)$ in this local frame :

$$\mathcal{F}^e(\mathbf{u}, \lambda) = \sigma_n^- \mathbf{n} + Proj_{C(\sigma_n^-)} \sigma_t, \tag{12}$$

where $\sigma = \sigma_n \mathbf{n} + \sigma_t$, $\sigma_n = \lambda_n + r u_n$, $\sigma_t = \lambda_t + r\delta\mathbf{u}_t$, $\sigma_n^- = min(\sigma_n, 0)$ and $C(\sigma_n^-)$ the Coulomb set $[\mu\sigma_n^-, -\mu\sigma_n^-]\mathbf{t})$ (where μ is the Coulomb coefficient and $\delta\mathbf{u}_t$ is a displacement increment). If the contact status is sliding, the tangent matrix of this operator is non symmetric and takes the tensorial form

$$\partial_\lambda \mathcal{F}^e(\mathbf{u}, \lambda) = (\mathbf{n} - \mu\mathbf{t}) \otimes \mathbf{n}, \quad \partial_u \mathcal{F}^e(\mathbf{u}, \lambda) = r(\mathbf{n} - \mu\mathbf{t}) \otimes \mathbf{n}.$$

For more complex contact elements, this type of local matrix is distributed on all contact nodes of target contactor areas.
We have chosen to treat both variables $\mathbf{u}$ and λ simultaneously through Newton's method. The system of equations is then split into two parts involving the pair $\mathbf{x} = (\mathbf{u}, \lambda)$, i.e. a differentiable elastic part G and a nondifferentiable frictional contact one $\mathcal{F}$

$$\mathbf{G}(\mathbf{x}) + \mathcal{F}(\mathbf{x}) = 0. \tag{13}$$

To overcome the nondifferentiability of the equation (13), Newton's method may be extended to the following iterative form [PC91]:

$$(\mathbf{K}^m + \mathbf{A}_c^m)\, \Delta\mathbf{x}^m = -\left(\mathbf{G}(\mathbf{x}^m) + \mathcal{F}(\mathbf{x}^m)\right) \quad \text{where} \quad \Delta\mathbf{x}^m = \mathbf{x}^{m+1} - \mathbf{x}^m, \tag{14}$$

to be solved at each iteration m by the previously introduced generalized Neumann-Neumann domain decomposition method. The matrix $\mathbf{K}^m = \partial\mathbf{G}(\mathbf{x}^m)$ is the usual elastic stiffness matrix and $\mathbf{A}_c^m \in \partial\mathcal{F}(\mathbf{x}^m)$ represents the generalized Jacobian of $\mathcal{F}$ at $\mathbf{x}^m$. The nonsymmetry of the matrix $\mathbf{A}_c^m$ is due to the friction terms. The contact interface is discretized by contact finite elements which yields elementary nonsymmetric tangent matrices if the contact status is "in friction situation".

"Multi-contact" structures

The efficiency of these different multilevel preconditioners will be assessed on two examples of "multicontact" structures :
- collections of deformable grains with contact interfaces between the grains.
- rolling shutters composed by many slats jointed by a hinge with play and eventually rotative friction.

Collection of deformable grains

Our motivation here is to study in granular media modelling the behaviour of a collection of deformable grains submitted to classical solicitations such as shear or compression. This problem is an interesting and delicate "multicontact" problem : the proportion of contact is very large. The interactions between the grains are governed by the frictional contact laws (Signorini unilateral contact law and Coulomb friction law).

At a discrete level, the interactions between grains are modelled by a frictional contact

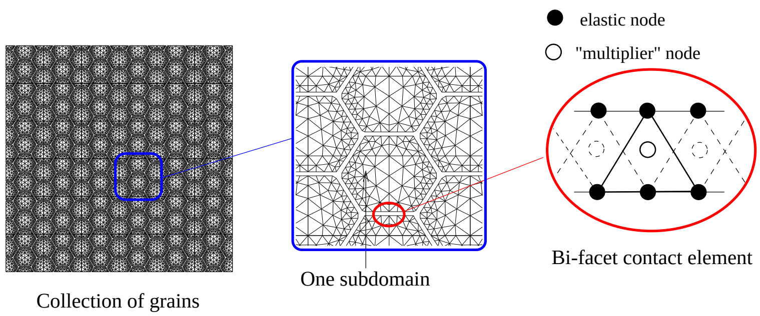

Figure 1: Deformable grains, one subdomain and a bi-facet contact element.

element (Figure 1) which takes into account large slip over the contact interface. This bi-facet contact element has 5 nodes : 4 elastic nodes which contain the displacement $\mathbf{u}$ (u_x, u_y) and a multiplier node containing the frictional contact forces λ. Moreover the contactor node can slip over two target facets. A generalization to more facets can be carried out easily.

