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30 Uniform Domain Decomposition for a Convection-Diffusion Problem

I. Boglaev[1]

Introduction

In this paper, for solving a singularly perturbed parabolic problem with a convection-dominated term, we present a finite difference domain decomposition algorithm based on a classical upwind difference approximation in a spatial variable and on the piecewise equidistant mesh of Shishkin-type [MOS96]. These meshes allow us to decompose a computational domain into subdomains outside boundary layers and inside them as well, and possess load balancing. This property is very important for implementation of iterative algorithms on parallel computers, since it avoids loss of efficiency due to one processor being idle. Our purpose is to construct and analyse a domain decomposition algorithm based on decomposition of boundary layers. We use a modification of the Schwarz alternating method proposed in [DDD91], in which the computational domain is partitioned into many nonoverlapping subdomains with interface Γ. Small interfacial subdomains are introduced near the interface Γ, and approximate boundary values computed on Γ are used for solving problems on the nonoverlapping subdomains. Thus, this approach may be considered as a variant of a block Gauss-Seidel iteration (or in the parallel context as a multicoloured algorithm) for the subdomains with a Dirichlet-Dirichlet coupling through the interface variables. This modification of the Schwarz method has been applied in [Bog98] for solving singularly perturbed reaction-diffusion problems.

In [Mat98], for singularly perturbed parabolic problems with convection-dominated terms, uniform convergent properties of some Schwarz-type methods based on continuous multidomain decomposition (i.e. without resort to discretization in the subdomains) have been studied. Here, we construct more accurate estimations of a contraction factor for the multidomain decomposition algorithm in a discrete form and additionally investigate this algorithm when the subdomains located inside the boundary layer.

We consider the following singularly perturbed parabolic problem:

$$\varepsilon u_{xx} + b(x,t)u_x - u_t = f(x,t,u), \quad (x,t) \in Q = \Omega \times (0,T], \tag{1}$$

$$\Omega = \{x : 0 < x < 1\}, \quad u(0,t) = u(1,t) = 0, \quad u(x,0) = u^0(x),\ x \in \Omega,$$

where ε is a positive parameter, functions $b(x,t),\ f(x,t,u)$ and $u^0(x)$ are sufficiently smooth. We assume that

$$b(x,t) \geq \beta_* = \text{const} > 0,\ \partial f/\partial u \geq 0,\ (x,t,u) \in Q \times (-\infty, +\infty).$$

Under suitable continuity and compatibility conditions on the data a unique solution $u(x,t)$ of (1) exists. For $\varepsilon \ll 1$ problem (1) is singularly perturbed and characterized by an exponential layer at $x = 0$.

[1]Institute of Fundamental Sciences, Massey University, Private Bag 11-222, Palmerston North, New Zealand, i.boglaev@massey.ac.nz. This work was supported in part by Marsden Fund MAU809 of the Royal Society of New Zealand.

Undecomposed Algorithm

Consider an implicit two-level time difference scheme which possesses an uniform in the perturbation parameter ε convergence.

On set $\bar{Q}$ introduce a rectangular mesh $\bar{\Omega}^h \times \bar{\Omega}^\tau$, where

$$\bar{\Omega}^h = \{x_i, i = 0, 1, \dots, N, x_0 = 0, x_N = 1, h_i = x_{i+1} - x_i\},$$

$$\bar{\Omega}^\tau = \{t_k = k\tau, k = 0, 1, \dots, N_\tau, N_\tau \tau = T\}.$$

For a mesh function $U(x,t)$ we use the following classical implicit difference scheme

$$\Lambda U(x,t) - \tau^{-1}[U(x,t) - U(x,t-\tau)] = f(x,t,U),\ (x,t) \in \Omega^h \times \Omega^\tau, \tag{2}$$

$$U(0,t) = U(1,t) = 0,\ t \in \bar{\Omega}^\tau, \quad U(x,0) = u^0(x),\ x \in \bar{\Omega}^h,$$

where $\Lambda U(x,t)$ is defined by

$$\Lambda U(x,t) = \varepsilon D_+ D_- U(x,t) + b(x,t) D_+ U(x,t),$$

$D_+ D_- U(x,t)$ and $D_+ U(x,t)$ are the central and forward difference approximations to the second and first derivatives in the x-direction, respectively.

