20 Overlapping Schwarz Waveform Relaxation for Convection Reaction Diffusion Problems

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Introduction

Overlapping Schwarz waveform relaxation is a long name for an algorithm which simply solves evolution problems in parallel. It got its name as follows: the distribution of the computation is achieved by partitioning the spatial domain into overlapping subdomains, like in the classical Schwarz method. However on subdomains, time dependent problems are solved in the iteration and thus the algorithm is also of waveform relaxation type. Hence the name overlapping Schwarz waveform relaxation. These algorithms have been introduced in [GK97] and independently in [GZ97] for the solution of evolution problems in a parallel environment with slow communication links, since they permit to solve over several time steps before communicating information to the neighboring subdomains. They are ideal when one wants to use large existing networks of PC's with a high latency network but reasonable throughput as a super-computer. An earlier analysis for first order hyperbolic problems of the same type of algorithm can be found in [Bjø95].

These algorithms stand in contrast to the classical approach in domain decomposition for evolution problems, where time is first discretized uniformly using an implicit discretization and then at each time step a problem in space only is solved using domain decomposition, see for example [Meu91] and [Cai91, Cai94]. The main disadvantage of the classical approach is that one is forced to use the same time step in all subdomains and thus looses one of the main features of domain decomposition, namely to treat subdomains numerically differently. A second disadvantage is that one needs to exchange information at each time step. Overlapping Schwarz waveform relaxation is a remedy for both problems.

In this paper we study overlapping Schwarz waveform relaxation for space decompositions in all generality for the linear convection reaction diffusion equation in n dimensions. We prove linear convergence of the algorithm on unbounded time intervals and state a theorem about superlinear convergence on bounded time intervals. Both results hold at the continuous level, which leads to algorithms that converge independently of the mesh size if the overlap is held constant.

Problem Description

We are interested to solve parabolic partial differential equations in n dimensions on a parallel computer with slow communication links. We consider as our guiding example the convection

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reaction diffusion equation on a bounded domain $\Omega \subset R^n$ with a smooth boundary $\partial\Omega$,

$$
\begin{array}{rcll}
\mathcal{L}(u) := -\dfrac{\partial u}{\partial t} + \nu \Delta u + \mathbf{a} \cdot \nabla u + cu & = & f(\mathbf{x}, t) & \mathbf{x} \in \Omega, \quad 0 < t < T, \\
u(\mathbf{x}, t) & = & g(\mathbf{x}, t) & \mathbf{x} \in \partial\Omega, \quad 0 < t < T, \\
u(\mathbf{x}, 0) & = & u_0(\mathbf{x}) & \mathbf{x} \in \Omega.
\end{array}
\tag{1}
$$

We assume that the initial condition $u_0(\mathbf{x})$ and the boundary condition $g(\mathbf{x}, t)$ are bounded piecewise continuous and $f(\mathbf{x}, t)$ is continuous. This gives existence and uniqueness of a solution to (1). In our analysis we will use the maximum principle satisfied by the solution $u(\mathbf{x}, t)$ of (1):

Theorem 1 (Maximum Principle) *Assume that $\mathcal{L}(u) \geq 0$ ($\mathcal{L}(u) \leq 0$). Let $M = \sup_\Omega u$ ($\inf_\Omega u$). Assume that $u = M$ at some interior point $(\mathbf{x}_0, t_0) \in \Omega$ and that one of the following holds:*

   1. $c = 0$ and M is arbitrary.

   2. $c \leq 0$ and $M \geq 0$ ($M \leq 0$).

   3. $M = 0$ and c is arbitrary.

Then $u = M$ on $\bar{\Omega} \times [0, t_0]$.

