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16 Substructuring techniques and Wavelets for Domain Decomposition

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Introduction

We consider in this paper a substructuring approach for preconditioning the linear system arising from the reduction to the interface unknown of the discrete three fields formulation of domain decomposition. In particular we concentrate on choosing the stabilization technique, needed to circumvent the otherwise very restrictive inf-sup conditions required for stability and convergence, in such a way that the stabilized method falls in the range for which the estimate on the preconditioner holds. For such preconditioner to work, it is in fact necessary that the stabilized bilinear form verifies continuity and coercivity *with respect to the same norm*. This leads us to choose a stabilization technique based on adding a residual term on the subdomain boundaries, measured in the natural norm of type $H^{1/2}$. The $H^{1/2}$ type scalar product can be cheaply realized in terms of a wavelet decomposition. Remark that wavelets are employed here as a tool for implementing stabilization and they do not need to be employed as discretization space.

A substructuring preconditioner for the three fields domain decomposition method

Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain. We will consider the following simple model problem: given $f \in L^2(\Omega)$, find u satisfying

$$-\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega. \tag{1}$$

In this paper we consider the *three fields domain decomposition* formulation of such a problem [BM94]. More precisely, considering for simplicity a geometrically conforming decomposition $\Omega = \cup_k \Omega_k$, with Ω_k quadrangles regular in shape, $\Gamma_k = \partial\Omega_k$, and letting $\Sigma = \cup_k \Gamma_k$, we introduce the following functional spaces

$$V = \prod_k H^1(\Omega_k), \qquad \Lambda = \prod_k H^{-1/2}(\Gamma_k),$$
$$\Phi = \{\varphi \in L^2(\Sigma) : \text{there exists } u \in H^1_0(\Omega),\ u = \varphi \text{ on } \Sigma\} = H^1_0(\Omega)|_\Sigma,$$

respectively equipped with the norms:

$$\|u\|_V^2 = \sum_k \|u^k\|^2_{H^1(\Omega_k)}, \qquad \|\lambda\|_\Lambda^2 = \sum_k \|\lambda^k\|^2_{H^{-1/2}(\Gamma_k)},$$

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and (see [Ber00a])

$$\|\varphi\|_{\Phi}^2 = \inf_{u \in H^1_0(\Omega): u=\varphi \text{ on } \Sigma} \|u\|^2_{H^1(\Omega)} \simeq \sum_k |\varphi|^2_{H^{1/2}(\Gamma_k)}.$$

We remark that here and in the following we will use the notation c and C to indicate several positive constants independent of any relevant parameter, like the mesh size or the size of the subdomains. The expression $A \simeq B$ will stand for $cA \leq B \leq CA$.

Let $a^k : H^1(\Omega_k) \times H^1(\Omega_k) \to \mathbb{R}$ denote the bilinear form corresponding to the Laplace operator:

$$a^k(w, v) = \int_{\Omega_k} \nabla w \nabla v.$$

The continuous three fields formulation of equation (1) is the following ([BM94]): find $(u, \lambda, \varphi) \in V \times \Lambda \times \Phi$ such that

$$\begin{cases} \forall k, \ \ \forall v^k \in H^1(\Omega_k), \ \ \forall \mu^k \in H^{-1/2}(\Gamma_k) : \\ \quad a^k(u^k, v^k) \ - \int_{\Gamma_k} v^k \lambda^k \qquad\qquad = \int_{\Omega_k} f v^k, \\ -\int_{\Gamma_k} u^k \mu^k \qquad\qquad + \int_{\Gamma_k} \mu^k \varphi \ = \ 0, \\ \\ \text{and } \forall \psi \in \Phi : \\ \qquad\qquad \sum_k \int_{\Gamma_k} \lambda^k \psi \qquad\qquad = \ 0. \end{cases} \tag{2}$$

