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4 A New Look at FETI

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Introduction

The Finite Element Tearing and Interconnecting (FETI) Method is usually formulated in terms of matrices and vectors (cf. [FR91], [MT96], [PJF97], [Tez98], [RF99], [MTF99], [KW01] and the references therein). In this paper we give a coordinate-free formulation of the FETI method and construct a new FETI preconditioner in terms of this formulation, which enable us to analyze it within the additive Schwarz framework. We will present the ideas for a second-order model problem on a polyhedral domain $\Omega \subset \mathbb{R}^3$. Details of the analysis can be found in [Bre00] (which deals with the 2D analog) and a forthcoming paper on the 3D FETI preconditioner.

Let $\Omega_1, \ldots, \Omega_J$ be tetrahedra which form a quasi-uniform triangulation of Ω with mesh-size H. Each of these subdomains is the union of tetrahedra from the quasi-uniform triangulation $\mathcal{T}$ of Ω, the mesh-size of which is denoted by h. Let $V(\Omega) \subset H_0^1(\Omega)$ be the P_1 finite element space associated with $\mathcal{T}$. The model problem is:

Find $u \in V(\Omega)$ such that

$$a(u,v) = \int_\Omega fv\,dx \qquad \forall\, v \in V(\Omega)\,, \tag{1}$$

where $f \in L_2(\Omega)$ and the variational form $a(\cdot,\cdot)$ is defined by

$$a(v,w) = \sum_{j=1}^{J} a_j(v,w) \quad \text{and} \quad a_j(v,w) = \alpha_j \int_{\Omega_j} \nabla v \cdot \nabla w\,dx\,. \tag{2}$$

The coefficients $\alpha_1, \ldots, \alpha_J$ in (2) are positive constants.

For simplicity we assume that $\partial\Omega_j \cap \partial\Omega$ is not zero-dimensional. We say that Ω_j is (i) *anchored* if $\partial\Omega_j \cap \partial\Omega$ contains a face of Ω_j, (ii) *hinged* if $\partial\Omega_j \cap \partial\Omega$ contains an edge of Ω_j but no faces, and (iii) *floating* if $\partial\Omega_j \cap \partial\Omega = \emptyset$.

Remark 1 *The construction and analysis of the 3D preconditioner in this paper can be applied (with modifications) to the general case where $\Omega_1, \ldots, \Omega_J$ are nonoverlapping polyhedral subdomains which do not necessarily form a triangulation of Ω and whose boundaries can intersect $\partial\Omega$ in zero-dimensional sets.*

A Coordinate-Free Formulation of FETI

Let $\Gamma_j = \partial\Omega_j \setminus \partial\Omega$ and $\Gamma = \bigcup_{j=1}^{J} \Gamma_j$ be the interface of the subdomains. The space $V(\Gamma)$ ($\subset V(\Omega)$) of discrete harmonic functions is the orthogonal complement of the space $\{v \in V(\Omega) : v = 0 \text{ on } \Gamma\}$ with respect to $a(\cdot,\cdot)$. By solving (in parallel) a discrete Poisson equation on each subdomain, the problem (1) can be reduced to the following problem on the interface:

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Find $\bar{u} \in V(\Gamma)$ such that

$$a(\bar{u}, v) = \int_\Omega f v \, dx \qquad \forall \, v \in V(\Gamma) \, . \tag{3}$$

Let $V(\Gamma_j)$ be the space of discrete harmonic functions on Ω_j which vanish on $\partial\Omega_j \cap \partial\Omega$, and

$$\tilde{V} = V(\Gamma_1) \times V(\Gamma_2) \times \cdots \times V(\Gamma_J) \, .$$

Let $\mathcal{N}_j$ (resp. $\mathcal{N}_e$ or $\mathcal{N}_F$) be the set of nodes on Γ_j (resp. the open edge e or the open face F of a subdomain) and $\mathcal{N} = \bigcup_{j=1}^J \mathcal{N}_j$. For each $p \in \mathcal{N}$ we define σ_p to be the index set of the subdomains neighboring p, i.e.,

$$\sigma_p = \{1 \le j \le J : \, p \in \partial\Omega_j\} \, ,$$

and for each $k, \ell \in \sigma_p$ we define $\mu_{p,k,\ell} \in \tilde{V}'$ (the dual space of $\tilde{V}$) by

$$\mu_{p,k,\ell}(\tilde{v}) = v_\ell(p) - v_k(p) \qquad \text{for} \quad \tilde{v} = (v_1, \ldots, v_J) \in \tilde{V} \, . \tag{4}$$

The subspace of $\tilde{V}'$ spanned by all such $\mu_{p,k,\ell}$'s is denoted by M_p and the space of Lagrange multipliers is $M = \sum_{p \in \mathcal{N}} M_p$.

In terms of M, which enforces the continuity along Γ, the interface problem (3) can be reformulated as:

Find $(\tilde{w}, \phi) \in \tilde{V} \times M$ such that

$$\sum_{j=1}^J a_j(w_j, v_j) + \langle \phi, \tilde{v} \rangle = \sum_{j=1}^J \int_{\Omega_j} f v_j \, dx \qquad \forall \, \tilde{v} \in \tilde{V} \, , \tag{5}$$

$$\langle \mu, \tilde{w} \rangle = 0 \qquad \forall \, \mu \in M \, , \tag{6}$$

where $\tilde{w} = (w_1, \ldots, w_J)$, $\tilde{v} = (v_1, \ldots, v_J)$ and $\langle \cdot, \cdot \rangle$ is the canonical bilinear form between a vector space and its dual space.

