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13 Discontinuous Hybrid Formulation turned to Domain Decomposition

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Introduction

We consider a macro hybrid primal finite element formulation turned to domain decomposition which produces a completely discontinuous approximation. The key point of the framework is an analogous of an argument already used in stabilization techniques for DDM with non matching grids, [BFMR97]. The resulting approximation is conforming and the convergence is established with no inspection of consistency error, nor inf-sup condition.

The finite element approximation of the second order elliptic equations has been investigated using several different approaches (see e.g. [Cia78] and the references therein). Previous analysis in primal formulation of these problems has been done for three types of approximation schemes : one which produces a continuous piecewise polynomial approximation, one which produces a piecewise polynomial approximation with a fixed number of continuous moments accross interelement edges (nonconforming approximation) and one which produces completely discontinuous polynomial approximation (interior penalty methods) [Arn82]. All these finite element methods have optimal order of convergence, assuming sufficient regularity. More recently, there has been growing interest in methods which can produce a completely discontinuous approximation for diffusion problems [JO98]. The motivation for developing these methods was the flexiblity afforded by discontinuous finite element spaces. Another advantage that has recently become apparent is the application of domain decomposition algorithms for the solution of the discrete solution.

1 Macro hybrid formulation for the model problem

Let Ω be a simply connected polygonal domain of $\mathbb{R}^d$, $d = 2$ or 3, and Γ its boundary. Let us perform a non overlaping domain decomposition on Ω,

$$\overline{\Omega} = \cup_{i=1}^{I} \overline{\Omega}_i$$

$$\Omega_i \cap \Omega_j = \emptyset, \;\; 1 \le i \ne j \le I.$$

We assume that each subdomain Ω_i is polygonal and set the following notations

$$\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j, \;\; \text{for} \;\; 1 \le i \ne j \le I,$$

$$\forall i \in \{1, \dots, I\}, \Gamma_i = \partial\Omega_i \setminus \Gamma$$

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and
$$\mathcal{M} = \{m = (i,j),\ 1 \le i \ne j \le I,\ \text{such that} \quad \text{meas}(\partial\Omega_i \cap \partial\Omega_j) \ne 0\}.$$

We consider, for simplicity, the Dirichlet problem for the Laplace equation :
$$-\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega = \Gamma. \tag{1}$$
where $f \in L^2(\Omega)$.
First, we introduce the following functional spaces
$$W_i = \{v \in H^1(\Omega_i);\ \frac{\partial u}{\partial n_i} \in L^2(\Gamma_i) \ \text{ and } \ v_{|\partial\Omega_i \cap \Gamma} = 0 \text{ if meas}(\partial\Omega_i \cap \Gamma) \ne 0\}$$
where $\frac{\partial u}{\partial n_i}$ is the outward normal derivative of u to the boundary $\Gamma_i, i = 1, \dots, I$.
$$\dot{W} = \prod_{i=1}^{I} W_i,$$
$$S = \{\phi = (\phi_i = v_{|\Gamma_i})_{1\le i \le I} \quad \text{with}\, v \in H_0^1(\Omega)\},$$
For $((\dot{u},\phi),(\dot{v},\psi)) \in (\dot{W} \times S)^2$, we define the product bilinear form
$$\dot{B}((\dot{u},\phi),(\dot{v},\psi)) = \sum_{i=1}^{I} \int_{\Omega_i} \nabla u_i \nabla v_i \, dx - < \frac{\partial u_i}{\partial n_i}, v_i - \psi_i >_{0,\Gamma_i} \tag{2}$$
$$+ < \frac{\partial v_i}{\partial n_i}, u_i - \phi_i >_{0,\Gamma_i} + \delta_i (u_i - \phi_i, v_i - \psi_i)_{0,\Gamma_i} \tag{3}$$
$$\tag{4}$$