The piecewise equidistant mesh of Shishkin-type from [MOS96] is formed by dividing interval $\bar{\Omega}$ into two parts $[0,\sigma],\ [\sigma,1]$, and in each part we use a uniform grid with $N/2+1$ mesh points. The step sizes of the mesh are defined by

$$h_i = h_\varepsilon = 2\sigma N^{-1}, i = 0, 1, \dots, N/2 - 1, \tag{3}$$

$$h_i = h = 2(1-\sigma) N^{-1}, i = N/2, \dots, N-1.$$

The transition point σ from [MOS96] is determined by $\sigma = \min\{2^{-1}, 2\varepsilon\beta_*^{-1} \ln N\}$. If $\sigma = 1/2$, then N^{-1} is very small relative to ε. This is unlikely in practice, and in this case the difference scheme (2) can be analyzed using standard techniques. We therefore assume that

$$\sigma = 2\varepsilon\beta_*^{-1} \ln N, \quad h_\varepsilon = 4\varepsilon\beta_*^{-1} N^{-1} \ln N, \quad N^{-1} < h < 2N^{-1}. \tag{4}$$

We note here that the size of the boundary layer is of order $O(\varepsilon|\ln\varepsilon|)$. Thus, for $\varepsilon \le N^{-1}$, the transition from the layer to the outside region is determined by the transition point σ which is located inside the boundary layer.

Theorem 1 *Let u(x,t) be the solution to problem (1). Then the solution of the difference scheme (2) on the mesh (3), (4) converges ε-uniformly to $u(x,t)$:*

$$\max_{(x,t) \in \bar{\Omega}^h \times \bar{\Omega}^\tau} |u(x,t) - U(x,t)| \le C(N^{-1} \ln N + \tau),$$

where N is the number of mesh points in the space direction, τ is the time step-size and constant C is independent of ε, N and τ.

Domain Decomposition Algorithm

We consider decomposition of domain $\bar{\Omega}$ into M nonoverlapping (adjoining) subdomains $\bar{\Omega}_m, m = 1, \dots, M$:

$$\Omega_m = (x_{m-1}, x_m), \quad \bar{\Omega}_m \cap \bar{\Omega}_{m+1} = x_m, \quad x_0 = 0, x_M = 1.$$

Additionally, we consider $M-1$ interfacial subdomains $\omega_m, m = 1, \dots, M-1$:

$$\omega_m = (x_m^b, x_m^e), \quad \omega_{m-1} \cap \omega_m = \emptyset, \quad x_m^b < x_m < x_m^e.$$

On $\bar{\Omega}_m, m = 1, \dots, M$ and $\bar{\omega}_m, m = 1, \dots, M-1$ we introduce meshes $\bar{\Omega}_m^h$, and $\bar{\omega}_m^h$, respectively, where

$$\bar{\Omega}_m^h = \{x_{mi}, i = 0, 1, \dots, N_m, x_{m0} = x_{m-1}, x_{N_m} = x_m, h_{mi} = x_{m,i+1} - x_{mi}\}, \quad (5)$$

$$\bar{\omega}_m^h = \{X_{mi}, i = 0, 1, \dots, N_{m\omega}, X_{m0} = x_m^b, X_{N_{m\omega}} = x_m^e, H_{mi} = X_{m,i+1} - X_{mi}\},$$

and suppose that $\bar{\Omega}^h = \bigcup \bar{\Omega}_m^h$, and the mesh points in $\bar{\omega}_m^h, m = 1, \dots, M-1$ coincide with the mesh points in $\bar{\Omega}^h$.

On each time-level t_k, we shall implement n_0 iterative steps of a domain decomposition algorithm. On each iterative step, firstly, we solve problems on the nonoverlapping subdomains $\bar{\Omega}_m^h, m = 1, \dots, M$ with Dirichlet boundary conditions passed from the previous iterate. Then Dirichlet data are passed from these subdomains to the interfacial subdomains $\bar{\omega}_m^h, m = 1, \dots, M-1$, and problems on the interfacial subdomains are computed. Finally, we impose continuity for piecing the solutions on the subdomains together.