Proof The proof can be found in [Lie96]. ∎

   To distribute the computation, we partition the domain Ω into overlapping subdomains. Such a partition can be obtained by first partitioning Ω into N non-overlapping subdomains $\widetilde{\Omega}_j$ with boundaries $\partial\widetilde{\Omega}_j$, $j = 1, 2, \ldots, N$. We denote the boundaries of the subdomain $\widetilde{\Omega}_j$ interior to the domain Ω by $\widetilde{\Gamma}_j$. Then we construct an overlapping decomposition Ω_j with boundary $\partial\Omega_j$ by enlarging each $\widetilde{\Omega}_j$ so that the boundaries of the new subdomains Γ_j interior to Ω are at least a distance δ away from $\widetilde{\Gamma}_j$. To solve the parabolic problem (1), the overlapping Schwarz waveform relaxation iteration constructs iteratively u_j^{k+1} on each subdomain Ω_j using as the boundary condition the values from the neighboring subdomains u_l^k at the previous iteration. To pass the boundary information, the boundary of Ω_j is decomposed into disjoint subsets Γ_{jl}, $l = 1, \ldots, N$ such that the Euclidean distance of $\mathbf{x} \in \Gamma_{jl}$ from the boundary of Ω_l is at least δ. This is possible because of the way the overlapping decomposition was constructed: we simply use the solutions obtained in Ω_l only within the smaller region $\widetilde{\Omega}_l$. Doing this for each subdomain, we define a complete approximation to the solution at step k on the whole of Ω which can be used at step $k + 1$ as boundary condition for the next subdomain solves. We denote also by Γ_{j0} the part of the boundary that subdomain Ω_j shares with the original domain Ω.

Linear Convergence for Unbounded Time Domains

For the convergence analysis, it suffices by linearity to consider the homogeneous problem, $f(\mathbf{x}, t) = g(\mathbf{x}, t) = u_0(\mathbf{x}) = 0$ in (1) and to analyze convergence to zero. We first consider the case where $T = \infty$ and hence restrict $c \leq 0$ to have bounded solutions. On each subdomain

Ω_j we solve at each step $k + 1$ of the overlapping Schwarz waveform relaxation iteration the subproblem

$$
\begin{array}{rclll}
\mathcal{L}(u^{k+1}) & = & 0 & \mathbf{x} \in \Omega_j, & 0 < t < T, \\
u_j^{k+1}(\mathbf{x}, t) & = & u_l^k(\mathbf{x}, t) & \mathbf{x} \in \Gamma_{jl}, & 0 < t < T, \\
u_j^{k+1}(\mathbf{x}, t) & = & 0 & \mathbf{x} \in \Gamma_{j0}, & 0 < t < T, \\
u_j^{k+1}(\mathbf{x}, 0) & = & 0 & \mathbf{x} \in \Omega_j,
\end{array}
\tag{2}
$$

for $j = 1, 2, \ldots, N$, using the boundary information from the neighboring subdomains at step k. This corresponds to an additive Schwarz or Jacobi iteration which can be done in parallel. One can also consider a multiplicative Schwarz or Gauss Seidel iteration which would need a special coloring of subdomains to remain a parallel algorithm.

We define the integer distance quantity m_j for each subdomain Ω_j to be the least number of subdomains one has to pass through to touch the boundary $\partial\Omega$, and also the maximum $m := \max_j m_j$. We further define the index sets $I_l := \{j : m_j = l\}$ so that the index set I_l contains the indices of all the subdomains which are within distance l of the boundary. Defining for bounded functions $g(\mathbf{x}, t) : \Omega \times [0, \infty) \to \mathbb{R}$ the norm

$$
||g(\cdot, \cdot)||_\infty := \sup_{\mathbf{x} \in \Omega, t > 0} |g(\mathbf{x}, t)|
$$

we have the following

Lemma 1 *The iterates of (2) satisfy for $T = \infty$ and $c \leq 0$ the estimate*

$$
\max_j ||u_j^{k+m+2}(\cdot, \cdot)||_\infty \leq \gamma(m, \delta) \max_j ||u_j^k(\cdot, \cdot)||_\infty
\tag{3}
$$

where $\gamma(m, \delta)$ is a number strictly less than one and independent of k.

Proof The idea of the proof is to construct a sequence of elliptic upper bounds on the iterates and then to apply the convergence analysis based on the maximum principle for the elliptic upper bounds in Lions [Lio88]. For k fixed we define $U^k := \max_j ||u_j^k(\cdot, \cdot)||_\infty$ and note that on each subdomain the solution $\tilde{u}_j^{k+1}$ of the elliptic problem

$$
\begin{array}{rclll}
\nu \Delta \tilde{u}_j^{k+1} + \mathbf{a} \cdot \nabla \tilde{u}_j^{k+1} + c\tilde{u}_j^{k+1} & = & 0 & \mathbf{x} \in \Omega_j, \\
\tilde{u}_j^{k+1}(\mathbf{x}) & = & U^k & \mathbf{x} \in \Gamma_{jl}, \\
\tilde{u}_j^{k+1}(\mathbf{x}) & = & 0 & \mathbf{x} \in \Gamma_{j0}
\end{array}
\tag{4}
$$