Rolling shutters composed by many hinged slats

The aim of this problem is to simulate the quasi-static behavior of such shutters submitted to strong winds [ABLM99]. A rolling shutter is a specific case of multi-contact structure. The rolling shutters for shops, stores and hangars are formed by a succession of slats jointed by a hinge [ABLM99]. Such a structure is then composed by an assembly of elastic structures (plates in flexion and torsion) which leads to consider a large number of contact zones. The edges of the slats are designed in such a way that the slats fit into each other. To facilitate the rolling of the shutters at the opening, the profile of the slat requires a gap or a play in the hinge. We must then develop a specific model which takes into account the play $(-g, +g)$ in the hinge and eventually the friction in the rotations $(\delta\beta)$ of the hinges between the slats. The contact and friction laws are more complicated than the usual case. For more details on the modelling, see [ABLM99].

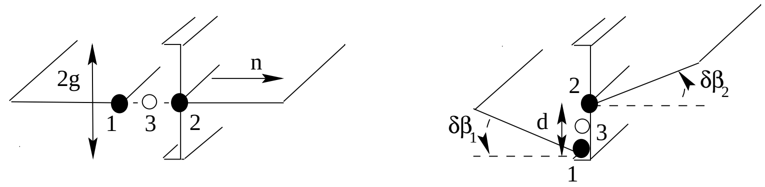

Figure 2: hinge contact element.

Substructuring strategy

One feature of this nonlinear nonsymmetric domain decomposition strategy consists in putting the numerical subdomain interfaces away from the physical contact interfaces [BAV01]. Contrary to current approaches we therefore suggest to treat the physical contact interfaces as internal surfaces : the contact interfaces (hinges for shutters and contact area for deformable grains) must be inside the subdomains and do not constitute decomposition interfaces. Thus, the decomposition is not forced to respect the geometry of its components; such a subdomain is shown in Figure 1. This allows a better balance of the size of the subdomains and leads to an optimal decomposition for parallel efficiency.

Numerical behaviour of Neumann-Neumann preconditioners

In this section, we analyse the convergence behaviour of the interface solver (GMRES) with the multi-level Neumann-Neumann preconditioners. We test their efficiency as a function of the friction coefficient and the number of subdomains (scalability properties). As previously observed, the nonsymmetry is due to our formulation of frictional contact problems. The considered preconditioners are :

- The standard Neumann-Neumann preconditioner with coarse space (2-level),
- The specific Neumann-Neumann preconditioner which uses a symmetrized matrix $\mathbf{S}^*$ (with a friction coefficient equal to zero),
- The new nonsymmetric Neumann-Neumann preconditioner introduced in this paper.

The first result, described in Figure 3, gives the evolution of average number of GMRES

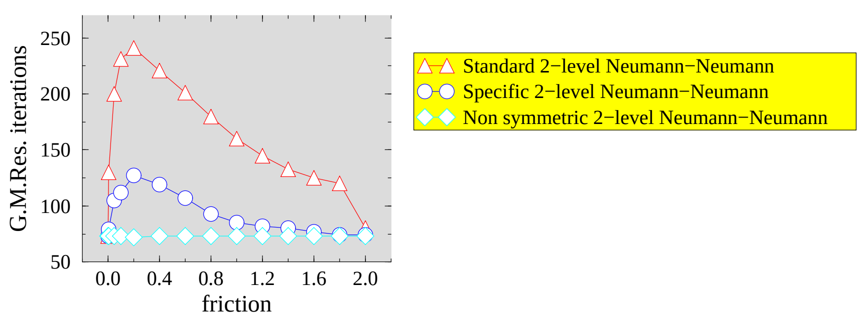

Figure 3: Influence of the friction coefficient on the preconditioners.

iterations (per Newton iterations) for different values of the friction coefficient varying from

0 to 2 for a rolling shutters with 16 slats and 30 subdomains (26 floating subdomains), respectively. We observe the inefficiency of the solver using the standard Neumann-Neumann preconditioner (curve $\triangle$) for values of friction coefficient close to $\mu = 0,2$. This is due to the large increase of the ratio of slip status and so to the large proportion of nonsymmetry. The first extension procedure (curve $\circ$) improves this dependance but does not cancel it. On the other hand, the new nonsymmetric preconditioner (curve $\diamond$) makes the interface solver insensitive to the nonsymetry.