The solution of (3) is related to $\tilde{w}$ by $\bar{u}\big|_{\Omega_j} = w_j$ for $1 \le j \le J$.

Remark 2 *Throughout this paper we always keep elements of $V(\Gamma_j)'$ or $\tilde{V}'$ on the left-hand side of the canonical bilinear form $\langle \cdot, \cdot \rangle$, and members of $V(\Gamma_j)$, $\tilde{V}$ or their quotient spaces on the right-hand side.*

The FETI method solves (5)–(6) in the following way.

Let the Schur complement operator $S_j : V(\Gamma_j) \longrightarrow V(\Gamma_j)'$ be defined by

$$\langle S_j z_1, z_2 \rangle = a_j(z_1, z_2) \qquad \forall \, z_1, z_2 \in V(\Gamma_j) \, . \tag{7}$$

Let $\mathrm{Ker} S_j = \{v \in V(\Gamma_j) : S_j v = 0\}$, $(\mathrm{Ker} S_j)^\perp = \{\psi \in V(\Gamma_j)' : \langle \psi, v \rangle = 0 \; \forall \, v \in \mathrm{Ker} S_j\}$, and the pseudo-inverse $S^+ : (\mathrm{Ker} S_j)^\perp \longrightarrow V(\Gamma)/\mathrm{Ker} S_j$ be defined by the following properties:

$$\begin{aligned} \langle \psi_1, S_j^+ \psi_2 \rangle &= \langle \psi_2, S_j^+ \psi_1 \rangle && \forall \, \psi_1, \psi_2 \in (\mathrm{Ker} S_j)^\perp \, , \\ S_j S_j^+ \psi &= \psi && \forall \, \psi \in (\mathrm{Ker} S_j)^\perp \, , \\ S_j^+ S_j z &= \pi_j z && \forall \, z \in V(\Gamma_j) \, , \end{aligned} \tag{8}$$

where $\pi_j : V(\Gamma_j) \longrightarrow V(\Gamma_j)/\mathrm{Ker} S_j$ is the canonical projection.

Remark 3 $\mathrm{Ker}\, S_j$ *is the space of constant functions for a floating* Ω_j *and* $\mathrm{Ker}\, S_j = \{0\}$ *for an anchored or a hinged* Ω_j*, in which case* $S_j^+ = S_j^{-1}$.

Let $\tilde{S} : \tilde{V} \longrightarrow \tilde{V}'$ be the product of the S_j's. Then

$$\mathrm{Ker}\tilde{S} = \mathrm{Ker}S_1 \times \cdots \times \mathrm{Ker}S_J$$

and the pseudo-inverse $\tilde{S}^+ : (\mathrm{Ker}\tilde{S})^\perp \longrightarrow \tilde{V}/\mathrm{Ker}\tilde{S}$ is the product of the S_j^+'s.

Let $\gamma_f \in \tilde{V}'$ be defined by

$$\langle \gamma_f, \tilde{v} \rangle = \sum_{j=1}^{J} \int_{\Omega_j} f v_j \, dx \qquad \forall\, \tilde{v} = (v_1, \ldots, v_J) \in \tilde{V}\,, \tag{9}$$

and let $\phi_0 \in M$ satisfy

$$\langle \phi_0, \tilde{v} \rangle = \langle \gamma_f, \tilde{v} \rangle \qquad \forall\, \tilde{v} \in \mathrm{Ker}\tilde{S}\,. \tag{10}$$

It follows from (5), (7), (9) and (10) that $\phi_* = \phi - \phi_0 \in M \cap (\mathrm{Ker}\tilde{S})^\perp$. Moreover, equation (5) can be written as

$$\tilde{S}\tilde{w} + \phi_* = \gamma_f - \phi_0\,. \tag{11}$$

Since $\gamma_f - \phi_0 \in (\mathrm{Ker}\tilde{S})^\perp$ by (10), we obtain from (8) and (11) the relation

$$\pi\tilde{w} + \tilde{S}^+\phi_* = \tilde{S}^+(\gamma_f - \phi_0)\,, \tag{12}$$

where $\pi : \tilde{V} \longrightarrow \tilde{V}/\mathrm{Ker}\tilde{S}$ is the product of the π_j's. Equation (6) and (12) then imply

$$\langle \psi, \tilde{S}^+\phi_* \rangle = \langle \psi, \tilde{S}^+(\gamma_f - \phi_0) \rangle \qquad \forall\, \psi \in M \cap (\mathrm{Ker}\tilde{S})^\perp\,. \tag{13}$$

Equation (13) is a symmetric positive definite (SPD) system on

$$F = M \cap (\mathrm{Ker}\tilde{S})^\perp$$

which determines ϕ_*. Once we have found ϕ_* (and hence $\phi = \phi_* + \phi_0$), then we can recover $\tilde{w}$ (and hence $\bar{u}$) in two steps. In the first step we determine (by a parallel solve) $\tilde{w}_* \in \tilde{V}$ with the property that

$$\pi\tilde{w}_* = \tilde{S}^+(\gamma_f - \phi)\,. \tag{14}$$

In the second step we find $\tilde{w}_0 \in \mathrm{Ker}\tilde{S}$ such that

$$\langle \mu, \tilde{w}_0 \rangle = -\langle \mu, \tilde{w}_* \rangle \qquad \forall\, \mu \in M\,. \tag{15}$$

Then $\tilde{w} = \tilde{w}_* + \tilde{w}_0$ and $\phi = \phi_* + \phi_0$ satisfy the system (5)–(6).