It is known that this problem admits a unique solution (u, λ, φ), where u is indeed the solution of (1) and such that $\lambda^k = \partial u^k / \partial \nu^k$ on Γ_k, and $\varphi = u$ on Σ, where ν^k denotes the outer normal derivative to the subdomain Ω_k. After choosing discretization spaces $V_h = \prod_k V_h^k \subset \prod_k H^1(\Omega_k)$, $\Lambda_h = \prod_k \Lambda_h^k \subset \prod_k H^{-1/2}(\Gamma_k)$ and $\Phi_h \subset \Phi$, equation (2) can be discretized by a Galerkin scheme. The linear system stemming from such an approximation takes the form

$$\begin{pmatrix} A & B^T & 0 \\ B & 0 & C^T \\ 0 & C & 0 \end{pmatrix} \cdot \begin{pmatrix} \underline{u}_h \\ \underline{\lambda}_h \\ \underline{\varphi}_h \end{pmatrix} = \begin{pmatrix} \underline{f} \\ 0 \\ 0 \end{pmatrix}, \tag{3}$$

($\underline{u}_h$, $\underline{\lambda}_h$, and $\underline{\varphi}_h$ being the vectors of the coefficients of u_h, λ_h and φ_h in the bases chosen for V_h, Λ_h and Φ_h respectively). By a Schur complement argument the solution of (3) can be reduced to a system in the unknown $\underline{\varphi}_h$, which takes the form

$$\mathbf{C}\mathbf{A}^{-1}\mathbf{C}^T \, \underline{\varphi}_h = -\mathbf{C}\mathbf{A}^{-1} \begin{pmatrix} \underline{f} \\ 0 \end{pmatrix}, \quad \mathbf{C} = [\ 0 \quad C\ ], \quad \mathbf{A} = \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}. \tag{4}$$

The matrix $S = \mathbf{C}\mathbf{A}^{-1}\mathbf{C}^T$ does not need to be assembled. The system (4) can rather be solved by an iterative technique (like for instance a conjugate gradient method) and therefore only the action of S on a given vector needs to be implemented. Multiplying by S implies the need for solving a linear system with matrix $\mathbf{A}$. This reduces, by a proper reordering of the unknowns, to independently solving a discrete Dirichlet problem with Lagrange multipliers in each subdomain.

Existence, uniqueness and stability of the solution of the discretized problem rely on the validity of two *inf-sup* conditions,

$$\inf_{\lambda_h \in \Lambda_h} \sup_{u_h \in V_h} \frac{\sum_k \int_{\Gamma_k} \lambda_h^k u_h^k}{\|u_h\|_V \|\lambda_h\|_\Lambda} \geq \beta_1 > 0, \qquad \inf_{\varphi_h \in \Phi_h} \sup_{\lambda_h \in \Lambda_h} \frac{\sum_k \int_{\Gamma_k} \lambda_h^k \varphi_h}{\|\varphi_h\|_\Phi \|\lambda_h\|_\Lambda} \geq \beta_2 > 0 \tag{5}$$

respectively coupling V_h with Λ_h, and Λ_h with Φ_h. Provided (5) holds, it is possible to prove that the bilinear form $s : \Phi_h \times \Phi_h \to \mathbb{R}$ corresponding to the Schur complement matrix S and defined by

$$s(u_h, v_h) = \underline{v}_h^T S \underline{u}_h,$$

is continuous and coercive with respect to the Φ norm:

$$s(\varphi_h, \psi_h) \leq M_0 \|\varphi_h\|_\Phi \|\psi_h\|_\Phi, \qquad s(\varphi_h, \varphi_h) \quad \geq \quad \alpha_0 \|\varphi_h\|_\Phi^2, \tag{6}$$

(M_0 and α_0 positive constants).

