15 Mortar spectral element discretization of Darcy’s equations

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Introduction

Darcy’s equations model the filtration of an incompressible viscous fluid in porous media. However, exactly the same equations are involved in the mixed formulation of the Laplace equation with Neumann boundary conditions and also in the projection–diffusion algorithm of Chorin [Cho68] and Temam [Tem68] for solving the time-dependent Navier–Stokes equations. So proposing discretizations of this problem which are both accurate and efficient, seems rather important. We first write its variational formulation, which involves the domain of the divergence operator, and prove that it is well-posed. We describe a spectral discretization of the problem that relies on the mortar domain decomposition technique introduced by Bernardi, Maday and Patera [BMP94], since it combines the accuracy of standard spectral methods with the advantage of handling complex geometries via the mortar algorithm. We prove the convergence of the discrete solution towards the exact one and derive error estimates.

Detailed proofs of the results presented in this paper can be found in [ABB03], and numerical experiments are under consideration.

Darcy’s equations and their variational formulation

Let Ω be a bounded connected domain in \(\mathbb{R}^d\), \(d = 2\) or \(3\), with a Lipschitz–continuous boundary, and let \(\mathbf{n}\) denote the unit normal vector outward to \(\Omega\). Darcy’s equations in this domain write

\[
\begin{align*}
\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\
\text{div } \mathbf{u} &= 0 \quad \text{in } \Omega, \\
\mathbf{u} \cdot \mathbf{n} &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where the unknowns are the velocity \(\mathbf{u}\) and the pressure \(p\). In order to write the variational formulation of problem (1), we first consider the space

\[
H(\text{div}, \Omega) = \left\{ \mathbf{v} \in L^2(\Omega)^d ; \text{div } \mathbf{v} \in L^2(\Omega) \right\},
\]

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provided with the natural norm
\[ \| \mathbf{v} \|_{H(\text{div}, \Omega)} = \left( \|\mathbf{v}\|_{L^2(\Omega)^d}^2 + \|\text{div} \mathbf{v}\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}. \]  
(3)

We note that \( H(\text{div}, \Omega) \) is a Hilbert space and we recall from [GR86](Chap. I, Thm 2.4) that the space \( \mathcal{D}(\overline{\Omega})^d \) of restrictions of infinitely differentiable functions on \( F^d \) to \( \overline{\Omega} \) is dense in \( H(\text{div}, \Omega) \). As a consequence, the trace operator: \( \mathbf{v} \mapsto \mathbf{v} \cdot \mathbf{n} \), defined from the formula
\[ \forall \varphi \in H^1(\Omega), \quad (\mathbf{v} \cdot \mathbf{n}, \varphi) = \int_{\Omega} (\mathbf{v} \cdot \text{grad} \varphi + (\text{div} \mathbf{v}) \varphi)(\mathbf{x}) \, d\mathbf{x} \]  
(4)

is continuous from \( H(\text{div}, \Omega) \) onto the dual space \( H^{-\frac{d}{2}}(\partial \Omega) \). So, we can now define the subspace
\[ H_0(\text{div}, \Omega) = \{ \mathbf{v} \in H(\text{div}, \Omega) ; \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial \Omega \}, \]  
(5)

which is also a Hilbert space and is the closure for the norm defined in (3) of the space \( \mathcal{D}(\overline{\Omega})^d \) of functions in \( \mathcal{D}(\overline{\Omega})^d \) with a compact support in \( \Omega \). Finally, we introduce the space
\[ L_0^2(\Omega) = \{ q \in L^2(\Omega) ; \int_{\Omega} q(\mathbf{x}) \, d\mathbf{x} = 0 \}. \]  
(6)

The variational formulation of problem (1) now reads

Find \((\mathbf{u}, p)\) in \( H_0(\text{div}, \Omega) \times L_0^2(\Omega) \) such that
\[ \forall \mathbf{v} \in H_0(\text{div}, \Omega), \quad a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = \int_{\Omega} f(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) \, d\mathbf{x}, \]  
\[ \forall q \in L_0^2(\Omega), \quad b(\mathbf{u}, q) = 0, \]  
(7)

where the bilinear forms \( a(\cdot, \cdot) \) and \( b(\cdot, \cdot) \) are defined by
\[ a(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \mathbf{u}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) \, d\mathbf{x}, \quad b(\mathbf{v}, q) = -\int_{\Omega} (\text{div} \mathbf{v})(\mathbf{x})q(\mathbf{x}) \, d\mathbf{x}. \]  
(8)