Next, we analyse the scalability properties of the different Neumann-Neumann preconditioners for the problems of rolling shutters and collections of deformable grains. For the rolling shutters (figure 4), we can verify that for a problem without friction ($\mu = 0$, symmetric problem), the 2-level Neumann-Neumann preconditioner has a classical behaviour : independence from subdomain number (curve $*$). But with friction, the standard procedure

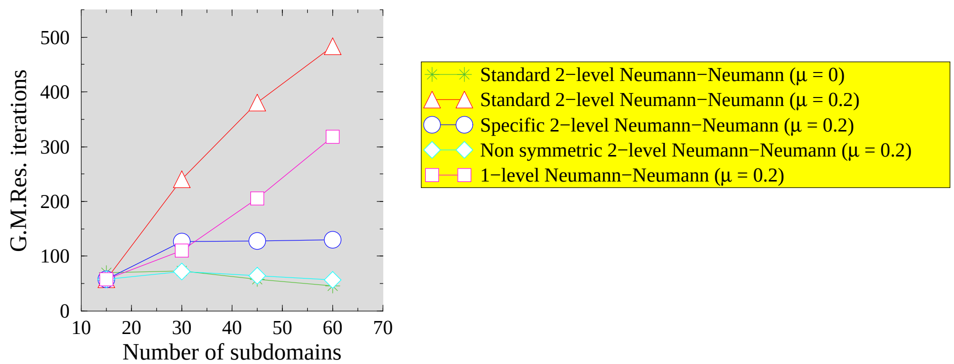

Figure 4: Numerical scalability of the preconditioners (rolling shutters).

leads to a high increase of the number of iterations (curve $\triangle$) with the number of subdomains. The results are even worse than without coarse solver (curve). The first extension strategy (curve $\circ$) improves the convergence but is not optimal. On the other hand, the 2-level nonsymmetric Neumann-Neumann preconditioner (curve $\diamond$) leads to a full recovery of the numerical scalability properties obtained with a symmetric problem.

We finally present for the collection of deformable grains the influence of the number of sub-domains (Figure 5) on the number of iterations. The good behaviour of the nonsymmetric preconditioner is confirmed when the number of floating subdomains increases. This nonsymmetric procedure is more efficient than the standard and specific balancing method specially in presence of shear. Indeed, the friction (and then the nonsymmetry) plays a more important role in shear than in compression (Figure 5). Thus the strategy developed in this paper extends to large scale nonsymmetric (frictional contact) problems.

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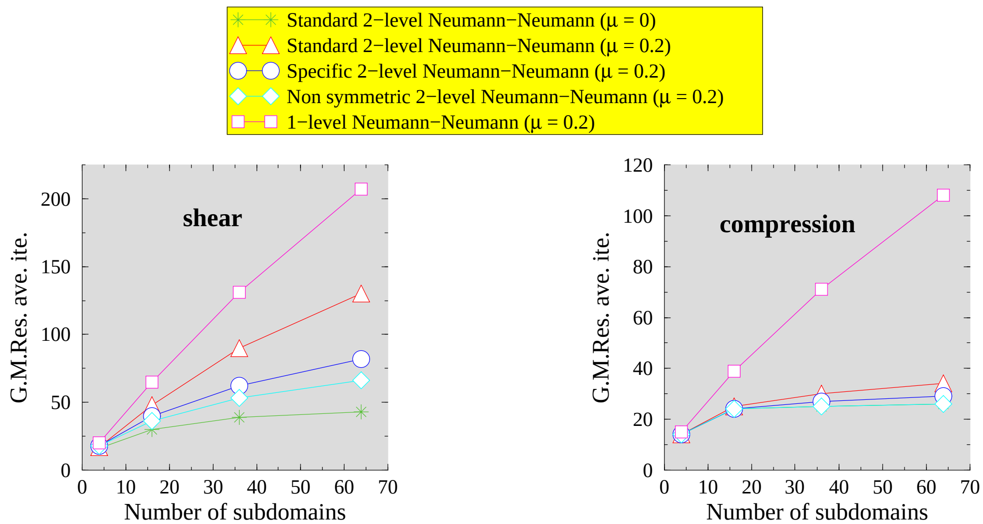

Figure 5: Numerical scalability of the preconditioners (deformable grains).

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