Equation (13) can be rewritten as

$$\mathbb{S}^+\phi_* = g_0\,,$$

where $\mathbb{S}^+ : F \longrightarrow F'$ and $g_0 \in F' = \tilde{V}/(M^\perp + \mathrm{Ker}\,\tilde{S})$ are defined by

$$\langle \psi, \mathbb{S}^+\eta \rangle = \langle \psi, \tilde{S}^+\eta \rangle \qquad \forall\, \psi, \eta \in F\,, \tag{16}$$

and $\langle \psi, g_0 \rangle = \langle \psi, \tilde{S}^+(\gamma_f - \phi_0) \rangle \quad \forall\, \psi \in F$. The operator $\mathbb{S}^+$ is therefore at the heart of the FETI method.

Remark 4 *The underdetermined system* (10) *is solvable since*

$$M^{\perp} \cap \mathrm{Ker}\, \tilde{S} = \{0\}\,. \tag{17}$$

The overdetermined system (15) *is solvable because* (13) *and* (14) *imply* $\langle \mu, \tilde{w}_* \rangle = 0\ \forall\, \mu \in M \cap (\mathrm{Ker}\, \tilde{S})^{\perp}$. *Its solution is also unique because of* (17).

Additive Schwarz FETI Preconditioners

Let $R_j : \tilde{V}' = V(\Gamma_1)' \times \cdots \times V(\Gamma_J)' \longrightarrow V(\Gamma_j)'$ be the restriction map. Then R_j maps $(\mathrm{Ker}\, \tilde{S})^{\perp}$ into $(\mathrm{Ker}\, S_j)^{\perp}$ and we can express (16) as

$$\mathbb{S}^{+} = \sum_{j=1}^{J} (R_j \Phi)^t S_j^{+} (R_j \Phi)\,,$$

where $\Phi : F \longrightarrow (\mathrm{Ker}\, \tilde{S})^{\perp}$ is the natural injection. It is therefore natural to precondition $\mathbb{S}^{+}$, which is a sum of SPD operators, by the sum of the "inverses" of the SPD operators, i.e., the FETI preconditioner $\mathbb{T} : F' \longrightarrow F$ should have the form

$$\mathbb{T} = \sum_{j=1}^{J} I_j S_j I_j^t\,, \tag{18}$$

where $I_j : (\mathrm{Ker}\, S_j)^{\perp} \longrightarrow F$ is "inverse" to $R_j \Phi$ in the sense that $\sum_{j=1}^{J} I_j R_j \Phi =$ the identity operator on F, i.e.,

$$\sum_{j=1}^{J} I_j R_j \phi = \phi \qquad \forall\, \phi \in F\,. \tag{19}$$

Remark 5 *It follows from* (18) *that* $\mathbb{T}$ *is an additive Schwarz preconditioner and hence can be analyzed by the well-known additive Schwarz theory (cf. [SBG96] and the references therein).*

We will define the operator I_j in terms of three operators.

For each $p \in \mathcal{N}_j$, we define $\delta_{p,j} \in V(\Gamma_j)$ by

$$\delta_{p,j}(q) = \begin{cases} 1 & \text{if } q = p\,, \\ 0 & \text{if } q \in \mathcal{N}_j \setminus \{p\}\,. \end{cases}$$

Note that $\{\delta_{p,j} : p \in \mathcal{N}_j\}$ is a basis of $V(\Gamma_j)$. For $t \in [1/2, \infty)$ we define $\mathbb{E}_j : V(\Gamma_j)' \longrightarrow M$ by

$$\mathbb{E}_j \psi = \sum_{p \in \mathcal{N}_j} \langle \psi, \delta_{p,j} \rangle \sum_{\ell \in \sigma_p} \alpha_\ell^t \mu_{p,\ell,j}\,, \tag{20}$$

and $\mathbb{D}_j : V(\Gamma_j)' \longrightarrow V(\Gamma_j)'$ by

$$\langle \mathbb{D}_j \eta, \delta_{p,j} \rangle = \langle \eta, \delta_{p,j} \rangle / \Big(\sum_{\ell \in \sigma_p} \alpha_\ell^t \Big) \qquad \forall\, p \in \mathcal{N}_j\,. \tag{21}$$

The operators $\mathbb{E}_j$ and $\mathbb{D}_j$ form a partition of unity with R_j on the space M:

$$\sum_{j=1}^{J} \mathbb{E}_j \mathbb{D}_j R_j \mu = \mu \qquad \forall\, \mu \in M\,. \tag{22}$$

Remark 6 *The operator $\mathbb{D}_j$ is a diagonal scaling operator which together with $\mathbb{E}_j$ forms an averaging process that yields (22) and also ensures the bound for the condition number of $\mathbb{T}\mathbb{S}^{+}$ is independent of the coefficients $\alpha_1, \dots, \alpha_J$. This scaling technique is well-known.*

Finally we note that if $\mathbb{Q} : M \longrightarrow F$ is a projection operator and the map $I_j : (\mathrm{Ker}\, S_j)^{\perp} \longrightarrow F$ is defined by

$$I_j v_j = \mathbb{Q}\mathbb{E}_j \mathbb{D}_j v_j \qquad \forall\, v_j \in (\mathrm{Ker}\, S_j)^{\perp}\,, \tag{23}$$

then (19) follows from (22). Hence the crucial step in defining the additive Schwarz FETI preconditioner $\mathbb{T}$ is the construction of the projection $\mathbb{Q}$.

