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27 Recent development on Aitken-Schwarz method

J. Baranger, M. Garbey, F. Oudin-Dardun [1]

Introduction

The idea of using Aitken acceleration [Hen64] [SB80], on the classical Schwarz additive domain decomposition method has been introduced in [GTD99]. For an elliptic operator with constant coefficient on a regular grid, this method is called Aitken-Schwarz procedure, and is a direct solver. This method has shown very good numerical performances, and has been used in more complex situations [GTD01].

In this work, we extend Aitken-Schwarz procedure to the case of a 2-D cartesian grid, **not necessarily regular**. The key idea is the replacement of the 1-D Fourier transform used on the regular space step discretization of the artificial interface grid by a transform using the eigenvectors of a suitable 1-D operator. For simplicity, this presentation is limited here to the Laplacian operator and to two subdomains. However, our method can be applied to the Helmholtz operator for example and one-dimensional domain decomposition with an arbitrary number of subdomains.

In section 2, we recall the basic idea of Aitken-Schwarz method on a regular grid. In section 3, we describe two extensions of the method on a general cartesian grid : one using all the eigenvectors, the other a limited number of them. Numerical experiments are described and analyzed in section 4.

Aitken-Schwarz method on a regular grid

We first recall the basic ideas of Aitken-Schwarz method as described in [GTD01].

Let us consider a linear problem

$$L[U] = f \text{ in } \Omega,\ U_{|\partial\Omega} = 0. \tag{1}$$

We partition the domain Ω into two overlapping strips : $\Omega = \Omega_1 \cup \Omega_2$; Γ_1 (resp.Γ_2) denotes the part of the boundary of Ω_1 (resp. Ω_2) which is not included in $\partial\Omega$ (boundary of Ω). The additive Schwarz algorithm is :

$$L[u_1^{n+1}] = f \text{ in } \Omega_1,\ u_{1|\Gamma_1}^{n+1} = u_{2|\Gamma_1}^{n}, \tag{2}$$

$$L[u_2^{n+1}] = f \text{ in } \Omega_2,\ u_{2|\Gamma_2}^{n+1} = u_{1|\Gamma_2}^{n}. \tag{3}$$

We observe that the operator T, defined by :

$$T : (u_{1|\Gamma_2}^{n} - U_{|\Gamma_2}, u_{2|\Gamma_1}^{n} - U_{|\Gamma_1}) \rightarrow (u_{1|\Gamma_2}^{n+1} - U_{|\Gamma_2}, u_{2|\Gamma_1}^{n+1} - U_{|\Gamma_1})$$

[1] MCS-ISTIL - University Lyon 1, 69622 Villeurbanne, France
{baranger, garbey, foudin}@mcs.univ-lyon1.fr

is linear.

Let us consider first the one-dimensional case $\Omega = (0,1)$; then, we have the following linear (affine) relation :

$$\left\{ \begin{array}{l} u^{n+1}_{1|\Gamma_2} - U_{|\Gamma_2} = \delta_1 (u^n_{2|\Gamma_1} - U_{|\Gamma_1}), \\ u^{n+1}_{2|\Gamma_1} - U_{|\Gamma_1} = \delta_2 (u^n_{1|\Gamma_2} - U_{|\Gamma_2}), \end{array} \right. \tag{4}$$

where δ_1 (resp. δ_2) is the amplification factor associated to the operator L in subdomain Ω_1 (resp. Ω_2). Consequently :

$$\left\{ \begin{array}{l} u^2_{1|\Gamma_2} - u^1_{1|\Gamma_2} = \delta_1 (u^1_{2|\Gamma_1} - u^0_{2|\Gamma_1}), \\ u^2_{2|\Gamma_1} - u^1_{2|\Gamma_1} = \delta_2 (u^1_{1|\Gamma_2} - u^0_{1|\Gamma_2}). \end{array} \right. \tag{5}$$

So, except if the initial boundary conditions $u^0_{1|\Gamma_2}$ or $u^0_{2|\Gamma_1}$ matches with the exact solution U at the interfaces Γ_i, the amplification factors δ_1 and δ_2 can be computed from (5). Then, if $\delta_1 \delta_2 \neq 1$, the limit $U_{|\Gamma_i}$, $i = 1, 2$, is obtained as the solution of the linear system (4).