The problem arises then to precondition the Schur complement matrix S. This can be done by a substructuring approach ([BPS86, Ber00b]). To this end we introduce a decomposition of the skeleton $\Sigma \setminus \partial\Omega = \cup_i e_i$ as the disjoint union of M macro-edges e_i, (each being the edge of two adjacent subdomains), and we split the discrete space Φ_h as the direct sum of a coarse space $\mathcal{L}_H$ of functions linear on each macro-edge of Σ,

$$\mathcal{L}_H = \{\varphi \in C^0(\Sigma) : \quad \forall i = 1, \dots, M, \ \varphi|_{e_i} \in \mathbb{P}_1(e_i), \ \varphi = 0 \text{ on } \partial\Omega\},$$

($\mathbb{P}_1$ denoting the space of polynomials of degree at most one) plus some local spaces (one per macro-edge) $\Phi_h^{0,i}$,

$$\Phi_h^{0,i} = \{\varphi_h \in \Phi_h : \ \varphi_h|_{\Sigma \setminus e_i} = 0\},$$

consisting in those functions in Φ_h vanishing outside the macro-edge e_i. Corresponding to such a decomposition we will consider a block-Jacobi type preconditioner. More precisely, it is possible to prove the following theorem.

Theorem 1 *Let $\hat{s}_H : \mathcal{L}_H \times \mathcal{L}_H \to \mathbb{R}$ and $\hat{s}_i : \Phi_h^{0,i} \times \Phi_h^{0,i} \to \mathbb{R}$ be symmetric bilinear forms satisfying*

$$\hat{s}_H(\varphi_H, \varphi_H) \simeq \|\varphi_H\|_\Phi^2 \ \ \forall \varphi_H \in \mathcal{L}_H, \qquad \textit{and} \qquad \hat{s}_i(\varphi_h, \varphi_h) \simeq \|\varphi_h\|_\Phi^2, \ \ \forall \varphi_h \in \Phi_h^{0,i},$$

and let $\hat{s} : \Phi_h \times \Phi_h \to \mathbb{R}$ be the bilinear form which, for $\varphi_h = \varphi_H + \sum_{i=1}^M \varphi_h^{0,i}$ and $\psi_h = \psi_H + \sum_{i=1}^M \psi_h^{0,i}$, is defined by

$$\hat{s}(\varphi_h, \psi_h) = \hat{s}_H(\varphi_H, \psi_H) + \sum_{i=1}^{M} \hat{s}_i(\varphi_h^{0,i}, \psi_h^{0,i}).$$

Then for all $\varphi_h \in \Phi_h$ it holds

$$c\|\varphi_h\|_\Phi^2 \leq \hat{s}(\varphi_h, \varphi_h) \lesssim \max_k \left(1 + \log \frac{H_k}{h_k}\right)^2 \|\varphi_h\|_\Phi^2,$$

where h_k and H_k are respectively the smallest mesh size of $\Phi_h|_{\Gamma_k}$ and the diameter of the subdomain Ω_k.

Thanks to (6), by a well known argument, Theorem 1 implies that we can derive the following corollary, where we denote by $\hat{S}$ the matrix corresponding to the Galerkin discretization of the bilinear forms $\hat{s}$, which has a block diagonal structure.

Corollary 1 *If (5) holds, then*

$$cond(\hat{S}^{-1}S) \lesssim \max_k \left(1 + \log \frac{H_k}{h_k}\right)^2 .$$

Wavelet stabilization

The need for the two inf-sup conditions (5) to hold, leads to discard several otherwise desirable choices for the three discretization spaces V_h, Λ_h and Φ_h. A possible remedy in this direction is to advocate a suitable *stabilization technique*, allowing to circumvent one or both inf-sup conditions. Several proposals have been made in this respect (see for instance [BFMR97]). In this particular context, we want however to choose the stabilization technique in such a way that the substructuring preconditioner briefly described in the previous section still applies. Therefore, the bilinear form corresponding to the Schur complement matrix deriving from the stabilized method needs to satisfy (6). A choice that fulfills such requirement is the *wavelet stabilization* proposed in [BK00]. This consists in introducing symmetric bilinear forms $[\cdot,\cdot]_{1/2,k} : H^{1/2}(\Gamma_k) \times H^{1/2}(\Gamma_k) \to \mathbb{R}$ satisfying the following bounds for all $\varphi_h, \psi_h \in \Phi_h|_{\Gamma_k}$ and for two suitable positive constants C_1 and c_1:

$$[\varphi_h, \psi_h]_{1/2,k} \leq C_1 |\varphi_h|_{H^{1/2}(\Gamma_k)} |\psi_h|_{H^{1/2}(\Gamma_k)}, \qquad [\varphi_h, \varphi_h]_{1/2,k} \geq c_1 |\varphi_h|^2_{H^{1/2}(\Gamma_k)}. \quad (7)$$