Let $\mu \in M$ and

$$\mathbb{Q}\mu = \mu - \mu_*\,. \tag{24}$$

Then $\mathbb{Q}$ is a projection from M onto F provided that $\mu \to \mu_*$ is linear and

$$\langle \mu_*, \tilde{v} \rangle = \langle \mu, \tilde{v} \rangle \qquad \forall\, \tilde{v} \in \mathrm{Ker}\, \tilde{S}\,, \tag{25}$$

$$\mu_* = 0 \qquad \text{if } \mu \in F\,. \tag{26}$$

Remark 7 *Once we have chosen a solution space for (25), we can take μ_* to be the member of the solution space that minimizes an appropriate inner product norm. This will automatically guarantee that $\mu \to \mu_*$ is linear and (26) is satisfied.*

A New 3D Preconditioner

Let $\mathcal{V}$ (resp. $\mathcal{E}$) be the set of vertices (resp. (open) edges) of floating subdomains and $M_{\mathcal{V}} = \sum_{p \in \mathcal{V}} M_p$. For $e \in \mathcal{E}$ we define

$$\sigma_e = \{1 \le j \le J : \; e \subset \partial\Omega_j\}\,,$$

and for $k, \ell \in \sigma_e$, we define

$$\mu_{e,k,\ell} = \frac{1}{|\mathcal{N}_e|} \sum_{p \in \mathcal{N}_e} \mu_{p,k,\ell}\,. \tag{27}$$

Let the space $M_e \subset M$ be generated by all such $\mu_{e,k,\ell}$'s and $M_{\mathcal{E}} = \sum_{e \in \mathcal{E}} M_e$.

The solution space of the projection equation (25) is then chosen to be $M_{\mathcal{W}} = M_{\mathcal{V}} + M_{\mathcal{E}}$, where the subscript $\mathcal{W}$ stands for *wire-basket*.

Remark 8 *Equation (25) is solvable in $M_{\mathcal{W}}$ because 0 is the only $\tilde{v} \in \mathrm{Ker}\, \tilde{S}$ annihilated by all $\mu_* \in M_{\mathcal{W}}$.*

According to Remark 7, we still need to introduce an appropriate inner product on $M_{\mathcal{W}}$ in order to complete the definition of $\mathbb{Q}$.

Let p (resp. e or F) be a vertex (resp. an open edge or an open face) of Ω_j, and $\tilde{1}_{p,j} \in \tilde{V}$ (resp. $\tilde{1}_{e,j} \in \tilde{V}$ or $\tilde{1}_{F,j} \in \tilde{V}$) be characterized by (i) the j-th component of $\tilde{1}_{p,j}$ (resp. $\tilde{1}_{e,j}$ or $\tilde{1}_{F,j}$) equals 1 at p (resp. the nodes in e or F) and vanishes at all the other nodes in $\mathcal{N}_j$, and (ii) all the other components of $\tilde{1}_{p,j}$ (resp. $\tilde{1}_{e,j}$ or $\tilde{1}_{F,j}$) vanish.

For $p \in \mathcal{V}$, $\mu_p \in M_p$ and $\tilde{v} = (c_1, \dots, c_J) \in \operatorname{Ker} \tilde{S}$, where the c_j's are constants, we have

$$\langle \mu_p, \tilde{v} \rangle = \langle \mu_p, \sum_{j \in \sigma_p} c_j \tilde{1}_{p,j} \rangle . \tag{28}$$

It is easy to see (cf. (2) and (44) below) that

$$a\big(\sum_{j \in \sigma_p} c_j \tilde{1}_{p,j}, \sum_{j \in \sigma_p} c_j \tilde{1}_{p,j} \big) \approx h \sum_{j \in \sigma_p} \alpha_j c_j^2 . \tag{29}$$

Remark 9 *To avoid the proliferation of constants, we use the notation $A \lesssim B$ to represent the statement that $A \leq$ constant $\times B$, where the constant is independent of the mesh-sizes, the number of subdomains and the coefficients $\alpha_1, \dots, \alpha_J$. The notation $A \approx B$ is equivalent to $A \lesssim B$ and $B \lesssim A$.*

In view of (28) and (29), it is natural to define

$$\|\mu_p\|_{M_p} = \sup_{\sum_{j \in \sigma_p} c_j^2 > 0} \frac{\langle \mu_p, \sum_{j \in \sigma_p} c_j \tilde{1}_{p,j} \rangle}{(h \sum_{j \in \sigma_p} \alpha_j c_j^2)^{1/2}} \qquad \forall\, \mu_p \in M_p . \tag{30}$$

Similarly, for $\mu_e \in M_e$ and $\tilde{v} = (c_1, \dots, c_J) \in \operatorname{Ker} \tilde{S}$, we have

$$\langle \mu_e, \tilde{v} \rangle = \langle \mu_e, \sum_{j \in \sigma_e} c_j \tilde{1}_{e,j} \rangle . \tag{31}$$

