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31 Domain decomposition methods for solving scattering problems by a boundary element method

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Introduction

Integral equation methods are widely used for the numerical solution of scattering problems. Among their advantages, we mention direct and simple dealing with the radiation condition, accuracy and reduction of the mesh only to the boundary. As a counterpart, this method generates large dense complex matrices and in the case of dielectric layers may need some extra auxiliary unknowns, namely the equivalent magnetic currents. Also, the repetition of some geometrical patterns can drastically increase the size of the final system to be solved in an artificial way. The aim of this paper is to show how these difficulties can be overcome by a suitable use of a nonoverlapping domain decomposition method while however keeping the advantages of the boundary integral equations solutions.

The main technique used to decompose the solution domain into smaller domains consists in expressing the usual matching of the Cauchy data of the problem (the equivalent currents as they are generally refered to in computational electromagnetics) in terms of some equivalent boundary conditions of impedance (also called Robin) type.

The method also applies to a conductor covered by a dielectric layer with now two advantages. First, at each step, the problem to be solved has for unknown the electric current only whereas the direct solution also involves the magnetic current as a supplemental unknown. Moreover, at each step, unknown interior and exterior currents are completely uncoupled.

Another interesting aspect of this method is to couple a finite element and a boundary element method. This approach has been investigated by several authors (e.g., [JN80], [Cos87], [dLB95], [Lan94], [HW92]). However, the resulting final system is generally large and difficult to solve because it involves equations coming both from the FEM and BEM formulations. On the contrary, the method proposed in this paper uncouples completely the two solution procedures.

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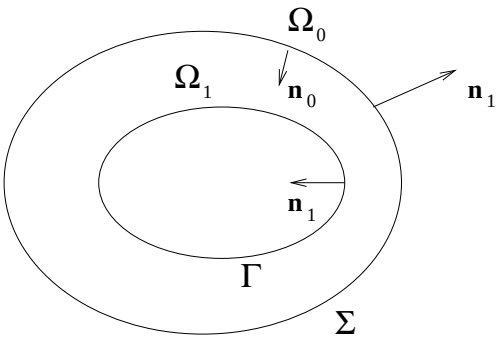

Figure 1: A typical geometry

Nonoverlapping domain decomposition method

To be specific, we consider the following problem related to the scattering of an TE wave by a coated perfectly conducting cylinder

$$\left\{\begin{array}{l} \text{find a sufficiently smooth } u \text{ such that} \\ \nabla\cdot(\frac{1}{\varepsilon}\nabla u)+k^2\frac{n^2}{\varepsilon}u=0 \quad \text{in } \Omega_1, \\ \Delta u+k^2u=0 \quad \text{in } \Omega_0, \\ \partial_{\mathbf{n_1}}u=0 \quad \text{on } \Gamma, \\ u_1=u_0, \quad \varepsilon^{-1}\partial_{\mathbf{n_1}}u_1=\partial_{\mathbf{n_1}}u_0 \quad \text{on } \Sigma, \\ \lim\limits_{|x|\to+\infty}|x|^{1/2}\left(\nabla(u-u^{\rm inc})\cdot\dfrac{x}{|x|}-ik(u-u^{\rm inc})\right)=0, \end{array}\right. \tag{1}$$

where $\mathbf{n_1}$ and $\mathbf{n_0}$ are respectively the unit normal to Σ outwardly directed to Ω_1 and to Ω_0 (fig. 1), k is the wave number, n and ε, respectively the index and the relative permittivity of the dielectric medium filling Ω_1. Superscript 1 and 0 indicate respective limits on Σ from within Ω_1 and Ω_0.

To uncouple the exterior problem solution in Ω_0 and the interior one in Ω_1, we use the methods initiated by P.-L. Lions [Lio90] and later developed for wave propagation problems by B. Després [Dep91] to write the transmission conditions on Σ in the following equivalent form

$$\left\{\begin{array}{l} \varepsilon^{-1}\partial_{\mathbf{n}_1}u_1+\eta Lu_1=-\partial_{\mathbf{n}_0}u_0+\eta Lu_0 \quad \text{on } \Sigma, \\ \partial_{\mathbf{n}_0}u_0+\eta Lu_0=-\varepsilon^{-1}\partial_{\mathbf{n}_1}u_1+\eta Lu_1 \quad \text{on } \Sigma. \end{array}\right. \tag{2}$$