For $i = 1, \dots, I$, let $\mathcal{T}_{h_i}$ be a regular triangulation of the subdomain Ω_i with triangular $(d = 2)$ or tetrahedral $(d = 3)$ finite elements whose diameters are less or equal than h_i and k_i be a positive integer. We assume that the triangulation is uniformily regular near Γ_i. We introduce the standard finite element space
$$V_{h_i} = \{v_{h_i} \in \mathcal{C}^0(\overline{\Omega}_i); \forall T \in \mathcal{T}_{h_i}, v_{{h_i}|T} \in P_{k_i}(T),\ v_{|\partial\Omega_i \cap \Gamma} = 0 \text{ if meas}(\partial\Omega_i \cap \Gamma) \ne 0\}$$
and we set
$$\dot{W}_h = \prod_{i=1}^{I} V_{h_i}.$$
Remark that V_{h_i} is a subspace of W_i, and so $\dot{W}_h$ is a subspace of $\dot{W}$.
Let us now proceed with the squeleton; For all $m = (i,j) \in \mathcal{M}$, let $\mathcal{T}_{h_m}$ be a regular subdivision $(d = 2)$ or triangulation $(d = 3)$ of $\Gamma_{i,j}$ by finite elements whose diameters are less or equal than h_m and k_m be a positive integer. We introduce related finite element space
$$S_{h_m} = \{\psi_{h_m} \in \mathcal{C}^0(\Gamma_{ij}) \,\text{such that}\, \forall T \in \mathcal{T}_{h_m},\ \psi_{h_m} \in P_{k_m}(T)\}$$
and we set the global related space
$$S_h = \{\phi_h = (\phi_i)_{1 \le i \le I} \in S; \forall m = (i,j) \in \mathcal{M}, \phi_{i|\Gamma_{ij}} \in S_{h_m},$$

$$\text{and } \phi_{i|\Gamma_i\cap\Gamma} = 0 \text{ if meas}(\Gamma_i \cap \Gamma) \neq 0\}$$

The discrete problem states then as,

$$\text{Find } (\dot{u}_h, \phi_h) \in \dot{W}_h \times S_h \text{ such that} \tag{5}$$

$$\dot{B}((\dot{u}_h, \phi_h), (\dot{v}_h, \psi_h)) = \sum_{i=1}^{I} \int_{\Omega_i} f v_{h_i} dx, \quad \forall (\dot{v}_h, \psi_h) \in \dot{W}_h \times S_h. \tag{6}$$

(7)

The functional space $\dot{W} \times S$ is equipped with the norm

$$\forall (\dot{v}, \psi) \in \dot{W} \times S, \quad \|(\dot{v}, \psi)\|^2 = \sum_{i=1}^{I} (|v_i|^2_{1,\Omega_i} + \delta_i \|v_i - \psi_i\|^2_{0,\Gamma_i}).$$

In the sequel, C is a generic constant independent of $\dot{h} = (h_i)_{i=1}^I$ and $\dot{\delta} = (\delta_i)_{i=1}^I$.

Lemma *The bilinear form $\dot{B}$ is continuous and coercive with respect to the $\dot{W} \times S$ norm in the following sense,*

$$\forall (\dot{v}_h, \psi_h) \in \dot{W}_h \times S_h, \quad \dot{B}((\dot{v}_h, \psi_h), (\dot{v}_h, \psi_h)) = \|(\dot{v}_h, \psi_h)\|^2;$$

$\forall ((\dot{v}, \psi), (\dot{w}_h, \zeta_h)) \in (\dot{W} \times S) \times (\dot{W}_h \times S_h),$

$$\dot{B}(((\dot{v}, \psi), (\dot{w}_h, \zeta_h)) \leq C \|(\dot{w}_h, \zeta_h)\| \{\|((\dot{v}, \psi)\|^2 + \sum_{i=1}^{I} \frac{1}{\delta_i} \|\frac{\partial v_i}{\partial n_i}\|^2_{0,\Gamma_i} + \frac{1}{h_i} \|v_i - \psi_i\|^2_{0,\Gamma_i}\}^{\frac{1}{2}}.$$

$\diamondsuit$

If u is the weak solution of the model problem (1) and $\phi = (\phi_i = u_{|\partial\Omega_i})_{i=1}^I$. such that $\dot{u} = (u_i := u_{|\Omega_i})_{i=1}^I \in \dot{W}$, then,

$$\forall (\dot{w}_h, \psi_h) \in \dot{W}_h \times S_h, \;\; \dot{B}((\dot{u}, \phi), (\dot{w}_h, \psi_h)) = \sum_{i=1}^{I} \int_{\Omega_i} f w_h dx.$$