It is again easy to see (cf. (2) and (44) below) that

$$a\big(\sum_{j \in \sigma_e} c_j \tilde{1}_{e,j}, \sum_{j \in \sigma_e} c_j \tilde{1}_{e,j} \big) \approx H \sum_{j \in \sigma_e} \alpha_j c_j^2 . \tag{32}$$

In view of (31) and (32), we define

$$\|\mu_e\|_{M_e} = \sup_{\sum_{j \in \sigma_e} c_j^2 > 0} \frac{\langle \mu_e, \sum_{j \in \sigma_e} c_j \tilde{1}_{e,j} \rangle}{(H \sum_{j \in \sigma_e} \alpha_j c_j^2)^{1/2}} \qquad \forall\, \mu_e \in M_e . \tag{33}$$

Since $\|\cdot\|_{M_p}$ and $\|\cdot\|_{M_e}$ are dual to inner product norms, they are also norms of inner products. If we define

$$\|\mu\|_{M_{\mathcal{W}}}^2 = \sum_{p \in \mathcal{V}} \|\mu_p\|_{M_p}^2 + \sum_{e \in \mathcal{E}} \|\mu_e\|_{M_e}^2 \qquad \forall\, \mu \in M_{\mathcal{W}} , \tag{34}$$

where $\mu = \sum_{p \in \mathcal{V}} \mu_p + \sum_{e \in \mathcal{E}} \mu_e$, $\mu_p \in M_p$ and $\mu_e \in M_e$, then $\|\cdot\|_{M_{\mathcal{W}}}$ is also an inner product norm.

We can now define the projection operator $\mathbb{Q}$ by (24), where $\mu_* \in M_{\mathcal{W}}$ is the solution of (25) with the minimum $M_{\mathcal{W}}$ norm. The preconditioner $\mathbb{T}$ is then given by (18), (20), (21) and (23).

Remark 10 *The computation of the minimum norm solution μ_* of (25) in the space $M_{\mathcal{W}}$ is the "coarse problem" that provides global communication among the subdomains.*

Note that both $\|\cdot\|_{M_p}$ and $\|\cdot\|_{M_e}$ can be computed without knowing the triangulation $\mathcal{T}$ if we use appropriate bases. The case of $\|\cdot\|_{M_e}$ is clear from (33) if we use the basis $\{\mu_{e,\ell_*,\ell} : \ell \in \sigma_e \setminus \{\ell_*\},\ \ell_* = \min \sigma_e\}$. For $\|\cdot\|_{M_p}$ we use the basis $\{h^{1/2}\mu_{p,\ell_*,\ell} : \ell \in \sigma_p \setminus \{\ell_*\},\ \ell_* = \min \sigma_p\}$. Let $\mu = \sum_{\ell \in \sigma_p \setminus \{\ell_*\}} \gamma_\ell (h^{1/2}\mu_{p,\ell_*,\ell})$. Using (4) we can rewrite the right-hand side of (30) as

$$\sup_{\sum_{j\in\sigma_p} c_j^2>0} \frac{h^{1/2}\sum_{\ell\in\sigma_p\setminus\{\ell_*\}} \gamma_\ell (c_\ell - c_{\ell_*})}{(h\sum_{j\in\sigma_p}\alpha_j c_j^2)^{1/2}} = \sup_{\sum_{j\in\sigma_p} c_j^2>0} \frac{\sum_{\ell\in\sigma_p\setminus\{\ell_*\}} \gamma_\ell (c_\ell - c_{\ell_*})}{(\sum_{j\in\sigma_p}\alpha_j c_j^2)^{1/2}} ,$$

which shows that $\|\cdot\|_{M_p}$ can indeed be computed without knowledge of $\mathcal{T}$.

Hence the coarse problem is mesh-independent and the coefficient matrix for the coarse problem can be computed and factorized once the Ω_j's and the α_j's are given. This process can be carried out prior to or simultaneous with the meshing of the subdomains, and the same factorization of the coarse problem can be applied to any triangulation of the subdomains.

Remark 11 *The complexity of the computation of the coefficient matrix for the coarse problem is the same as that of a finite element stiffness matrix, where each floating subdomain corresponds to a node and two such nodes are neighbors if they share a common vertex.*

Remark 12 *By construction, we have*

$$\|\mu_*\|_{M_{\mathcal{W}}} \le \|\mu_\dagger\|_{M_{\mathcal{W}}} , \tag{35}$$

where $\mu_\dagger \in M_{\mathcal{W}}$ is any solution of (25). In Section 4 we will construct a solution $\mu_\dagger = \sum_{p\in\mathcal{V}} \mu_{\dagger,p} + \sum_{e\in\mathcal{E}} \mu_{\dagger,e}$, where each $\mu_{\dagger,p} \in M_p$ (resp. $\mu_{\dagger,e} \in M_e$) depends only on the restriction of μ to $\bigcup_{j\in\sigma_p} \Gamma_j$ (resp. $\bigcup_{j\in\sigma_e} \Gamma_j$). Then $\|\mu_\dagger\|_{M_{\mathcal{W}}}$ provides local estimates for μ_* (and hence $\mathbb{Q}\mu$) which ensure that the condition number estimate for $\mathbb{T}\mathbb{S}^+$ is independent of J.