The Aitken acceleration procedure gives the exact limit of the sequence on the interface Γ_i based on two successive Schwarz iterates $u^n_{i|\Gamma_i}$, $n = 1, 2$, and the initial condition $u^0_{i|\Gamma_i}$. An additional solve of each subproblem (2), (3) with boundary conditions $u^\infty_{i|\Gamma_i}$, gives the solution of (4). The Aitken acceleration thus transforms the additive Schwarz procedure into an exact solver regardless of the speed of convergence of the original Schwarz method.

Let us consider now the 2-D Poisson problem $u_{xx} + u_{yy} = f$ in the square $\Omega = (0,1)^2$, with Dirichlet boundary conditions.

We introduce the regular discretization in the y-direction : $y_i = (i-1)h$, $h = \dfrac{1}{N-1}$, and central second-order finite differences approximation of the u_{yy} derivative.

Let us denote by $\hat{u}_i$ (resp. $\hat{f}_i$) the coefficient of the sine expansion of u (resp. f). The Poisson problem decomposes then into N independent semi-discretised equations corresponding to sinus waves $\sin(iy)$, $i = 1, ..., N$:

$$\hat{u}_{i,xx} - 4/h^2 \sin^2(ih\pi/2)\hat{u}_i = \hat{f}_i. \tag{6}$$

These N problems are linear, 1-D and independent. We can apply to each of them the Aitken acceleration procedure described above. Then, the algorithm becomes :

- step 1 : compute two iterates of Schwarz algorithm with solver of 2-dimensional subdomain problem of choice;
- step 2 : compute the sine expansion of the traces on the artificial interfaces, for the two iterates and the initial condition ; apply Aitken acceleration separately to each wave coefficients ; re-compose the trace ;
- step 3 : compute an other iteration of Schwarz algorithm.

This method gives satisfactory results. Its parallel implementation share the same communication pattern than the parallel CFD test case [ETL99]. The analysis of its parallel efficiency is therefore well known. However, this method is limited to grid with constant space step in y-direction. We proceed then with a generalization of this method to arbitrary space step grid in y direction.

Method on a general cartesian grid

Let us consider the Poisson problem $u_{xx} + u_{yy} = f$ in the square $\Omega = (0,1)^2$, with Dirichlet boundary conditions. We consider a P_1 finite element approximation on the triangles obtained by cutting each element of the cartesian mesh of Ω :

$$\Omega = \bigcup_{i,j} K_{i,j},$$

where $K_{i,j} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, and $h_i = x_i - x_{i-1}$, $k_j = y_j - y_{j-1}$.

Let φ_{ij} be the P_1 basis function associated to the node (x_i, y_j) ; then, the finite element approximation of u is defined by :

$$u_h = \sum_{i,j} u_{ij}\varphi_{ij},$$

where the unknowns u_{ij} satisfied the following equations :

$$\begin{array}{l} \left\{ -\frac{1}{h_i}u_{i-1j} + (\frac{1}{h_i} + \frac{1}{h_{i+1}})u_{ij} - \frac{1}{h_{i+1}}u_{i+1j} \right\} \frac{k_j + k_{j+1}}{2} + \\ \left\{ -\frac{1}{k_j}u_{ij-1} + (\frac{1}{k_j} + \frac{1}{k_{j+1}})u_{ij} - \frac{1}{k_{j+1}}u_{ij+1} \right\} \frac{h_i + h_{i+1}}{2} = f_{ij}, \end{array} \tag{7}$$

for $i = 1, ..., N$ and $j = 1, ..., M$, with boundary conditions. Let us define the vector $u_i = (u_{ij})_j$; these equations appear in matrix form:

$$-\frac{1}{h_i}Ku_{i-1} + ((\frac{1}{h_i} + \frac{1}{h_{i+1}})K + \frac{h_i + h_{i+1}}{2}\overline{K})u_i - \frac{1}{h_{i+1}}Ku_{i+1} = f_i,$$

where the matrix K and $\overline{K}$ are defined by : $K = diag(\frac{1}{k_j} + \frac{1}{k_{j+1}})$ and $\overline{K} = tridiag(-\frac{1}{k_j}, (\frac{1}{k_j} + \frac{1}{k_{j+1}}), -\frac{1}{k_{j+1}})$.