where L is positive self-adjoint inversible operator, $\eta=-ik(\mathcal{R}+i\mathcal{X})$ with $\mathcal{R}>0$ and $\mathcal{X}\geq 0$. Therefore, the computation of the solution consists in solving the following two problems separately at each step n

$$\left\{\begin{array}{l} \nabla\cdot(\frac{1}{\varepsilon}\nabla u_1^{(n+1)})+k^2\frac{n^2}{\varepsilon}u_1^{(n+1)}=0 \quad \text{in } \Omega_1, \\ \partial_{\mathbf{n}_1}u_1^{(n+1)}=0 \quad \text{on } \Gamma, \end{array}\right. \tag{3a}$$

$$\frac{1}{\varepsilon}\partial_{\mathbf{n}_1}u_1^{(n+1)}+\eta Lu_1^{(n+1)}=-\partial_{\mathbf{n}_0}u_0^{(n)}+\eta Lu_0^{(n)} \quad \text{on } \Sigma, \tag{3b}$$

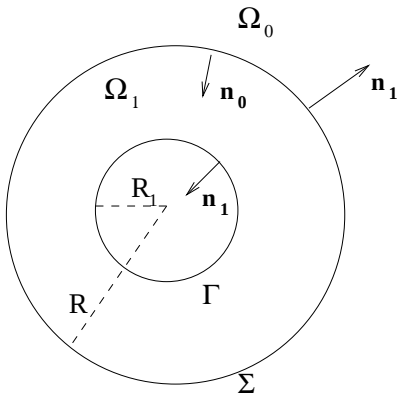

Figure 2: A circular geometry

$$\begin{cases} \Delta u_0^{(n+1)} + k^2 u_0^{(n+1)} = 0 \quad \text{in } \Omega_0, \\ \lim\limits_{|x|\to+\infty} |x|^{1/2} \left(\nabla(u_0^{(n+1)} - u^{\text{inc}}) \cdot \dfrac{x}{|x|} - ik(u_0^{(n+1)} - u^{\text{inc}}) \right) = 0, \end{cases} \tag{4a}$$

$$\partial_{\mathbf{n}_0} u_0^{(n+1)} + \eta L u_0^{(n+1)} = -\frac{1}{\varepsilon} \partial_{\mathbf{n}_1} u_1^{(n)} + \eta L u_0^{(n)} \quad \text{on } \Sigma. \tag{4b}$$

It is well-known (e.g., [CZ92]) that both problems (1), (3) and (4) are well-posed in an appropriate functional setting. Observe that the direct solution of problem (1) requires the determination of the following coupled Cauchy data $\lambda_\Sigma = u_0 = u_1$ on Σ, $p_\Sigma = \varepsilon^{-1}\partial_{\mathbf{n}_1} u_1 = \partial_{\mathbf{n}_1} u_0$ on Σ and $\lambda_\Gamma = u_1$ on Γ (e.g., [BS94]), whereas, the solution of problem (3) requires the determination of $\lambda_1^{(n+1)}$ and $\lambda_\Gamma^{(n+1)}$ and that of problem (4), the determination of $\lambda_0^{(n+1)}$ only.

Convergence of the domain decomposition method

For $\Re(\eta) = 0$, i.e. $\mathcal{X} = 0$, the theoretical convergence of the algorithm (3) and (4) is well known [Dep91], [CGJ00]. However, plots of the residual in figure 3 clearly indicates that the discrete version of the algorithm converges for $\mathcal{X} > 0$ only. It seems that only variational schemes like finite element methods can keep the convergence properties of the algorithm at the discrete level (e.g., [Dep91], [dLBFM$^+$98]). Boundary element method is not based on such a principle and thus results in a non convergent scheme for $\mathcal{X} = 0$.

For $\mathcal{X} > 0$, the proof of convergence seems to be out of reach for the general case. This is probably due to a lack of a suitable way to handle propagative and evanescent parts of the solution separately. However, for all cases when a decomposition of the solution in propagative and evanescent modes can be done, we are able to prove that the algorithm with $\mathcal{X} > 0$ has a better behaviour than with $\mathcal{X} = 0$. The following example rather strikingly illustrates this claim.