From the density of \( \mathcal{D}(\Omega)^d \) in \( H_0(\text{div}, \Omega) \), it is readily checked that problem (7) is equivalent to problem (1). Problem (7) is of saddle-point type, and the arguments for proving its well-posedness are given in [GR86] (Chap. I, Thm 4.1). First, the bilinear forms \( a(\cdot, \cdot) \) and \( b(\cdot, \cdot) \) are continuous on \( H_0(\text{div}, \Omega) \times H_0(\text{div}, \Omega) \) and \( H_0(\text{div}, \Omega) \times L_0^2(\Omega) \), respectively. Second, let \( V \) stand for the kernel
\[ V = \{ \mathbf{v} \in H_0(\text{div}, \Omega) ; \forall q \in L_0^2(\Omega), \quad b(\mathbf{v}, q) = 0 \}, \]  
(9)

or, equivalently,
\[ V = \{ \mathbf{v} \in H_0(\text{div}, \Omega) ; \text{div} \mathbf{v} = 0 \text{ in } \Omega \}. \]  
(10)

The following ellipticity property is then obvious
\[ \forall \mathbf{v} \in V, \quad a(\mathbf{v}, \mathbf{v}) = \|\mathbf{v}\|_{H(\text{div}, \Omega)}^2. \]  
(11)
Third, the following inf-sup condition, for a constant $\beta > 0$,

$$\forall q \in L^2_0(\Omega), \quad \sup_{\nu \in H_0(\text{div},\Omega)} \frac{b(\nu,q)}{\|\nu\|_{H(\text{div},\Omega)}} \geq \beta \|q\|_{L^2(\Omega)}, \quad (12)$$

is derived by taking $\nu$ equal to $\text{grad} \varphi$, where $\varphi$ is the solution of the Laplace equation with data $q$ and homogeneous Neumann boundary conditions. Combining all this leads to the following statement.

**Proposition 1.** For any data $f$ in $L^2(\Omega)^d$, problem (7) has a unique solution $(u,p)$ in $H_0(\text{div},\Omega) \times L^2_0(\Omega)$.

Unfortunately, even for smooth data, the solution of problem (7) is not very regular. For any data $f$ in $L^2(\Omega)^d$ such that $\text{curl} f$ belongs to $L^2(\Omega)^{2d-3}$, the solution $(u,p)$ belongs to $H^s(\Omega)^d$ for $s = \frac{1}{2}$ in the general case, $s = 1$ if $\Omega$ is convex and some $s > \frac{1}{2}$ if $\Omega$ is a polygon or polyhedron (we refer to [Cos90], [Dau92] and [ABDG98] for these results).

**Remark:** Another variational formulation of problem (1) exists, where the spaces $H_0(\text{div},\Omega)$ and $L^2_0(\Omega)$ are replaced by $L^2(\Omega)^d$ and $H^1(\Omega) \cap L^2_0(\Omega)$, respectively. Then, the boundary conditions in (1) are enforced in a variational way. However this second formulation does not seem appropriate when Darcy’s system appears in the discretization of the Stokes problem, since the pressure in this problem does not belong to $H^1(\Omega)$ in most cases when $\Omega$ is a non convex polygon or polyhedron.

**The mortar spectral element discrete problem**

From now on, in view of applying the mortar element method to our problem, we assume that $\Omega$ admits a disjoint decomposition into a finite number of (open) rectangles in dimension $d = 2$, rectangular parallelepipeds in dimension $d = 3$:

$$\Omega = \bigcup_{k=1}^K \overline{\Omega}_k \quad \text{and} \quad \Omega_k \cap \Omega_{k'} = \emptyset, \quad 1 \leq k \neq k' \leq K. \quad (13)$$

We make the further assumption that the intersection of each $\partial \Omega_k$ with $\partial \Omega$, if not empty, is a corner, a whole edge or a whole face of $\Omega_k$. We denote by $\mathbf{n}_k$, $1 \leq k \leq K$, the unit normal vectors outward to $\Omega_k$. We introduce the skeleton $\mathcal{S}$ of the decomposition, $\mathcal{S} = \bigcup_{k=1}^K \partial \Omega_k \setminus \partial \Omega$. According to the ideas in [BMP94], we choose a disjoint decomposition of this skeleton into mortars:

$$\mathcal{S} = \bigcup_{m=1}^M \overline{\gamma}_m \quad \text{and} \quad \gamma_m \cap \gamma_{m'} = \emptyset, \quad 1 \leq m \neq m' \leq M, \quad (14)$$