The *stabilized three fields formulation* of problem (1) reads: find u_h, λ_h and φ_h such that

$$\begin{cases} \forall k, \ \ \forall v_h^k \in V_h^k, \ \ \forall \mu_h^k \in \Lambda_h^k : \\ a^k(u_h^k, v_h^k) + \gamma[u_h^k, v_h^k]_{1/2,k} - \int_{\Gamma_k} v_h^k \lambda_h^k \quad - \gamma[\varphi_h, v_h^k]_{1/2,k} = \int_{\Omega_k} f v_h^k, \\ -\int_{\Gamma_k} u_h^k \mu_h^k \qquad + \int_{\Gamma_k} \mu_h^k \varphi_h = 0, \\ \text{and } \forall \psi_h \in \Phi_h : \\ -\sum_k \gamma[u_h^k, \psi_h]_{1/2,k} + \sum_k \int_{\Gamma_k} \lambda_h^k \psi_h + \sum_k \gamma[\varphi_h, \psi_h]_{1/2,k} = 0, \end{cases} \quad (8)$$

where $\gamma > 0$ is a parameter independent of the choice of the discretization spaces. Such formulation is consistent with the original continuous problem, that is by substituting in (8) the solution (u, λ, φ) of (2) at the place of $(u_h, \lambda_h, \varphi_h)$ we obtain an identity. The linear system stemming from such a problem takes this time the following form:

$$\begin{pmatrix} \check{A} & B^T & -\gamma D^T \\ B & 0 & C^T \\ -\gamma D & C & \gamma E \end{pmatrix} \cdot \begin{pmatrix} \underline{u}_h \\ \underline{\lambda}_h \\ \underline{\varphi}_h \end{pmatrix} = \begin{pmatrix} \underline{f} \\ 0 \\ 0 \end{pmatrix}, \quad (9)$$

with $\check{A} = A + \gamma F$, the matrices D, E and F deriving from the stabilizing terms. Again, the solution of (9) can be reduced to a system in the unknown $\underline{\varphi}_h$, this time taking the form

$$\check{S}\underline{\varphi}_h := \left(\mathbf{D}\check{\mathbf{A}}^{-1}\mathbf{D}^T + \gamma E\right)\underline{\varphi}_h = -\mathbf{D}\check{\mathbf{A}}^{-1} \begin{pmatrix} \underline{f} \\ 0 \end{pmatrix}$$

with

$$\check{\mathbf{A}} = \begin{pmatrix} \check{A} & B^T \\ B & 0 \end{pmatrix}, \qquad \mathbf{D} = [\ -\gamma D \quad C\ ].$$

Once again we let $\check{s} : \Phi_h \times \Phi_h \to \mathbb{R}$ be the bilinear form corresponding to the Schur complement matrix $\check{S}$

$$\check{s}(\varphi_h, \psi_h) = \underline{\psi}_h{}^T \check{S} \underline{\varphi}_h,$$

and, if the space V_h and Λ_h satisfy the first of the two inf-sup conditions (5), also the bilinear form $\check{s}$ is continuous and coercive with respect to the Φ norm:

$$\check{s}(\varphi_h, \psi_h) \leq M_1 \|\varphi_h\|_\Phi \|\psi_h\|_\Phi, \qquad \check{s}(\varphi_h, \varphi_h) \geq \alpha_1 \|\varphi_h\|_\Phi^2.$$

Also for the bilinear form $\check{s}$, Theorem 1 yields then the corollary

Corollary 2 *It holds*

$$cond(\hat{S}^{-1}\check{S}) \lesssim \max_k \left(1 + \log \frac{H_k}{h_k}\right)^2.$$