On subdomains $\bar{\Omega}_m^h, m = 1, \dots, M$, introduce mesh functions $v_m^{(n)}(x, t_k), m = 1, \dots, M$ (here the index n stands for a number of iterative steps, and $n = 1, \dots, n_0$) satisfying the following implicit difference schemes

$$\Lambda v_m^{(n)}(x, t_k) - (1/\tau)[v_m^{(n)}(x, t_k) - V(x, t_{k-1})] = f(x, t_k, v_m^{(n)}(x, t_k)), \ x \in \Omega_m^h, \quad (6)$$

$$v_m^{(n)}(x, t_k) = V^{(n-1)}(x, t_k), \ x = x_{m-1}, x_m, \quad v_1^{(n)}(0, t_k) = 0, \quad v_M^{(n)}(1, t_k) = 0.$$

On the interfacial subdomains $\bar{\omega}_m^h, m = 1, \dots, M-1$, we solve the difference problems

$$\Lambda z_m^{(n)}(x, t_k) - (1/\tau)[z_m^{(n)}(x, t_k) - V(x, t_{k-1})] = f(x, t_k, z_m^{(n)}(x, t_k)), \ x \in \omega_m^h, \quad (7)$$

$$z_m^{(n)}(x_m^b, t_k) = v_m^{(n)}(x_m^b, t_k), \quad z_m^{(n)}(x_m^e, t_k) = v_{m+1}^{(n)}(x_m^e, t_k).$$

The mesh function $V^{(n)}(x, t_k)$ is determined in the form

$$V^{(n)}(x, t_k) = \begin{cases} v_m^{(n)}(x, t_k), & x \in \overline{\Omega_m^h \setminus (\omega_{m-1}^h \cup \omega_m^h)}, m = 1, \dots, M; \\ z_m^{(n)}(x, t_k), & x \in \overline{\omega}_m^h, m = 1, \dots, M-1, \end{cases} \quad (8)$$

where we introduce the following notations

$$V(x, t_k) = V^{(n_0)}(x, t_k), \quad V^{(0)}(x, t_k) = V(x, t_{k-1}), \ k \geq 1,$$

$$V(x, 0) = u^0(x), \ x \in \bar{\Omega}^h.$$

Algorithm (6)-(8) can be carried out by parallel processing, since on each iterative step n the M problems (6) for $v_m^{(n)}(x,t_k), m = 1,\dots,M$ and the $M-1$ problems (7) for $z_m^{(n)}(x,t_k), m = 1,\dots,M-1$ can be implemented concurrently.

On a mesh $\bar{\Omega}_*^h = \{x_i, i = 0,1,\dots,N_*; x_0 = x_a, x_{N_*} = x_b\}$, consider the difference problems

$$\Lambda\Phi^s(x) - \tau^{-1}\Phi^s(x) = 0,\ x \in \Omega_*^h,\ s = 1,2, \tag{9}$$

$$\Phi^1(x_0) = 1,\ \Phi^1(x_{N_*}) = 0, \quad \Phi^2(x_0) = 0,\ \Phi^2(x_{N_*}) = 1.$$

Introduce the notations

$$q_m^b = \Phi_m^1(x_m^b) + \Phi_m^2(x_m^b),\ q_m^e = \Phi_{m+1}^1(x_m^e) + \Phi_{m+1}^2(x_m^e),\ m = 1,\dots,M-1,$$

$$q_m = q_m^b\Phi_{m,\omega}^1(x_m) + q_m^e\Phi_{m,\omega}^2(x_m),\ m = 1,\dots,M-1,$$

where $\Phi_m^{1,2}(x)$ and $\Phi_{m,\omega}^{1,2}(x)$ are the solutions to (9) on $\bar{\Omega}_m^h$ and $\bar{\omega}_m^h$, respectively.

Theorem 2 *Algorithm (6)- (8) on mesh (3), (4) converges to the solution of (1) with the following rate:*

$$\max_{(x,t)\in\bar{\Omega}^h\times\bar{\Omega}^\tau} |u(x,t) - V(x,t)| \le C((N\tau)^{-1}\ln N + T)(N^{-1}\ln N + \tau + q^{n_0}),$$

$$q = \max_{1\le m\le M-1} q_m, \tag{10}$$

where the contraction coefficient $q \in (0,1)$ and constant C is independent of ε, h, τ and q.

Theorem 2 guarantees that the domain decomposition algorithm (6)- (8) converges for any initial guesses. From Theorem 2, it follows that asymptotically one would expect to choose the number of mesh points N in the space direction such that $N \approx N_\tau$. If $N \approx N_\tau$, then we conclude the following estimate

$$\max_{(x,t)\in\bar{\Omega}^h\times\bar{\Omega}^\tau} |u(x,t) - V(x,t)| \le C(N^{-1}\ln N + \tau + q^{n_0}),$$

where constant C is independent of ε, N, τ and q.