where each $\gamma_m$ is a whole edge in dimension $d = 2$, face in dimension $d = 3$, of a subdomain $\Omega_k$, denoted by $\Omega_{k(m)}$. To describe the discrete spaces, for each nonnegative integer $n$, we define on each $\Omega_k$, resp. on each edge or face $\Gamma$ of $\Omega_k$, the space $P_n(\Omega_k)$, resp. $P_n(\Gamma)$, of restrictions to $\Omega_k$, resp. $\Gamma$, of polynomials with $d$, resp. $d-1$, variables and degree $\leq n$ with
respect to each variable. The discretization parameter $\delta$ is then a $K$-tuple $(N_1, \ldots, N_K)$ of integers $N_k \geq 2$. We first introduce the space $M_\delta(\Omega)$ of discrete pressures:

\[ M_\delta(\Omega) = \{ q_\delta \in L^2_0(\Omega); \quad q_\delta|_{\Omega_k} \in \mathcal{P}_{N_k-2}(\Omega_k), \quad 1 \leq k \leq K \}. \]  

(15)

Next, in analogy with the standard definition of the mortar approximation of $H^1(\Omega)$ [BMP94], we define the discrete space $X_\delta(\Omega)$ which approximates $H_0(\text{div}, \Omega)$. It is the space of functions $v_\delta$ such that:

- their restrictions $v_\delta|_{\Omega_k}$ to each $\Omega_k$, $1 \leq k \leq K$, belong to $\mathcal{P}_{N_k}(\Omega_k)^d$, 
- their normal traces $v_\delta \cdot \textbf{n}$ vanish on $\partial \Omega$, 
- the mortar function $\varphi$ being defined on each $\gamma_m$, $1 \leq m \leq M$, by

\[ \varphi|_{\gamma_m} = v_\delta|_{\Omega_{k(m)}} \cdot \textbf{n}_{k(m)}, \]  

(16)

the following matching condition holds on each edge or face $\Gamma$ of $\Omega_k$, $1 \leq k \leq K$, which is not a mortar:

\[ \forall \chi \in \mathcal{P}_{\nu_{k-2}}(\Gamma), \quad \int_{\Gamma} (v_\delta|_{\Omega_k} \cdot \textbf{n}_k + \varphi(\tau) \chi(\tau)) \, d\tau = 0. \]  

(17)

**Remark:** The space $X_\delta$ is not contained in $H(\text{div}, \Omega)$ since the matching conditions on the normal derivative through the interfaces are only enforced in a weak way. So the discretization is nonconforming. Starting from the standard Gauss–Lobatto formula on $]-1,1[$, we define on each $\Omega_k$ and in each direction:

- the nodes $x^k_i$ and $y^k_i$, and the weights $\rho_i^{x,k}$ and $\rho_i^{y,k}$, $0 \leq i \leq N_k$, in the case of dimension $d = 2$, 
- the nodes $x^k_i$, $y^k_i$, and $z^k_i$, and the weights $\rho_i^{x,k}$, $\rho_i^{y,k}$ and $\rho_i^{z,k}$, $0 \leq i \leq N_k$, in the case of dimension $d = 3$.

A discrete product is then introduced on each $\Omega_k$, according if $d = 2$ or $3$, by

\[ (u_\delta, v_\delta)_\delta = \begin{cases} 
\sum_{i=0}^{N_k} \sum_{j=0}^{N_k} u_\delta(x^k_i, y^k_j) v_\delta(x^k_i, y^k_j) \rho_i^{x,k} \rho_j^{y,k} \\
\sum_{i=0}^{N_k} \sum_{j=0}^{N_k} \sum_{p=0}^{N_k} u_\delta(x^k_i, y^k_j, z^k_p) v_\delta(x^k_i, y^k_j, z^k_p) \rho_i^{x,k} \rho_j^{y,k} \rho_p^{z,k}.
\end{cases} \]  

(18)

The global discrete product on $\Omega$:

\[ (u_\delta, v_\delta)_\delta = \sum_{k=1}^{K} (u_\delta, v_\delta)_\delta, \]  

(19)

coincides with the scalar product of $L^2(\Omega)$ for all functions $u_\delta$ and $v_\delta$ such that each product $(u_\delta v_\delta)|_{\Omega_k}$, $1 \leq k \leq K$, belongs to $\mathcal{P}_{2N_k-2}(\Omega_k)$. The discrete problem is now built from the variational formulation (7). For any continuous data $f$ on $\Omega$, it reads