For a circular geometry (fig. 2) with $\Omega_0 = \{x \in \mathbb{R}^2; \quad |x| > R\}$, $\Omega_1 = \{x \in \mathbb{R}^2; \quad R_1 < |x| < R\}$, we can decompose the error in modes from a Fourier-Hankel series

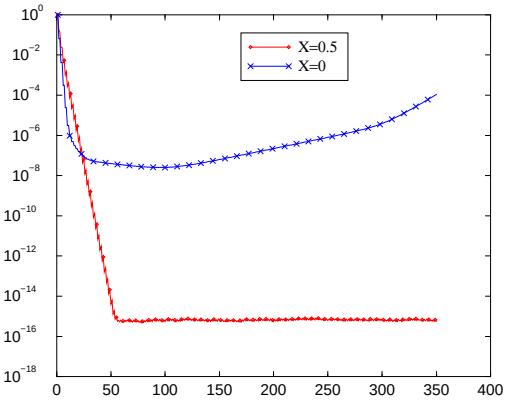

Figure 3: Behaviour of the residuals

expansion and analyze separately the convergence of the propagative and evanescent parts of the wave. Setting

$$u_0(r,\theta) = \sum_{m=-\infty}^{+\infty} u_0^{(m)}(r)e^{im\theta}, \qquad u_1(r,\theta) = \sum_{m=-\infty}^{+\infty} u_1^{(m)}(r)e^{im\theta}, \tag{5}$$

problems (3), (4) are reduced to the following one-dimensional problems

$$\begin{cases} \dfrac{1}{r}\partial_r(r\partial_r u_0^{(m)}) - \dfrac{m^2}{r^2}u_0^{(m)} + k^2 u_0^{(m)} = 0, \quad r > R, \\ \lim\limits_{r\to+\infty} r^{1/2}\left(\partial_r u_0^{(m)} - iku_0^{(m)}\right) = 0, \end{cases} \tag{6a}$$

$$-\partial_r u_0^{(m)} + \eta L_m u_0^{(m)} = g_0^{(m)}, \quad r = R, \tag{6b}$$

$$\begin{cases} \dfrac{1}{r}\partial_r(r\partial_r u_1^{(m)}) - \dfrac{m^2}{r^2}u_1^{(m)} + k^2 u_1^{(m)} = 0, \quad R_1 < r < R, \\ -\partial_r u_1^{(m)} = 0, \quad r = R_1, \end{cases} \tag{7a}$$

$$\varepsilon^{-1}\partial_r u_1^{(m)} + \eta L_m u_1^{(m)} = g_1^{(m)}, \quad r = R. \tag{7b}$$

We have assumed that the operator L is diagonal relatively to the Fourier series expansion. Solutions to problems (6), (7) are respectively obtained by $u_0^{(m)} = \alpha_m H_m^{(1)}(kr)$ and $u_1^{(m)} = \beta_m N_m(knr)$ where $H_m^{(1)}$ represents the Hankel function of the first kind and $N_m(knr)$ is a solution of the Bessel equation of order m which can be expressed by a linear combination of the Bessel J_m and Neumann Y_m functions of order m such that $N_m^{'}(knR_1) = 0$. The iteration operator is characterized by the matrices

$$\mathcal{S} = \begin{pmatrix} 0 & \mathcal{S}_m^{(0)} \\ \mathcal{S}_m^{(1)} & 0 \end{pmatrix}, \tag{8}$$

where $\mathcal{S}_m^{(0)}$ et $\mathcal{S}_m^{(1)}$ are defined by

$$\mathcal{S}_m^{(0)} g_0^{(m)} = -1 + 2\eta L_m u_0^{(m)}(R), \qquad \mathcal{S}_m^{(1)} g_1^{(m)} = -1 + 2\eta L_m u_1^{(m)}(R). \tag{9}$$

First, we give a criterion characterizing the convergence of the algorithm.

Theorem 1 *The domain decomposition algorithm converges if and only if for all* m $\rho(\mathcal{S}_m) < 1$, $\rho(\mathcal{S}_m)$ *being the spectral radius of matrix* S_m.

Proof Let $g = (g_0, g_1)^T$. One possible definition of the norm in $H^{-s}(\Sigma) \times H^{-s}(\Sigma)$ is given by $\|g\|_{-s}^2 = \sum_{-\infty}^{+\infty} (1+m^2)^{-s} \left|g^{(m)}\right|^2$, where $g^{(m)}$ is defined by $g = \sum_{-\infty}^{+\infty} g^{(m)} e^{im\theta}$ and $g^{(m)} = (g_0^{(m)}, g_1^{(m)})^T$. The convergence of the method will be established if we can show that $\lim_{n\to+\infty} \|\mathcal{S}^n g\|_{-s} = 0$ with

$$\|\mathcal{S}^n g\|_{-s}^2 = \sum_{-\infty}^{+\infty} (1+m^2)^{-s} \left|(\mathcal{S}_m)^n g^{(m)}\right|^2. \tag{10}$$