Estimates on Rate of Convergence

The interfacial subdomains outside the boundary layer. Consider algorithm (6)- (8) with the interfacial subdomains $\omega_m^h,\ m = 1,\dots,M-1$, located outside the boundary layer. Suppose for simplicity that the centre of the discrete interval $\bar{\omega}_m^h$ is located at x_m, i.e. in (5) $x_m = I_{m\omega}h,\ N_{m\omega} = 2I_{m\omega}$. For sufficiently small values of ε, we can approximate q in (10) uniformly in ε by

$$q \approx \exp[-I_\omega \ln(1 + h(\beta_{**}\tau)^{-1})],\ \beta_{**} = \max_{(x,t)\in\bar{Q}} b(x,t).$$

We compare this estimate with the convergence rate of the Schwarz alternating method obtained in [Mat98]:

$$\max_{(x,t)}|u^{(n+1)}-u|\le\hat{q}\max_{(x,t)}|u^{(n)}-u|,\ \hat{q}=\exp(-\alpha d\tau^{-1/2}),\tag{11}$$

where $u^{(n)}$ is the Schwarz iterate, $d>0$ measures the overlap between two subdomains and $\alpha>0$ is independent of τ. Outside the boundary layer $d=O(I_\omega h)$, the contraction factor $\hat{q}$ is approximated by

$$\hat{q}\approx\exp(-\bar{\alpha}I_\omega h\tau^{-1/2}),$$

where $\bar{\alpha}>0$. From Theorem 2, one would expect to choose $\tau\approx h$, and asymptotically we get

$$q\approx\exp[-I_\omega\ln(1+\beta_{**}^{-1})],\ \hat{q}\approx\exp(-\bar{\alpha}I_\omega\tau^{1/2})\to 1,\tau\to 0,\ I_\omega\le N(2M)^{-1}.$$

It follows that the estimate of the convergence rate from [Mat98] is impractical.

The interfacial subdomains inside the boundary layer. Suppose that N is divisible by $2M$ and M is even, we decompose the boundary layer $[0,\sigma]$ and the region outside the layer $[\sigma,1]$ into $M/2$ equal subdomains, respectively, where σ from (4). We note that each subdomain $\bar{\Omega}_m^h$ contains the same number of mesh points $2I+1, I=N/(2M)$. From (5), we have

$$\begin{aligned}\bar{\Omega}_m^h&=\{x_{mi},x_{mi}=x_{m-1}+ih_\varepsilon,i=0,1,\dots,2I\},\\ x_{m-1}&=2(m-1)Ih_\varepsilon,\ m=1,\dots,M/2,\\ \bar{\Omega}_m^h&=\{x_{mi},x_{mi}=x_{m-1}+ih,i=0,1,\dots,2I\},\\ x_{m-1}&=\sigma+2(m-M/2-1)Ih,m=M/2+1,\dots,M,\end{aligned}\tag{12}$$

where h, h_ε are the uniform step sizes outside and inside the boundary layer. We choose the interfacial subdomains in the following forms:

$$\begin{aligned}\bar{\omega}_m^h&=\{X_{mi},X_{mi}=x_m^b+ih_\varepsilon,i=0,1,\dots,2I_\omega\},\\ x_m^b&=x_m-I_\omega h_\varepsilon,\ m=1,\dots,M/2-1,\\ \bar{\omega}_{M/2}^h&=\{X_{M/2,i},X_{M/2,i}=x_{M/2}^b+ih_\varepsilon,i=0,1,\dots,I_\omega;\\ &\quad X_{M/2,i}=\sigma+ih,i=I_\omega+1,\dots,2I_\omega\},\\ x_{M/2}^b&=\sigma-I_\omega h_\varepsilon,\\ \bar{\omega}_m^h&=\{X_{mi},X_{mi}=x_m^b+ih,i=0,1,\dots,2I_\omega\},\\ x_m^b&=x_m-I_\omega h,\ m=M/2+1,\dots,M-1.\end{aligned}$$

Here the interfacial subdomains $\bar{\omega}_m^h, m=1,\dots,M-1$ contain the same number of mesh points $2I_\omega+1$, and the centre of the discrete interval $\bar{\omega}_m^h$ is located at x_m. We suppose $1\le I_\omega\le I$, such that $\omega_{m-1}^h\cap\omega_m^h=\emptyset, m=2,\dots,M-1$. On this domain decomposition, we can approximate the contraction factor q in (10) by

$$q\approx\varepsilon\tau h^{-2}+[(2+\sqrt{2})/2]^{-\frac{N}{2M}}+\exp[-I_\omega\ln(1+\beta_{**}^{-1})],\ I_\omega\le N(2M)^{-1}.$$

If in (11) $d=O(I_\omega h_\varepsilon)$, then $\hat{q}$ is approximated by $\hat{q}\approx\exp(-\bar{\alpha}I_\omega h_\varepsilon\tau^{-1/2})$. In the case of the maximal size of the interfacial subdomains $I_\omega=N(2M)^{-1}$, we get

$$\hat{q}\approx\exp(-\bar{\alpha}\varepsilon\ln N(M\sqrt{\tau})^{-1}).$$

Again, we conclude that the estimate of the convergence rate from (11) is impractical for the proposed domain decomposition.