We need at this point to provide bilinear forms $[\cdot,\cdot]_{1/2,k}$ with the required characteristics. Following the proposal of [BK00], these are designed by means of a wavelet decomposition. For simplicity, let us assume that the subdomains are squares (otherwise we would need to map them onto a square). Since the $H^{1/2}(\Gamma_k)$ seminorm is invariant under changes of scale, we can rescale the subdomain in such a way that $|\Gamma_k| = 1$ (that is $H = 1/4$). For simplicity, let us concentrate on the case in which the skeleton Σ is discretized by means of P1 finite elements, and let us assume that on each macro-edge e_i the grid is uniform, with L_i elements, L_i being a power of two:

$$L_i = 2^{j_i} \qquad \text{for some } j_i \geq 1,$$

so that for all k, $\Phi_h|_{\Gamma_k} \subset V_{j_k+2}$, with $j_k = \max_{i:|e_i \cup \Gamma_k|>0} j_i$, where, for $j > 0$, V_j denotes the space of 1-periodic P1 finite elements on the uniform grid with mesh size $1/2^j$.

The sequence $\{V_j\}_{j\geq 0}$ forms a so called *multiresolution analysis* of $L^2(\Gamma_k)$ and it is well known (see for example [CDF92]) that there exists several wavelet bases associated with such a multiresolution analysis. More precisely there exist several P1 compactly supported functions $\theta \in C^0(\mathbb{R})$ defined on the uniform grid of mesh size 1 and integer nodes, such that, if we define *wavelets* $\theta_{m,\ell}$ by $\theta_{m,\ell} = \sum_{n=-\infty}^{+\infty} 2^{m/2}\theta(2^m(x-n)-\ell)$, all functions $\eta \in V_j$ can be written as

$$\eta = \eta_0 + \sum_{m=0}^{j-1} \sum_{\ell=1}^{2^m} \eta_{m,\ell}\theta_{m,\ell}, \qquad \eta_0 \text{ constant},$$

and such that

$$\eta \in V_j \quad \Longrightarrow \quad |\eta|^2_{H^{1/2}(\Gamma_k)} \simeq \sum_{m=0}^{j-1} \sum_{\ell=1}^{2^m} 2^m |\eta_{m,\ell}|^2.$$

If, for $\zeta, \xi \in L^2(\Gamma_k)$, we express in terms of the wavelet basis $\{\theta_{m,\ell}\}$ the respective $L^2(\Gamma_k)$ projections $\Pi_k(\zeta)$ and $\Pi_k(\xi)$ onto V_{j_k+2},

$$\Pi_k(\zeta) = \zeta_0 + \sum_{m=0}^{j_k+1} \sum_{\ell=1}^{2^m} \zeta_{m,\ell}\theta_{m,\ell}, \qquad \Pi_k(\xi) = \xi_0 + \sum_{m=0}^{j_k+1} \sum_{\ell=1}^{2^m} \xi_{m,\ell}\theta_{m,\ell},$$

we can define the bilinear form $[\cdot,\cdot]_{1/2,k}$ as

$$[\zeta,\xi]_{1/2,k} = \sum_{m=0}^{j_k+1} \sum_{\ell=1}^{2^m} 2^m \zeta_{m,\ell}\xi_{m,\ell}.$$

It is possible to prove ([BK00]) that the bilinear forms thus defined satisfies (7).

With this definition, the computation of $[u_h^k - \varphi_h, v_h^k - \psi_h]_{1/2,k}$ essentially reduces to first computing the nodal values of $\Pi_{j_k}(u_h^k)$, $\Pi_{j_k}(v_h^k)$, $\Pi_{j_k}(\varphi_h)$ and $\Pi_{j_k}(\psi_h)$ respectively and then applying a *Fast Wavelet Transform*.