Remark 1 : In the case of a uniform mesh : $h_i = k_j = h$, the equations (7) becomes:

$$-u_{i+1j} - u_{i-1j} + 4u_{ij} - u_{ij-1} - u_{ij+1} = f_{ij}.$$

If we consider the sine expansion of u_{ij} :

$$u_{ij} = \sum_{k=1}^{M} \hat{u}_{ik} \sin(kjh),$$

and if we notice that : $\sum_{j=1}^{M} \sin(kjh)\sin(ljh) = \frac{M+1}{2}\delta_{kl}$, then we obtain the discrete analogue of (6) :

$$-\hat{u}_{i+1l} - \hat{u}_{i-1l} + d_l\hat{u}_{il} = \hat{f}_{il}, \tag{8}$$

for $i = 1, ..., N$, $l = 1, ..., M$ and with $d_l = 2(1 + 2\sin^2(lh/2))$. So the Aitken procedure must be applied separately on each mode •

For a general cartesian grid, coming back to (7), we define in place of the Fourier transform as in Remark 1, a new transformation based on vector Φ_l to be chosen later:

$$u_{ij} = \sum_{l=1}^{M} \hat{u}_{il} \Phi_{lj}.$$

For technical reasons, which will be clear later, we introduce weights d_i ; by multiplication of each line of equations (7) by $(\frac{2}{k_j + k_{j+1}} d_j^2 \Phi_{mj})$ and summation on the parameter j, we obtain :

$$\begin{aligned}
&\sum_{l=1}^{M} \left\{ \left(-\frac{1}{h_i}\hat{u}_{i-1l} - \frac{1}{h_{i+1}}\hat{u}_{i+1l} + (\frac{1}{h_i} + \frac{1}{h_{i+1}})\hat{u}_{il} \right) \sum_{j=1}^{M} d_j \Phi_{lj} d_j \Phi_{mj} \right. \\
&\left. + \hat{u}_{il} \frac{h_i + h_{i+1}}{2} \sum_{j=1}^{M} d_j \left(-\frac{1}{k_j}\Phi_{lj-1} - \frac{1}{k_{j+1}}\Phi_{lj+1} + (\frac{1}{k_j} + \frac{1}{k_{j+1}})\Phi_{lj} \right) \frac{2}{k_j + k_{j+1}} d_j \Phi_{mj} \right\} \\
&= \sum_{j=1}^{M} \frac{2}{k_j + k_{j+1}} f_{ij} d_j^2 \Phi_{mj}.
\end{aligned} \tag{9}$$

We use the following notations (with $D = diag(d_j)$) :

- $\tilde{c}_{lm} = \sum_{j=1}^{M} d_j \Phi_{lj} d_j \Phi_{mj} = (D\Phi_l, D\Phi_m)$;
- $\tilde{d}_{lm} = \sum_{j=1}^{M} d_j \left(-\frac{1}{k_j}\Phi_{lj-1} - \frac{1}{k_{j+1}}\Phi_{lj+1} + (\frac{1}{k_j} + \frac{1}{k_{j+1}})\Phi_{lj} \right) \frac{2}{k_j + k_{j+1}} d_j \Phi_{mj}$
  $= (D\tilde{K}\Phi_l, D\Phi_m)$,
  where $(.,.)$ denotes the discrete scalar product, $\Phi_l = (\Phi_{lj})_j$ and
  $\tilde{K} = tridiag(\frac{-2}{k_j(k_j + k_{j+1})}, \frac{2}{k_j k_{j+1}}, \frac{-2}{k_{j+1}(k_j + k_{j+1})})$.