Numerical Results

As a test problem, consider the following problem

$$\varepsilon u_{xx} + u_x - u_t = 0,\ (x,t) \in (0,1) \times (0,T],$$

$$u(0,t) = 1,\ u(1,t) = 0,\ u(x,0) = 0.$$

with $b(x,t) = 1$. Note that in the new variable $\tilde{u}(x,t) = u(x,t) + (x-1)$, this problem becomes (1) with $f(x,t,u) = 1$ and $u^0(x) = x - 1$.

On each time-level, we implement n_0 iterates of algorithm (6)-(8) to satisfy the stopping criterion

$$\max_{x \in \bar{\Omega}^h} |V^{(n_0)}(x,t_k) - U(x,t_k)| \le \delta,\ \delta = \max(N^{-1} \ln N, \tau),$$

where $U(x,t_k)$ is the solution of the undecomposed algorithm (2) at time-level t_k.

Consider the domain decomposition (12) with the interfacial subdomains inside the boundary layer. In Table 1, for $\tau = 10^{-2}, 5 \cdot 10^{-3}, 10^{-3}$ and various values of ε, M, we give the average (over ten time-levels) number of iterations n_0 with $N = 64$ and the maximal size of the interfacial subdomains $I_\omega = N(2M)^{-1}$. From the data, it follows that for M fixed, n_0 is a monotone increasing function with respect to the time mesh spacing τ, and for $\varepsilon \le 10^{-3}$, n_0 is independent of the perturbation parameter. We notice that the number of iterations approaches 1 as $\tau \to 0$. These results substantiate the theoretical convergent estimates.

M	n_0		
2	2; 2; 2	1.4; 1.4; 1	1; 1; 1
4	2; 2; 2	1.4; 1.4; 1	1; 1; 1
8	2; 2; 2	1.4; 1.4; 1	1; 1; 1
16	2.4; 2; 2	1.4; 1.4; 1	1; 1; 1
32	8.2; 5; 2	1.4; 1.4; 1	1; 1; 1
ε	0.1	0.01	0.001

Table 1: Average number of iterations n_0 for $N = 64$, $\tau = 10^{-2}, 5 \cdot 10^{-3}, 10^{-3}$.

M	n_0				
2	2	2	2	2	2
4	2	2	2	2	2
8	7.2	3.7	2.5	2	2
16	11.2	5.6	3.8	3	3
I_ω	1	2	3	4	$N(2M)^{-1}$

Table 2: Average numbers of iterations n_0 for $N = 128, \tau = 10^{-2}, \varepsilon = 10^{-1}$.

In Table 2, for various numbers M and sizes I_ω of the interfacial subdomains, we represent the average number of iterations with $N = 128, \tau = 10^{-2}, \varepsilon = 10^{-1}$. Note that the last column in the table corresponds to the interfacial subdomains with the maximal size. The average number of iterations as a function of the size of the interfacial subdomains is a monotone

decreasing function, and this is in agreement with our theoretical estimates. Another notable feature is that this function varies very quickly for small values of I_ω, and relatively small sizes of the interfacial subdomains are needed to essentially reduce the number of iterations.

Conclusion

We summarise our discussion concerning the theoretical results and numerical experiments.

1. We emphasise here the domain decomposition algorithm (6)-(8) on the piecewise uniform mesh (3), (4) possesses uniform in the perturbation parameter convergence. Thus, the proposed algorithm keeps the main property of the most effective undecomposed algorithms for singular perturbation problems.

2. In the context of parallel computing, the proposed uniform decomposition (12) guarantees us load balancing of a multiprocessor computer.

3. The numerical experiments confirm effectiveness of the proposed domain decomposition algorithm. Algorithm (6)-(8) requires few iterations on each time-level and sufficiently small sizes of the interfacial subdomains and still maintains stable approximation.

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