Condition Number Estimates

There is a simple estimate for $\lambda_{\min}(\mathbb{T}\mathbb{S}^+)$. Let $\phi \in F$ be arbitrary. We have

$$\langle \phi, \tilde{S}^+\phi\rangle = \sum_{j=1}^{J} \langle R_j\phi, S_j^+ R_j\phi\rangle$$

and hence, in view of (16) and (19),

$$\langle \phi, \mathbb{S}^+\phi\rangle = \sum_{j=1}^{J} \langle \nu_j, S_j^+\nu_j\rangle , \tag{36}$$

where $\nu_j = R_j\phi \in (\operatorname{Ker} S_j)^\perp$ and $\sum_{j=1}^J I_j\nu_j = \phi$. It then follows from (18), (36) and the additive Schwarz theory that

$$\lambda_{\min}(\mathbb{T}\mathbb{S}^+) \ge 1 . \tag{37}$$

The following are useful formulas (cf. [MT96]) for $\langle\cdot, \tilde{S}^+\cdot\rangle$ and $\langle\cdot, S_j^+\cdot\rangle$.

Lemma 1 *The following estimates hold*:

$$\langle \mu, \tilde{S}^{+}\mu\rangle^{1/2} \approx \sup_{\tilde{v}\in\tilde{V}\setminus\mathrm{Ker}\,\tilde{S}} \frac{\langle \mu, \tilde{v}\rangle}{\left(\sum_{j=1}^{J}\alpha_j |v_j|^2_{H^1(\Omega_j)}\right)^{1/2}} \qquad \forall\, \mu \in (\mathrm{Ker}\,\tilde{S})^{\perp},$$

where $\tilde{v} = (v_1, v_2, \ldots, v_J)$, *and*

$$\langle \nu_j, S_j^{+}\nu_j\rangle^{1/2} \approx \sup_{v_j\in V(\Gamma_j)\setminus\mathrm{Ker}\,S_j} \frac{\langle \nu_j, v_j\rangle}{\alpha_j^{1/2}|v_j|_{H^1(\Omega_j)}} \qquad \forall\, \nu_j \in (\mathrm{Ker}\,S_j)^{\perp}.$$

Remark 13 *If* Ω_j *is a floating subdomain, then without loss of generality we may assume the* v_j *in Lemma 1 satisfy* $\int_{\Omega_j} v_j\, dx = 0$.

Lemma 1 enables us to employ the following estimates, which have been established in the study of 3D domain decomposition preconditioners (cf. [Dry88], [BPS89], [BX91], [DSW94], [DW95], [KW01] and the references therein).

Lemma 2 *Let* D *be a regular tetrahedron of diameter* H *and* v *be any discrete harmonic function with respect to a quasi-uniform subdivision of* D *with mesh-size* h. *Let* p (*resp.* e *or* F) *be a vertex* (*resp. an open edge or an open face*) *of* D, *and* v_p (*resp.* v_e *or* v_F) *be the discrete harmonic function that coincides with* v *at* p (*resp. the nodes in* e *or* F) *and vanishes at all the other nodes on* ∂D.

Then the following estimates hold provided v *vanishes on one of the faces or* $\int_D v\, dx = 0$:

$$\|v_e\|^2_{L_2(e)} \lesssim [1+\ln(H/h)]\,|v|^2_{H^1(D)}, \tag{38}$$

$$|v_p|^2_{H^1(D)} \lesssim |v|^2_{H^1(D)}, \tag{39}$$

$$|v_e|^2_{H^1(D)} \lesssim [1+\ln(H/h)]\,|v|^2_{H^1(D)}, \tag{40}$$

$$|v_F|^2_{H^1(D)} \lesssim [1+\ln(H/h)]^2|v|^2_{H^1(D)}. \tag{41}$$

If v *vanishes on one of the edges, then* (38) *and* (41) *remain valid, and the following estimates hold*:

$$|v_p|^2_{H^1(D)} \lesssim [1+\ln(H/h)]\,|v|^2_{H^1(D)}, \tag{42}$$

$$|v_e|^2_{H^1(D)} \lesssim [1+\ln(H/h)]^2|v|^2_{H^1(D)}. \tag{43}$$

In the special case where v *is the constant function* 1, *we have*

$$|1_p|^2_{H^1(D)} \approx h\,,\ |1_e|^2_{H^1(D)} \approx H \quad \textit{and} \quad |1_F|^2_{H^1(D)} \approx H[1+\ln(H/h)]. \tag{44}$$