is an upper bound on the modulus of u_j^{k+1}. Now $\tilde{u}_j^{k+1}$ satisfies a maximum principle and for $j \in I_0$ we have $\tilde{u}_j^{k+1} < U^k$ in the interior of $\widetilde{\Omega}_j$, since $\tilde{u}_j^{k+1}$ satisfies on part of the boundary of Ω_j a homogeneous boundary condition. Note that for $j \notin I_0$ we have $\tilde{u}_j^{k+1}$ not necessarily strictly less than U^k since $\tilde{u}_j^{k+1}$ might have the value U^k on all its boundaries and thus by the maximum principle $\tilde{u}_j^{k+1} \equiv U^k$. Define

$$
U^{k+1} := \sup_{\mathbf{x} \in \widetilde{\Omega}_l, l \in I_0} \tilde{u}_l^{k+1} \leq \gamma_1(\delta) U^k
$$

for some constant $\gamma_1(\delta) < 1$. Note that γ_1 depends on the size of the overlap, but not on k since $\tilde{u}_j^{k+1}$ is a linear function of the boundary condition. Now for the next iteration by

definition part of the boundary of subdomains Ω_j with $j \in I_1$ lie strictly within $\widetilde{\Omega}_l$ with $l \in I_0$ and therefore for $j \in I_1$ the solution $\tilde{u}_j^{k+2}$ of the elliptic problem

$$
\begin{aligned}
\nu\Delta\tilde{u}_j^{k+2} + \mathbf{a}\cdot\nabla\tilde{u}_j^{k+2} + c\tilde{u}_j^{k+2} &= 0 && \mathbf{x}\in\Omega_j, \\
\tilde{u}_j^{k+2}(\mathbf{x}) &= U^k && \mathbf{x}\in\Gamma_{jl},\ l\notin I_0, \\
\tilde{u}_j^{k+2}(\mathbf{x}) &= U^{k+1} && \mathbf{x}\in\Gamma_{jl},\ l\in I_0
\end{aligned}
\tag{5}
$$

is an upper bound on the modulus of u_j^{k+2}. Since $U^{k+1} \le \gamma_1(\delta)U^k$ we have by the maximum principle $\tilde{u}_j^{k+2} < U^k$ in $\widetilde{\Omega}_j$ and defining U^{k+2} similarly to U^{k+1} before, we find $U^{k+2} \le \gamma_2(\delta)U^k$ for some constant $\gamma_1(\delta) \le \gamma_2(\delta) < 1$ independent of k. By induction we find at step $k+m+1$ for the iterate in the subdomains Ω_j with $j \in I_m$ the elliptic upper bound

$$
\begin{aligned}
\nu\Delta\tilde{u}_j^{k+m+1} + \mathbf{a}\cdot\nabla\tilde{u}_j^{k+m+1} + c\tilde{u}_j^{k+m+1} &= 0 && \mathbf{x}\in\Omega_j, \\
\tilde{u}_j^{k+m+1}(\mathbf{x}) &= U^k && \mathbf{x}\in\Gamma_{jl},\ l\notin I_{m-1}, \\
\tilde{u}_j^{k+m+1}(\mathbf{x}) &= U^{k+m} && \mathbf{x}\in\Gamma_{jl},\ l\in I_{m-1}
\end{aligned}
\tag{6}
$$

and $\tilde{u}_j^{k+m+1} < U^k$ in $\widetilde{\Omega}_j$. Defining U^{k+m+1} as before we find $U^{k+m+1} \le \gamma_{m+1}(\delta)U^k$ for some constant $\gamma_1(\delta) \le \gamma_2(\delta) \le \ldots \le \gamma_{m+1}(\delta) < 1$ independent of k. Now for the next iteration step $k+m+2$ all the u_j^{k+m+2} have boundary values less than or equal to $U^{k+m+1} \le \gamma_{m+1}(\delta)U^k$, since they come from iteration step $k+m+1$ in the interior of neighboring subdomains. Defining $\gamma(m,\delta) := \gamma_{m+1}(\delta)$ the result follows. ■

Theorem 2 (Linear Convergence) *For $c \le 0$ the overlapping Schwarz waveform relaxation algorithm (2) converges on unbounded time intervals $t \in [0, T = \infty)$ at least at the linear rate*

$$
\max_j ||u_j^{k(m+2)}(\cdot,\cdot)||_\infty \le (\gamma(m,\delta))^k \max_j ||u_j^0(\cdot,\cdot)||_\infty
\tag{7}
$$

where $\gamma(m,\delta) < 1$ as in Lemma 1.