If it exists m_0 such that $\rho(\mathcal{S}_{m_0}) > 1$, clearly the method does not converge. So, we can restrict the discussion to the case where $\rho(\mathcal{S}_m) < 1$ for all m. The matrix $\mathcal{S}_m$ has two distinct eigenvalues $\lambda_m = \pm\sqrt{\mathcal{S}_m^{(0)}\mathcal{S}_m^{(1)}}$. So, it can be put in a diagonal form by $\mathcal{S}_m = P_m D_m P_m^{-1}$, D_m being a diagonal matrix. Therefore, we obtain

$$\left\|\mathcal{S}_m^n g^{(m)}\right\| \leq \|P_m\| \, \|P_m^{-1}\| \, \|g^{(m)}\| \, \left(\rho(\mathcal{S}_m)\right)^n.$$

The most important point in the proof is that the condition number $\|P_m\| \, \|P_m^{-1}\|$ of matrix S_m remains uniformly bounded. Elementary arguments then permit to end the proof. ■

The previous characterization establishes that the method converges if $|\mathcal{S}_m^{(0)}\mathcal{S}_m^{(1)}| < 1$ for each m to obtain the convergence of the method. Solving problems (6), (7), we get

$$\mathcal{S}_m^{(0)} = \frac{-\mathcal{Z}_0 + i(\mathcal{R} + i\mathcal{X})}{\mathcal{Z}_0 + i(\mathcal{R} + i\mathcal{X})}, \quad \mathcal{S}_m^{(1)} = \frac{-\mathcal{Z}_1 + i(\mathcal{R} + i\mathcal{X})}{\mathcal{Z}_1 + i(\mathcal{R} + i\mathcal{X})}, \tag{11}$$

where $\mathcal{Z}_0 = H_m^{(1)'}(kR)/L_m H_m^{(1)}(kR)$, $\quad \mathcal{Z}_1 = -N_m'(knR)/L_m N_m(knR)$, which are well defined because both the two problems are well posed.

Proposition 1

- *For both evanescent and propagative modes,* $|\mathcal{S}_m^{(0)}| < 1$.
- *If* m *corresponds to an evanescent mode, that is,* $m \geq m_0$, *for* m_0 *large enough, then* $|\mathcal{S}_m^{(1)}| < 1$.

Proof Let $\mathcal{Z}_0 = -x_m + iy_m$. Clearly, it is enough to show that x_m and y_m are both > 0 to prove that $|\mathcal{S}_m^{(0)}| < 1$. Signs of x_m and y_m are respectively that of $\Im(H_m^{(1)'}(kR)\overline{H_m^{(1)}(kR)})$ and $-\Re(H_m^{(1)'}(kR)\overline{H_m^{(1)}(kR)})$. From [CK92], it is well-known that $\Im(H_m^{(1)'}(kR)\overline{H_m^{(1)}(kR)})$ is equal to the Wronskian $W(J_m(kR), Y_m(kR)) = 2/\pi kR$. Since $kR > 0$, from [CK92] we get that $y_m > 0$. The property $x_m > 0$ uses a more difficult argument. First, we remark that

$\Re(H_m^{(1)'}(t)\overline{H_m^{(1)}(t)})_{t=kR} = \frac{1}{2}(|H_m^{(1)}(t)|^2)'_{t=kR}$. Using Nicholson's formula [Wat22], we get that function $|H_m^{(1)}(kR)|^2$ is a strictly decreasing function, so the quantity $\Re(H_m^{(1)'}(kR)\overline{H_m^{(1)}(kR)})$ is negative and then $x_m > 0$. We conclude that for all $\mathcal{R} > 0$ and $\mathcal{X} > 0$, $|\mathcal{S}_m^{(0)}| < 1$.

For the problem in the bounded domain, the previous sign determination can be more easily obtained from coerciveness estimates. Let $\mathcal{Z}_1 = -x_m + iy_m$. The variational formulation of problem (7) gives

$$\Re\left(Ru_1^{(m)'}(R)\overline{u_1^{(m)}}(R)\right) = \int_{R_1}^{R}\left\{r|u_1^{(m)'}|^2 + (\frac{m^2}{r^2} - k^2n^2)|u_1^{(m)}|^2 r\right\}dr,$$

and then if m is large enough, using coerciveness property, we get that

$$\Re(Ru_1^{(m)'}(R)\overline{u_1^{(m)}}(R)) > 0. \tag{12}$$

Definition of $u_1^{(m)}$ and $\mathcal{Z}_1$ yields $x_m > 0$. Since we have considered that the material filling Ω_1 is without losses ($\Im(\eta) = 0$) and perfectly reflecting boundary condition on Γ, we are led the most severe case $y_m = 0$. Indeed, in this case

$$\mathcal{S}_m^{(1)} = \frac{(\mathcal{X} - x_m) + i\mathcal{R}}{(\mathcal{X} + x_m) + i\mathcal{R}}.$$