In order to obtain an uncoupled system for the unknowns vectors $\hat{u}_l = (\hat{u}_{il})_{i=1,\ldots,M}$, we need to diagonalise simultaneously the matrices $(\tilde{c}_{lm})$ and $(\tilde{d}_{lm})$. The matrix $\tilde{K}$ is non symmetric, but the choice

$$d_j = \sqrt{\frac{k_{j+1} + k_j}{k_2 + k_1}} \tag{10}$$

implies that the matrix $D\tilde{K}D^{-1}$ is symmetric.

We now choose the vectors $D\Phi_l$ as the orthogonal family of eigenvectors of the matrix $D\tilde{K}D^{-1}$, and denote by λ_l the eigenvalue associated to $D\Phi_l$:

$$D\tilde{K}D^{-1}(D\Phi_l) = \lambda_l D\Phi_l. \tag{11}$$

Then :

$$\tilde{c}_{lm} = |D\Phi_l|^2 \delta_{lm} \tag{12}$$

and :

$$\tilde{d}_{lm} = \lambda_l |D\Phi_l|^2 \delta_{lm}. \tag{13}$$

We can now state :

Theorem 1 *With d_j defined by (10) and (λ_l, Φ_l) defined by (11), the system (9) is uncoupled in $\hat{u}_l$:*

$$\left\{ -\frac{1}{h_i}\hat{u}_{i-1l} - \frac{1}{h_{i+1}}\hat{u}_{i+1l} + ((\frac{1}{h_i} + \frac{1}{h_{i+1}}) + \frac{h_i + h_{i+1}}{2}\lambda_l)\hat{u}_{il} \right\} |D\Phi_l|^2 = \sum_{j=1}^{M} \frac{2}{k_j + k_{j+1}} f_{ij} d_j^2 \Phi_{lj},$$

for $i = 1, ..., M$.

Each small system in $\hat{u}_l$ is the analog for the general grid of the discrete version of (6) for the regular grid that is (8).

We denote by $\hat{u}_{n_1 l}$ (resp. $\hat{u}_{n_2 l}$) the transformed unknowns corresponding to the boundary Γ_1 (resp. Γ_2) ; $(\hat{u}_{n_1 l})_l$ and $(\hat{u}_{n_2 l})_l$ satisfy a relation analogous to (4). Aitken acceleration procedure is applied to each of them.

Finally, we remark that :

$$(Du_i, D\Phi_l) = \sum_m \hat{u}_{im}(D\Phi_m, D\Phi_l) = \hat{u}_{il}|D\Phi_l|^2,$$

which allows to compute the l-component of u_i for the decomposition into the vectors basis Φ_l. We can now summarize the Aitken-Schwarz algorithm extended to grid in y direction with arbitrary space step:

Algorithm 1

- Step 1 : computation of the matrix D and $D\tilde{K}D^{-1}$;
- Step 2 : research of an orthogonal family of eigenvectors of $D\tilde{K}D^{-1}$, noted $(U_l)_l$;
- Step 3 : decomposition in the base $(\Phi_l)_l$ (with $\Phi_l = D^{-1}U_l$) of u_{n_1} and u_{n_2} (which compute vectors $(\hat{u}_{n_1 m})_m$ and $(\hat{u}_{n_2 m})_m$) ;
- Step 4 : acceleration of all modes, using (3) and (4) ;
- Step 5 : recomposition of the trace, using $u_{n_i} = \sum_l \hat{u}_{n_i l}\Phi_l$ $(i = 1, 2)$.

We notice that in case of Poisson solve with multiple right-hand sides as in Pressure solve for the time integration of the unsteady Navier Stokes equation with the projection method, Step 1 and 2 can be done once for all. The arithmetic complexity of the method is then dominated by the subdomain problem solves. Otherwise the arithmetic complexity of step 1 and 2 is of order N^3, and therefore slightly higher than a fast Poisson solver. But such fast Poisson solver does not work anyway with tensorial product grid exhibiting an arbitrary space step in one of the spatial direction.