Proof The proof follows by induction from Lemma 1. ■

The convergence result we derived on unbounded time domains depends on the number of subdomains, as one can see explicitly from the dependence of γ on m. The more subdomains one uses, the longer it takes for information to propagate from the outer boundary of Ω to the inner subdomains. This is because the steady state solution is limiting the convergence rate, and the steady state solution does not see the zero initial condition. This is different if the algorithm is analyzed over a bounded time interval. This analysis, which is beyond the scope of this short paper, leads to a superlinear convergence result for the algorithm. Defining for bounded functions $g(\mathbf{x},t) : \Omega \times [0,T] \to \mathbb{R}$ the norm

$$
||g(\cdot,\cdot)||_T := \sup_{\mathbf{x}\in\Omega, 0<t<T} |g(\mathbf{x},t)|
$$

we have the following

Theorem 3 (Superlinear Convergence) *For $c \le 0$ the overlapping Schwarz waveform relaxation algorithm converges superlinearly on bounded time intervals $t \in [0, T < \infty)$ in the infinity norm,*

$$
\max_j ||u_j^k(\cdot,\cdot)||_T \le \left(2n\cosh(\delta\bar{a}/(2\nu\sqrt{n}))\right)^k erfc(\frac{k\delta}{2\sqrt{nT}}) \max_j ||u_j^0(\cdot,\cdot)||_T.
\tag{8}
$$

There are two interesting facts to note about this theorem: first the convergence rate is independent of the number of subdomains, there is no dependence on a parameter m related to the number of subdomains as in Theorem 2. second the superlinear convergence rate is faster than the superlinear convergence rate found for classical waveform relaxation algorithms. The classical result gives a contraction governed by a factorial [MN87] with asymptotic expansion

$$\frac{(CT)^k}{k!} = \left(\frac{1}{\sqrt{2\pi}} + O(k^{-1})\right) e^{-k\ln k + (1+\ln(CT))k - \frac{1}{2}\ln k} \sim e^{-k\ln k}$$

whereas the new result (8) gives a contraction with asymptotic expansion

$$C_1^k \text{erfc}(\frac{C_2 k}{\sqrt{T}}) = \left(\frac{\sqrt{T}}{C_2\sqrt{\pi}} + O(k^{-2})\right) e^{-\frac{C_2^2}{T}k^2 + \ln(C_1)k - \ln k} \sim e^{-k^2}.$$

Numerical Experiments

We perform all our experiments on the two dimensional model problem

$$\frac{\partial u}{\partial t} = \nu \Delta u + \mathbf{a} \cdot \nabla u + cu, \quad (x_1, x_2) \in [0, 1] \times [0, 1], \ t \in [0, T]. \tag{9}$$

The convection is chosen to be diagonal, $\mathbf{a} := (1, 1)$ and the other parameters are $c = 0$ and $\nu = 1/10$. We decompose the domain into smaller squares with equal size and overlap both in the x_1 and x_2 direction and simulate directly the error equations. In space we discretize using central finite differences and in time using backward Euler. To see linear convergence the problem is integrated over a relatively long time interval $t \in [0, 10]$ and to see superlinear convergence the problem is integrated over a shorter time interval $t \in [0, 0.5]$. Our analysis showed that for both the linear and superlinear convergence the convergence rate depends on the size of the overlap as usual. Increasing the overlap, the error decays faster, as shown in Figure 1 on the left for a long time interval and on the right for a short time interval. We used $\delta = 0.1$ and $\delta = 0.06$ for the overlap parameter and 2×2 subdomains.