Let $\nu_j \in (\mathrm{Ker}\,S_j)^{\perp}$ and $\phi = \sum_{j=1}^{J} I_j\nu_j$. From (23), (24) and Lemma 1 we have

$$\langle \phi, \tilde{S}^{+}\phi\rangle \lesssim \mathcal{S}_1 + \mathcal{S}_2, \tag{45}$$

$$\mathcal{S}_1 = \sup_{\tilde{v}\in\tilde{V}\setminus\mathrm{Ker}\,\tilde{S}} \frac{\langle \sum_{j=1}^{J}\mathbb{E}_j\mathbb{D}_j\nu_j, \tilde{v}\rangle^2}{\sum_{j=1}^{J}\alpha_j|v_j|^2_{H^1(\Omega_j)}}, \tag{46}$$

$$\mathcal{S}_2 = \sup_{\tilde{v}\in\tilde{V}\setminus\mathrm{Ker}\,\tilde{S}} \frac{\langle \mu_*, \tilde{v}\rangle^2}{\sum_{j=1}^{J}\alpha_j|v_j|^2_{H^1(\Omega_j)}}, \tag{47}$$

where $\mu_* \in M_{\mathcal{W}}$ is the minimum $M_{\mathcal{W}}$ norm solution of (25) with

$$\mu = \sum_{j=1}^{J} \mathbb{E}_j \mathbb{D}_j \nu_j \, . \tag{48}$$

The term $\mathcal{S}_1$ can be estimated by (20), (21), Remark 13, (39)–(43) and (46). The result is

$$\mathcal{S}_1 \lesssim [1 + \ln(H/h)]^2 \sum_{j=1}^{J} \langle \nu_j, S_j^+ \nu_j \rangle \, . \tag{49}$$

In view of Remark 12, we will estimate $\mathcal{S}_2$ by constructing a local solution of (25). For $e \in \mathcal{E}$ we define $\mu_{\dagger,e} \in M_e$ by

$$\langle \mu_{\dagger,e}, \tilde{1}_{e,j} \rangle = \langle \mu, \tilde{1}_{e,j} + \frac{1}{3} \sum_{F \in \mathcal{F}_{e,j}} \tilde{1}_{F,j} \rangle \qquad \forall\, j \in \sigma_e \, , \tag{50}$$

where μ is given by (48) and $\mathcal{F}_{e,j}$ is the set of the two faces of Ω_j which have e as an edge. We also define $\mu_{\dagger,p} = \mu_p$ for $p \in \mathcal{V}$, or equivalently,

$$\langle \mu_{\dagger,p}, \tilde{1}_{p,j} \rangle = \langle \mu, \tilde{1}_{p,j} \rangle \qquad \forall\, j \in \sigma_p \, . \tag{51}$$

Note that, for a floating subdomain Ω_j, the j-th component of the sum of $\tilde{1}_{e,j} + \frac{1}{3} \sum_{F \in \mathcal{F}_{e,j}} \tilde{1}_{F,j}$ over all six edges of Ω_j and $\tilde{1}_{p,j}$ over all four vertices of Ω_j is exactly the constant function 1. Hence $\mu_\dagger = \sum_{p \in \mathcal{V}} \mu_{\dagger,p} + \sum_{e \in \mathcal{E}} \mu_{\dagger,e}$ satisfies (25).

Remark 14 *Since μ_e and μ belong to the space M and the functions $\sum_{j \in \sigma_e} \tilde{1}_{e,j}$ and $\sum_{j \in \sigma_e} \big(\tilde{1}_{e,j} + \sum_{F \in \mathcal{F}_{e,j}} \frac{1}{3} \tilde{1}_{F,j}\big)$ are continuous on the interface Γ, we have*

$$\langle \mu_e, \sum_{j \in \sigma_e} \tilde{1}_{e,j} \rangle = 0 = \langle \mu, \sum_{j \in \sigma_e} \big(\tilde{1}_{e,j} + \sum_{F \in \mathcal{F}_{e,j}} \frac{1}{3} \tilde{1}_{F,j}\big) \rangle \, .$$

Therefore the overdetermined system (50) is consistent and, in view of (4) and (27), has a unique solution.

It follows from (20), (21), (30), (33), Lemma 1, (44), (48), (50), and (51) that

$$\begin{aligned} \|\mu_{\dagger,p}\|_{M_p}^2 &\lesssim \sum_{\partial\Omega_j \ni p} \langle \nu_j, S_j^+ \nu_j \rangle \, , \\ \|\mu_{\dagger,e}\|_{M_e}^2 &\lesssim [1 + \ln(H/h)] \sum_{\partial\Omega_j \supset e} \langle \nu_j, S_j^+ \nu_j \rangle \, , \end{aligned}$$

and hence, by (34),

$$\|\mu_\dagger\|_{M_{\mathcal{W}}}^2 \lesssim [1 + \ln(H/h)] \sum_{j=1}^{J} \langle \nu_j, S_j^+ \nu_j \rangle \, . \tag{52}$$

Let $\tilde{v} = (v_1, \ldots, v_J) \in \tilde{V}$ be arbitrary. Definitions (27), (30), (33), (34) and the Cauchy-Schwarz inequality imply

$$\langle \mu_*, \tilde{v} \rangle \lesssim \Big(\sum_{p \in \mathcal{V}} \|\mu_{*,p}\|_{M_p}^2 \Big)^{1/2} \Big[h \sum_{p \in \mathcal{V}} \sum_{j \in \sigma_p} \alpha_j v_j(p)^2 \Big]^{1/2} + \Big(\sum_{e \in \mathcal{E}} \|\mu_{*,e}\|_{M_e}^2 \Big)^{1/2} \Big[H \sum_{e \in \mathcal{E}} \sum_{j \in \sigma_e} \alpha_j \bar{v}_{e,j}^2 \Big]^{1/2}, \tag{53}$$

where $\bar{v}_{e,j} = |\mathcal{N}_e|^{-1} \sum_{p \in \mathcal{N}_e} v_j(p)$ is the mean nodal value of v_j on e. It follows from Remark 13, (38), (39), (42), (44) and the Cauchy-Schwarz inequality that

$$h v_j(p)^2 \lesssim |(v_j)_p|_{H^1(\Omega_j)}^2 \lesssim [1 + \ln(H/h)] \, |v_j|_{H^1(\Omega_j)}^2 \,, \tag{54}$$

$$H \bar{v}_{e,j}^2 \lesssim \|(v_j)_e\|_{L_2(e)}^2 \lesssim [1 + \ln(H/h)] \, |v_j|_{H^1(\Omega_j)}^2 \,. \tag{55}$$