We can improve the efficiency of our method by accelerating only the P first modes ($P < M$) corresponding to the P smallest eigenvalues. The efficiency of this method depends then on how small is the damping factor for the remaining higher order modes, and how good is the truncated representation of the trace on the interface. All these question are well known for the Fourier case, but less clear for grid with arbitrary space steps.

This second algorithm writes:

Algorithm 2

- Step 1 : computation of the matrix D and $D\tilde{K}D^{-1}$;
- Step 2 : research of an orthogonal family of P eigenvectors (corresponding to the P first eigenvalues) of $D\tilde{K}D^{-1}$, noted $(U_l)_{l=1,\dots,P}$;
- Step 3 : decomposition in the base $(\Phi_l)_{l=1,\dots,P}$ (with $\Phi_l = D^{-1}U_l$) of u_{n_1} and u_{n_2} (which compute vectors $(\hat{u}_{n_1 m})_m$ and $(\hat{u}_{n_2 m})_m$) ;
- Step 4 : acceleration of the P first modes, using (3) and (4) ;
- Step 5 : recomposition of the trace, using :

$$u_{n_i} = \sum_{l=1}^{P} \hat{u}^{\infty}_{n_i l}\Phi_l + \sum_{l=P+1}^{M} \hat{u}^{2}_{n_i l}\Phi_l, \quad i = 1, 2,$$

(where $(\hat{u}^{\infty}_{n_i l})_l$ denotes the accelerated vector, and $(\hat{u}^{2}_{n_i l})_l$, the last iterated vector).

The level of truncation P should be decided adaptively, comparing for example acceleration with P modes and $P+1$ modes. We proceed now with few numerical experiments illustrating the method.

Numerical results

We consider, on the domain $\Omega =]0,1[\times]0,1[$, the Poisson problem : $-(u_{xx} + u_{yy}) = f$, with $u = g$ on $\partial\Omega$, such that the exact solution is : $u(x,y) = 150x(x-1)y(y-1)(y-1/2)$. We use a cartesian grid of Ω with 31×31 elements, uniform in x, randomize in y, (see Figure 1) and an overlap on one element.

The algorithm using all the modes gives the error and residue shown in Figure 2. The error is then of order 10^{-6}, after one Aitken acceleration. However, one can apply the method twice to reduce this error to machine precision level. Figure 3 show compares the performance of Algorithm 2 depending on the number of modes that are accelerated. Several cycles of Schwarz Aitken acceleration are applied. We are using a direct solver for the subdomains problems and P has then a marginal impact on the number of flops. For $P \leq 20$, further cycles of Aitken Schwarz acceleration does not improve the situation because the higher modes left out from the acceleration process are the limiting factor of convergence.

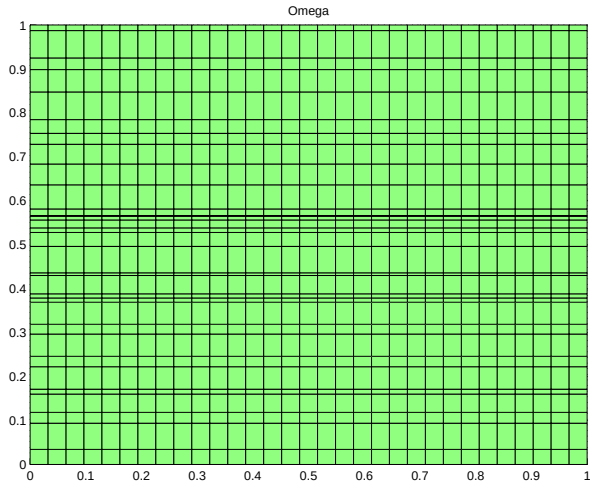

Figure 1: Mesh

Conclusion

We have shown a generalization of the so-called Aitken-Schwarz algorithm to the Poisson problem discretised on tensorial product grid with arbitrary space step in each direction. The arithmetic complexity of the method is slightly more expensive than in the case of constant space step. We expect that the Steffensen-Schwarz analogue of this method will be numerically efficient for nonlinear problem that are perturbation of the Laplace operator as the Bratu problem. However the generalization of this method to fully unstructured grid remains the interesting challenge.