For $\mathcal{X} = 0$ (Despré's algorithm [Dep91]) $|\mathcal{S}_m^{(1)}| = 1$ and so $|\mathcal{S}_m^{(0)}\mathcal{S}_m^{(1)}| < 1$. The algorithm converges as expected from the study for the general case [Dep91]. Observe however that parameter $\mathcal{S}_m^{(1)}$ has no influence upon the convergence of the algorithm and $\mathcal{S}_m^{(0)}$ gives a less effective damping of the evanescent modes. The interesting point is that taking $\mathcal{X} > 0$ also gives $|\mathcal{S}_m^{(1)}| < 1$ for all m except a finite number generally corresponding to propagative modes. But since for $\mathcal{X} = 0$ $|\mathcal{S}_m^{(0)}\mathcal{S}_m^{(1)}| < 1$, it is sufficient to tune $\mathcal{X}$ for each of these exceptional mode to obtain a maximal value for $\mathcal{X}$ insuring the convergence of the algorithm. ■

Numerical results

At each step, problem (4) in the unbounded domain Ω_0 has been solved by a boundary element method [BBC00] and problem (3) in the bounded domain Ω_1 by an usual nodal finite element method. The exterior problem in Ω_0 is solved by a BEM following the approach introduced in [Ver99]. The solution is represented as a superposition of a single- and a double-layer potentials

$$u_0(x) = u^{\text{inc}} + \int_\Sigma G(x,y)p(y)d\Sigma(y) - \int_\Sigma \partial_{\mathbf{n}_y}G(x,y)\lambda(y)d\Sigma(y), \tag{13}$$

where the unknown densities p and λ are linked by the following relation induced by the impedance condition

$$p + \eta L = 0. \tag{14}$$

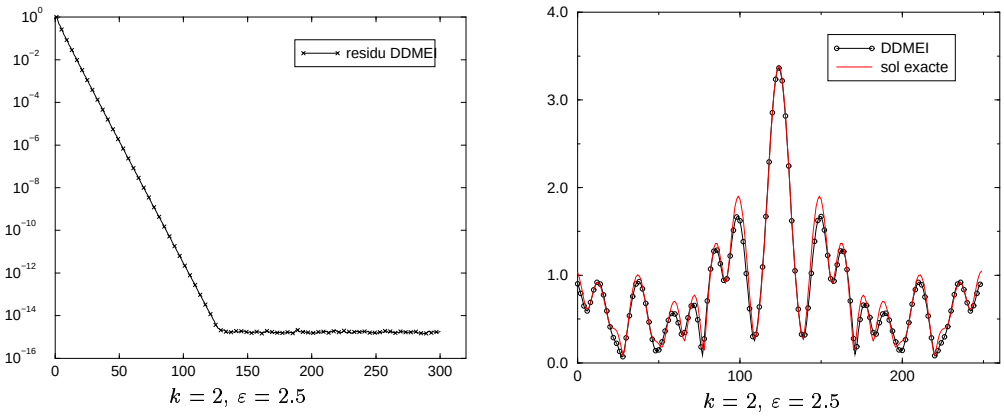

Figure 4: Coupling FEM and BEM

The boundary condition can then be expressed variationnally as

$$\int_\Sigma \left(\partial_{\mathbf{n}_0} u_0 \lambda' - u_0 p'\right) d\Sigma = \int_\Sigma g_0 \lambda' d\Sigma, \tag{15}$$

with p' and λ' are linked by the same relation that p and λ. Formulating these constrains through a Lagrange multiplier, both the latter and the magnetic currents p and p' can be eliminated at the element level when all the unknowns are approximated by a $\mathbb{P}_1$-continuous BEM, (see [Ver99] for more details).

Plots in figure 4 give the residual and comparison between exact and computed electric current on Σ. The incident wave is a plane wave propagating along the x-axis.

The interesting point is that now, with $\mathcal{X} > 0$, the discrete algorithm converges using either a nodal finite element or a boundary element method.

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