Combining (53), (54) and (55), we find

$$\langle \mu_*, \tilde{v} \rangle^2 \lesssim [1 + \ln(H/h)] \, \|\mu_*\|_{M_{\mathcal{W}}}^2 \sum_{j=1}^{J} \alpha_j |v_j|_{H^1(\Omega_j)}^2 \,,$$

which together with (35), (47) and (52) yield

$$\mathcal{S}_2 \lesssim [1 + \ln(H/h)]^2 \sum_{j=1}^{J} \langle \nu_j, S_j^+ \nu_j \rangle \,. \tag{56}$$

Finally we conclude from (16), (45), (49) and (56),

$$\langle \phi, \mathbb{S}^+ \phi \rangle \lesssim [1 + \ln(H/h)]^2 \sum_{j=1}^{J} \langle \nu_j, S_j^+ \nu_j \rangle \tag{57}$$

whenever $\nu_j \in (\mathrm{Ker}\, S_j)^\perp$ for $1 \le j \le J$ and $\phi = \sum_{j=1}^{J} I_j \nu_j$. It then follows from (18), (57) and the additive Schwarz theory that

$$\lambda_{\max}(\mathbb{T}\mathbb{S}^+) \lesssim [1 + \ln(H/h)]^2. \tag{58}$$

Combining (37) and (58), we have the following theorem on the condition number $\kappa(\mathbb{T}\mathbb{S}^+)$.

Theorem 1 *There exists a positive constant C, independent of h, H, J and the α_j's, such that*

$$\kappa(\mathbb{T}\mathbb{S}^+) \le C[1 + \ln(H/h)]^2.$$

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References

[BPS89]James H. Bramble, Joseph E. Pasciak, and Alfred H. Schatz. The construction of preconditioners for elliptic problems by substructuring, IV. *Math. Comp.*, 53:1–24, 1989.

[Bre00]Susanne C. Brenner. An additive Schwarz preconditioner for the FETI method. Technical Report IMIP 2000:08, University of South Carolina, Department of Mathematics, 2000.

[BX91]James H. Bramble and Jinchao Xu. Some estimates for a weighted L^2 projection. *Math. Comp.*, 56:463–476, 1991.

[Dry88]Maksymilian Dryja. A method of domain decomposition for 3-D finite element problems. In Roland Glowinski, Gene H. Golub, Gérard A. Meurant, and Jacques Périaux, editors, *First International Symposium on Domain Decomposition Methods for Partial Differential Equations*, pages 43–61, Philadelphia, PA, 1988. SIAM.

[DSW94]Maksymilian Dryja, Barry F. Smith, and Olof B. Widlund. Schwarz analysis of iterative substructuring algorithms for elliptic problems in three dimensions. *SIAM J. Numer. Anal.*, 31(6):1662–1694, December 1994.

[DW95]Maksymilian Dryja and Olof B. Widlund. Schwarz methods of Neumann-Neumann type for three-dimensional elliptic finite element problems. *Comm. Pure Appl. Math.*, 48(2):121–155, February 1995.

[FR91]Charbel Farhat and Francois-Xavier Roux. A Method of Finite Element Tearing and Interconnecting and its Parallel Solution Algorithm. *Int. J. Numer. Meth. Engng.*, 32:1205–1227, 1991.

[KW01]Axel Klawonn and Olof B. Widlund. FETI and Neumann–Neumann Iterative Substructuring Methods: Connections and New Results. *Comm. Pure Appl. Math.*, 54:57–90, January 2001.

[MT96]Jan Mandel and Radek Tezaur. Convergence of a Substructuring Method with Lagrange Multipliers. *Numer. Math.*, 73:473–487, 1996.

[MTF99]Jan Mandel, Radek Tezaur, and Charbel Farhat. A Scalable Substructuring Method by Lagrange Multipliers for Plate Bending Problems. *SIAM J. Numer. Anal.*, 36(5):1370–1391, 1999.

[PJF97]K.C. Park, M.R. Justino, and C.A. Felippa. An algebraically partitioned FETI method for parallel structural analysis: algorithm description. *Int. J. Num. Meth. Engng.*, 40:2717–2737, 1997.

[RF99]Daniel Rixen and Charbel Farhat. A simple and efficient extension of a class of substructure based preconditioners to heterogeneous structural mechanics problems. *Int. J. Numer. Meth. Engng.*, 44:489–516, 1999.

[SBG96]Barry F. Smith, Petter E. Bjørstad, and William Gropp. *Domain Decomposition: Parallel Multilevel Methods for Elliptic Partial Differential Equations*. Cambridge University Press, 1996.

[Tez98]Radek Tezaur. *Analysis of Lagrange Multiplier Based Domain Decomposition*. PhD thesis, University of Colorado at Denver, Department of Mathematics, 1998.