It is a trivial consequence of Lax-Milgram lemma that the discrete problem (5) has the unique solution $(\dot{u}_h, \phi_h) \in \dot{W}_h \times S_h$. Moreover by standard arguments and for $\delta_i = \dfrac{1}{h_i}, i = 1, \ldots, I,$

$$\|(\dot{u} - \dot{u}_h, \phi - \phi_h)\| \leq C \inf_{(\dot{v}_h, \zeta_h) \in \dot{W}_h \times S_h} \{\|((\dot{u} - \dot{v}_h, \psi - \zeta_h)\|^2 + \sum_{i=1}^{I} h_i \|\frac{\partial(u_i - v_{h_i})}{\partial n_i}\|^2_{0,\Gamma_i}\}^{\frac{1}{2}}.$$

The main result states then as,

Theorem *Let $(\dot{u}_h, \phi_h) \in \dot{W}_h \times S_h$ be the solution of discrete problem (2), u be the weak solution of the model problem and $\phi = (\phi_i := u_{|\partial\Omega_i})_{1\leq i\leq I} \in S$. We assume that $\dot{u} = (u_i :=$*

$u_{|\Omega_i})_{1\le i\le I} \in \prod_{i=1}^{I} H^{\sigma_i}(\Omega_i)$, $\dfrac{3}{2} < \sigma_i \le k_i + 1,\ i = 1,...,I$, and for all $m = (i,j) \in \mathcal{M}$, $\phi_m := \phi_{i|\Gamma_{ij}} \in H^{\sigma_m}(\Gamma_{ij})$, $\frac{1}{2} \le \sigma_m \le k_m + 1$. Assume moreover that

$$\forall m = (i,j) \in \mathcal{M}, \quad h_m \le C \text{ min}\,(h_i, h_j),$$

and

$$\forall i = 1,...,I, \ \ \delta_i = \frac{C}{h_i}.$$

Then the following estimate holds,

$$||\dot{u} - \dot{u}_h, \phi - \phi_h||^2 \le C\{\sum_{i=1}^{I} h_i^{2(\sigma_i - 1)} \|u_i\|^2_{\sigma_i,\Omega_i} + \sum_{m=(i,j)\in\mathcal{M}} h_m^{2\sigma_m - 1} \|\phi_m\|^2_{\sigma_m,\Gamma_{ij}}\}.$$

$\diamondsuit$

The proof of the theorem requires the following technical lemma, given as an appendix.

Lemma *Let T be a regular triangle (tetrahedron), and e an edge (face) of T. For all $v \in H^{1+\sigma}(T)$ with $1/2 < \sigma \le 1$, we have*

$$h_T^{\frac{1}{2}} \| \frac{\partial v}{\partial \nu_e} \|_{0,e} \le C(h_T^{\sigma} |v|_{1+\sigma,T} + |v|_{1,T}).$$

Proof As usually, let $\hat{T}$ be a reference triangle (tetrahedron), and $F(\hat{x}) = B\hat{x} + b$, the affine application defined from $\hat{T}$ onto T such that $F(\hat{T}) = T$. First, we have

$$\| \frac{\partial v}{\partial \nu_e} \|_{0,e} \le \|\nabla v\|_{0,e},$$

then

$$\| \frac{\partial v}{\partial \nu_e} \|_{0,e} \le (\frac{\text{meas}(\hat{e})}{\text{meas}(e)})^{\frac{1}{2}} \|B^{-1}\| \ \ \|\nabla \hat{v}\|_{0,\hat{e}}$$

with $\hat{v} = v \circ F$ and $\hat{e} = F^{-1}(e)$.

Using the trace theorem applied to $\nabla \hat{v}$ on $\hat{e}$, we obtain

$$\text{meas}(e)^{\frac{1}{2}} \| \frac{\partial v}{\partial \nu_e} \|_{0,e} \le C \|B^{-1}\| (|\hat{v}|_{1,\hat{T}} + |\hat{v}|_{1+\sigma,\hat{T}}).$$

Then

$$\text{meas}(e)^{\frac{1}{2}} \| \frac{\partial v}{\partial \nu_e} \|_{0,e} \le C \|B^{-1}\| \ \ \|B\| \ \ |detB|^{-\frac{1}{2}} (|v|_{1,T} + \|B\|^{\sigma} (\|B\|^{d} \ |detB|^{-1})^{s} |v|_{1+\sigma,T})$$

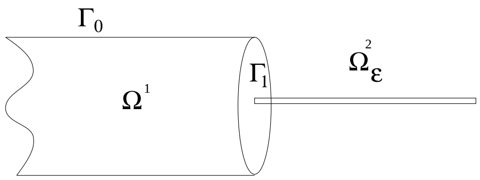
with $s = 0$ if $\sigma = 1$ and $s = \dfrac{1}{2}$ otherwise.