Theorem 2 shows for the linear convergence regime that the decay of the error depends on the number of subdomains; the parameter m appears in equation (7), which is similar to the results found for the heat equation in [GS98]. Thus for a long time interval, the overlapping Schwarz waveform relaxation algorithm does not scale with respect to the number of subdomains. This is illustrated in Figure 2 on the left for $\mathbf{a} = (1, 1)$, $c = 0$, $\nu = 1/20$, overlap parameter $\delta = 0.04$ and $t \in [0, 5]$. Note how initially the algorithm does not exhibit convergence, the information needs to be propagated first from the domains connected to the boundary towards the interior, as we saw in the analysis. In the superlinear convergence regime however for the same problem parameters and $t \in [0, 0.1]$ the convergence rate is independent of the number of subdomains, as stated in Theorem 3. This is confirmed in the numerical experiments shown in Figure 2 on the right and corresponds to the result found earlier for the heat equation in [GZ97]. Note how the error reduction in the superlinear convergence regime is considerably faster than the one in the linear convergence regime. Note also that the error reduction in the superlinear convergence regime is considerably faster than the one in the linear convergence regime.
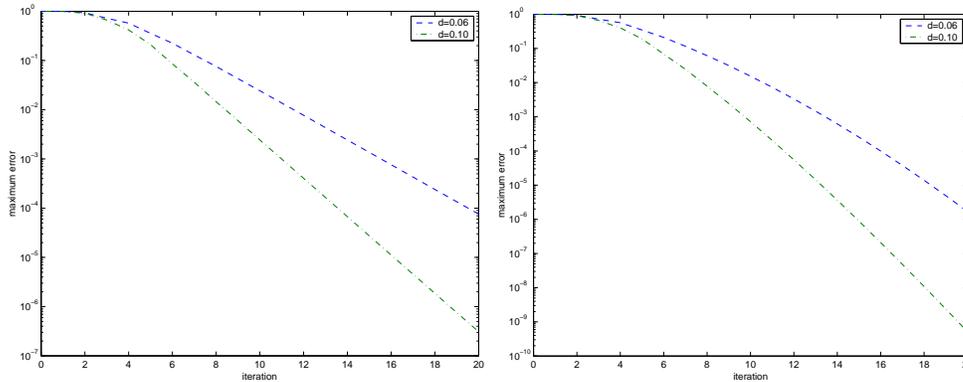
Figure 1: Two dimensional problem with four subdomains and different size of overlap for a long time interval on the left, where the algorithm is in the linear convergence regime and for a short time interval on the right, where the algorithm is in the superlinear convergence regime.

Conclusions

We have shown that the overlapping Schwarz waveform relaxation algorithm for general linear convection reaction diffusion equations with very general domain decomposition exhibits two different types of convergence regimes: on unbounded time intervals the algorithm converges at least at a linear rate depending on the size of the overlap, the problem parameters and the number of subdomains. On bounded time intervals however the convergence is superlinear. The convergence rate depends on the overlap and the diffusion coefficient, but is independent of the number of subdomains and the other problem parameters.

The main interest of the algorithm are the following three points:

1. The original problem is solved on subdomains in space-time and thus one can refine both in space and time independently on each subdomain.

2. Communication is not necessary at each time step, each processor continues to solve over a whole time window before it needs to communicate.

3. Theorem 3 shows that algorithm converges superlinearly and independently of the number of subdomains, so there is no coarse grid needed for scalability.

For a given hardware configuration, it remains to find the best length of time windows so that the convergence speed of the algorithm is balanced with the communication cost. Longer time windows lead to slower convergence, but they require less often communication which makes them faster.

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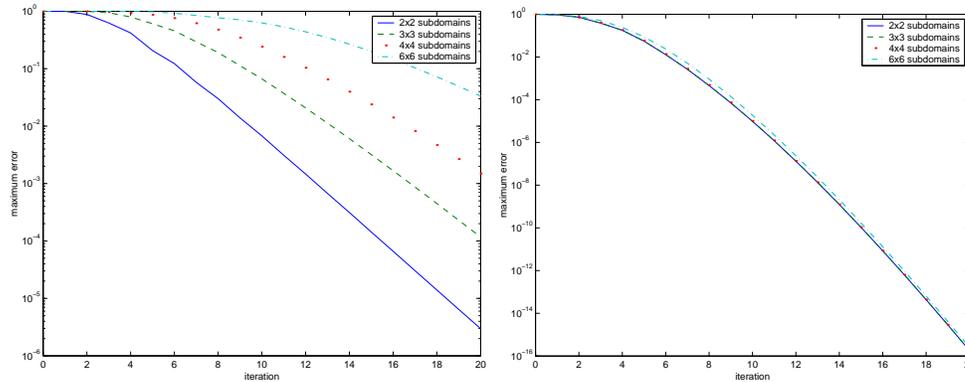
Figure 2: The effect of the number of subdomains on the algorithm. On the left for long time intervals where the linear convergence rate depends on the number of subdomains and on the right for short time intervals where the superlinear convergence rate is independent of the number of subdomains.

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