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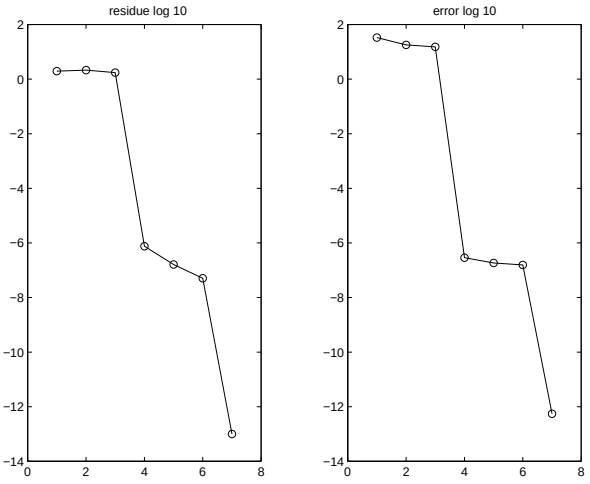

Figure 2: Error and residue - all modes

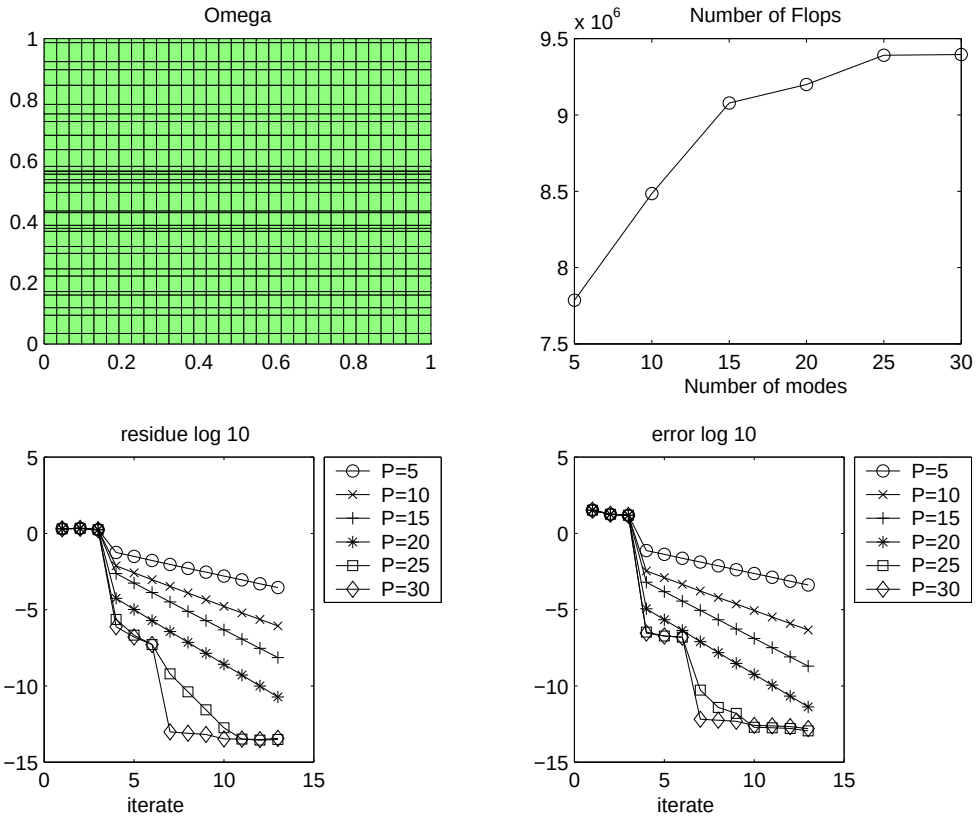

Figure 3: Error and residue - first modes