Find $(u_\delta, p_\delta)$ in $X_\delta(\Omega) \times M_\delta(\Omega)$ such that

\[ \forall v_\delta \in X_\delta(\Omega), \quad a_\delta(u_\delta, v_\delta) + b_\delta(v_\delta, p_\delta) = (f, v_\delta)_\delta, \]  

\[ \forall q_\delta \in M_\delta(\Omega), \quad b_\delta(u_\delta, q_\delta) = 0, \]  

(20)

where the bilinear forms $a_\delta(\cdot, \cdot)$ and $b_\delta(\cdot, \cdot)$ are defined by

\[ a_\delta(u_\delta, v_\delta) = (u_\delta, v_\delta)_\delta, \quad b_\delta(v_\delta, q_\delta) = -\langle \text{div} v_\delta, q_\delta \rangle_\delta. \]  

(21)
Note however that, thanks to the exactness property of the quadrature formula, we have
\[ b_{\delta}(\mathbf{v}_{\delta}, q_{\delta}) = b_{\delta}(\mathbf{v}, q_{\delta}). \]

To check the well-posedness of problem (20), we first state the discrete analogue of the inf-sup condition in (12), its proof combines the arguments in [ABG94] and [BBCM00]. It involves the “broken” norm

\[ \|\nabla\|_{H(div; \Omega_k)} = \left( \sum_{k=1}^{K} \|\nabla\|_{H(div; \Omega_k)}^2 \right)^{\frac{1}{2}}. \]  

**Lemma 2.** There exists an integer \( N_D \) and a positive constant \( \beta_D \), both depending on the decomposition of \( \Omega \) but independent of \( \delta \), such that, if all the \( N_k \) are \( \geq N_D \), the following inf-sup condition holds

\[ \forall q_{\delta} \in M_\delta(\Omega), \quad \sup_{\mathbf{v}_{\delta} \in X_\delta(\Omega)} \frac{b_{\delta}(\mathbf{v}_{\delta}, q_{\delta})}{\|\mathbf{v}_{\delta}\|_{H(div; \Omega_k)}} \geq \beta_D \|q_{\delta}\|_{L^2(\Omega)}, \]  

**Proposition 3.** For any continuous data \( f \) on \( \bar{\Omega} \) and if all the \( N_k \) are \( \geq N_D \), problem (20) has a unique solution \((\mathbf{u}_{\delta}, p_{\delta})\) in \( X_\delta(\Omega) \times M_\delta(\Omega) \).

**Proof:** Problem (20) results into a square linear system, so that it has a unique solution if and only if the only solution for \( f = 0 \) is \((0, 0)\). So we take \( f \) equal to 0. Choosing \( \mathbf{v}_{\delta} \) equal to \( \mathbf{u}_{\delta} \) in (20) yields that \( a_{\delta}(\mathbf{u}_{\delta}, \mathbf{u}_{\delta}) \) is zero and, since the weights of the Gauss–Lobatto formula are positive, this implies that \( \mathbf{u}_{\delta} \) vanishes in the \((N_k + 1)^d\) nodes of a tensorized grid on each \( \Omega_k \), hence is zero. Then, \( b_{\delta}(\mathbf{v}_{\delta}, q_{\delta}) \) is equal to zero for all \( \mathbf{v}_{\delta} \) in \( X_\delta \), hence \( p_{\delta} \) is zero due to (24).

**A priori analysis**

The main difficulty for evaluating the error on the velocity comes from the fact that the form \( a_{\delta}(\cdot, \cdot) \) is no longer uniformly elliptic with respect to the norm \( \|\cdot\|_{H(div; \Omega_k)} \) on the discrete kernel

\[ V_{\delta} = \{ \mathbf{v} \in X_{\delta}(\Omega); \forall q_{\delta} \in M_\delta(\Omega), \quad b_{\delta}(\mathbf{v}_{\delta}, q_{\delta}) = 0 \}, \]  

since \( V_{\delta} \) is not made of exactly divergence-free functions. So the usual arguments for bounding the error do not hold, and we must evaluate “by hand” the quantity \( \|\mathbf{u} - \mathbf{u}_{\delta}\|_{L^2(\Omega)^d} \). It involves three terms:

- the approximation error, which is easy to evaluate in dimension \( d = 2 \) but requires some further conformity assumptions in dimension \( d = 3 \),
- the error issued from numerical integration,
- the consistency error, which gives rise to a term of type (here, \([\cdot]\) denotes the jump through \( S \) with the appropriate sign)

\[ \inf_{q_{\delta} \in M_\delta(\Omega)} \sup_{\mathbf{w}_{\delta} \in X_\delta(\Omega)} \frac{\int_S (\mathbf{w}_{\delta} \cdot \mathbf{n})(\tau) [p-q_{\delta}](\tau) d\tau}{\|\mathbf{w}_{\delta}\|_{H(div; \Omega_k)}}, \]  

and it seems that using an inverse inequality is unavoidable to bound this term, which leads to non optimal estimates. Once the error on the velocity is evaluated, the error on the pressure is
derived from the inf-sup condition (24). Let \( \mu_\delta \) denote the maximal ratio \( N_k/N_{k'} \) for all pairs of subdomains \( \Omega_k \) and \( \Omega_{k'} \), \( 1 \leq k \neq k' \leq K \), such that \( \partial \Omega_k \cap \partial \Omega_{k'} \) has a positive measure in \( S \).

**Theorem 4.** In dimension \( d = 2 \), assume the data \( f \) such that each \( f_{\Omega_k} \), \( 1 \leq k \leq K \), belongs to \( H^{s_k}(\Omega_k)^2 \), \( \sigma_k > 1 \), and the solution \( (u, p) \) of problem (1) such that each \( (u|_{\Omega_k}, p|_{\Omega_k}) \), \( 1 \leq k \leq K \), belongs to \( H^{s_k}(\Omega_k)^2 \times H^{s_k+1}(\Omega_k) \), \( s_k > 0 \). If all the \( N_k \) are \( \geq N_P \), the following error estimate holds between this solution \( (u, p) \) and the solution \( (u_\delta, p_\delta) \) of problem (20):

\[
\|u - u_\delta\|_{L^2(\Omega)^2} + \|p - p_\delta\|_{L^2(\Omega)} \\
\leq c \sum_{k=1}^{K} (\mu_\delta N_k^{-s_k} (\|u\|_{H^{s_k}(\Omega_k)^2} + \|p\|_{H^{s_k+1}(\Omega_k)})) + \sum_{k=1}^{K} (\mu_\delta N_k^{-s_k} \|f\|_{H^{s_k}(\Omega_k)^2}).
\]

(27)

This estimate is not optimal, however the same lack of optimality appears in several finite element discretizations of Darcy’s equations (for instance, when Crouzeix–Raviart finite elements are employed for the approximation of the velocity). Moreover, if the parameter \( \mu_\delta \) is bounded independently of \( \delta \), the convergence of the method can be derived form this estimate in all polygons, thanks to the regularity results stated in Section 1.

To conclude, we recall that the decomposition (13) of \( \Omega \) is said to be conforming if the intersection of all \( \partial \Omega_k \) and \( \partial \Omega_{k'} \), \( 1 \leq k < k' \leq K \), if not empty, is a whole edge in dimension \( d = 2 \), a whole face in dimension \( d = 3 \), of both \( \Omega_k \) and \( \Omega_{k'} \). The mortar element method does not require the conformity of the decomposition. However, if the decomposition is conforming, an approximation \( q_\delta \) of the pressure \( p \) can be constructed in \( M_\ell(\Omega) \cap H^1(\Omega) \), which means that the quantity in (26) vanishes for this \( q_\delta \). So the error estimate is optimal in this case.

**Corollary 5.** If all assumptions of Theorem 4 hold and if, moreover, (i) the decomposition (13) of \( \Omega \) is conforming, (ii) in dimension \( d = 3 \), for \( 1 \leq m \leq M \), \( N_{k(m)} \) is \( \geq N_k \), where \( \gamma_m \) is the intersection of this \( \Omega_k \) and \( \Omega_{k(m)} \), the following error estimate holds between the solution \( (u, p) \) of problem (1) and the solution \( (u_\delta, p_\delta) \) of problem (20):

\[
\|u - u_\delta\|_{L^2(\Omega)^2} + \|p - p_\delta\|_{L^2(\Omega)} \\
\leq c \sum_{k=1}^{K} (N_k^{-s_k} (\|u\|_{H^{s_k}(\Omega_k)^2} + \|p\|_{H^{s_k+1}(\Omega_k)})) + \sum_{k=1}^{K} (N_k^{-s_k} \|f\|_{H^{s_k}(\Omega_k)^2}).
\]

(28)

**Conclusion**

As a conclusion, the mortar spectral element discretization of problem (1) is fully optimal in the case of a conforming decomposition. It is not for a nonconforming decomposition,
however estimate (27) can be improved in this case by taking into account the local properties of conformity. It can also be noted that, for smooth data, the solution of problem (1) is regular outside a neighbourhood of the corners and edges of $\Omega$, so that enforcing the conformity of the decomposition is more important in a neighbourhood of $\partial\Omega$ than elsewhere.

References