Numerical results

We will consider problem (1) with $f = 1$ and $\Omega =]0,1[^2$. We consider an uniform decomposition of Ω in $K = 4 \times 4$ equal square subdomains of size $H \times H$, $H = 1/4$. In each subdomain Ω_k we take an uniform mesh composed by $N_k \times N_k$ equal square elements of size $\delta_k \times \delta_k$, $\delta_k = H/N_k = 1/(4N_k)$. We then define V_h^k to be the corresponding space of Q1 finite elements. The value of N_k is randomly assigned in such a way that for about one third of the subdomains $N_k = 5$, for about another third $N_k = 10$, and for the remaining subdomains $N_k = 15$. The multiplier space Λ_h^k is then defined as the trace on Γ_k of V_h^k. With such a choice it is possible to prove that the spaces Λ_h and V_h satisfy the first of the two inf-sup conditions needed for stability. The space Φ_h is chosen to be a P1 finite element space corresponding to a uniform grid on Σ with mesh size $1/(4 \cdot 2^J)$. As J increases, the second inf-sup condition – coupling Φ_h and Λ_h – fails. The consequent instability clearly appears in Figure 1, where on top we plot the solution φ_h obtained by the unstabilized formulation (2) for $J = 3$ (on the left) and $J = 5$ (on the right). On the bottom, we plot the solution φ_h obtained by the stabilized formulation (8) for the same values of J and for $\gamma = .05$. The stabilizing effect of the correction is evident. We next show, for different values of the stabilization parameter γ, the performance of the block Jacobi type preconditioner introduced in Section **??**, where the bilinear forms $\hat{s}_H$ and $\hat{s}_i$ are chosen according to [BPS86, Ber00b]. While the stabilized system is better preconditioned then the unstabilized one (first column in the table), apparently the stabilization parameter influences its performance, so its correct choice is important.

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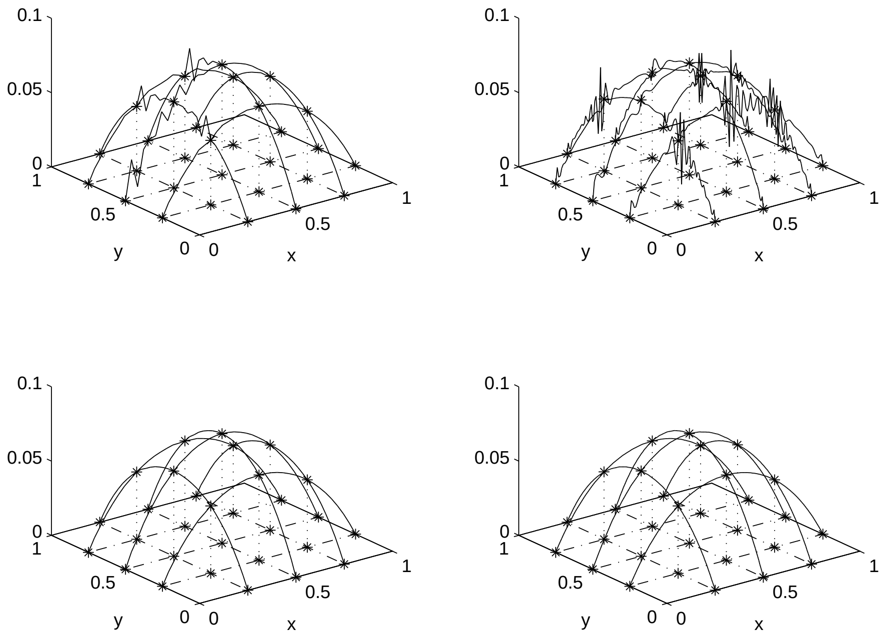

Figure 1: Effect of the stabilization: on top we display the results of the plain formulation and at the bottom the ones obtained by adding the stabilization term

2^J	$\gamma = 0$	$\gamma = .00125$	$\gamma = .05$	$\gamma = .25$
4	11	11	11	11
8	40	44	15	16
16	—	57	17	25
32	—	59	21	41

Table 1: Number of CG iterations needed to reduce the residual of a factor 10^{-5}. For $\gamma = 0$ and $J \geq 4$ the conjugate gradient procedure did not converge in the maximum number of iteration (which was set to 100).

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