Since T is regular, we obtain the required inequality,

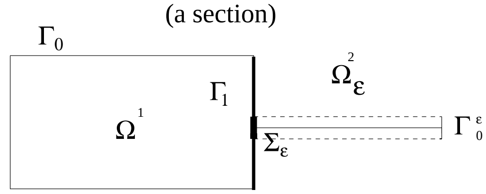
$$h_T^{\frac{1}{2}} \| \frac{\partial v}{\partial \nu_e} \|_{0,e} \le C(h_T^{\sigma} |v|_{1+\sigma,T} + |v|_{1,T}).$$

$\diamondsuit$

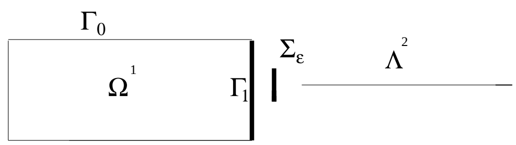
The physical model



Two dimensional related model
(a section)



The strategy



2 Application to heterogeneous domain decomposition

Let us turn now to the motivation of this study. This methodology has been first investigated for the treatmant of models of elastic multi-structures. Consider the junction of two elastic bodies Ω^1 and Ω^2_ϵ, with $\epsilon << 1$. In Ω^2_ϵ the model is a 2-dimensional model derived from a thin 3-dimensional linearly elastic plate using variational methods [SA99]. In 2D the related model reduces to a formulation on the union of a macro-element Ω^1, a patch element Σ_ϵ and a one-dimensional element Λ^2. Internal domain decomposition can be performed on each element.

Since the methodology is intended for PDEs arising from general elastic multi-structures models, we present it here for the Laplace equation.

We consider the two dimensional (a section) global model problem,

$$\left\{\begin{array}{rcl} -\Delta u^\epsilon &=& f \quad \text{in } \Omega^1 \cup \Omega^2_\epsilon \\ u^\epsilon &=& 0 \quad \text{on } \Gamma_0 \\ \dfrac{\partial u^\epsilon}{\partial n} &=& 0 \quad \text{on } \Gamma_1 \setminus \Sigma_\epsilon \\ u &=& 0 \quad \text{on } \Gamma^\epsilon_0 \\ \dfrac{\partial u^\epsilon}{\partial n} &=& 0 \quad \text{on } \partial\Omega^2_\epsilon \setminus \{\Sigma_\epsilon \cup \Gamma^\epsilon_0\} \end{array}\right.$$

where f is no longer dependent on the y-variable in the domain Ω^2_ϵ. The asymptotic problem (the strategy) states as

$$\left\{\begin{array}{rcl} -\Delta u &=& f \quad \text{in } \Omega^1 \\ u &=& 0 \quad \text{on } \Gamma_0 \\ \dfrac{\partial u}{\partial n} &=& 0 \quad \text{on } \Gamma_1 \end{array}\right. \qquad \left\{\begin{array}{rcl} -w" &=& g \qquad \text{on } \Lambda^2 \;=\; (0,1) \\ w(1) &=& 0 \end{array}\right.$$

$$\begin{cases} u_{|\Sigma_\epsilon} &= w(0) \\ w'(0) &= \dfrac{1}{\epsilon} \displaystyle\int_{\Sigma_\epsilon} \frac{\partial u}{\partial n}\, d\sigma \end{cases}$$

Let us set the adapted hybrid functional framework : the space
$\dot{W} = \{(u,w) \in\ H^1(\Omega^1) \times H^1(\Lambda^2)\ ;\ u_{|\Gamma_0} = 0,\ w(1) = 0,\ \dfrac{\partial u}{\partial n}_{|\Sigma_\epsilon}\ \in\ L^2(\Sigma_\epsilon)\}$,
equipped with the norm $\|(u,w)\|_\delta^2 = (|u|^2_{1,\Omega^1} + \epsilon|w|^2_{1,\Lambda^2} + \delta\|u - w(0)\|^2_{0,\Sigma_\epsilon}$
and the bilinear form $B_\delta((u,w),(v,z)) = \displaystyle\int_{\Omega^1} \nabla u \nabla v dx\ +\ \epsilon\ \int_{\Lambda^2} w'\ z'\ dx\ -$
$\displaystyle\int_{\Sigma_\epsilon} \frac{\partial u}{\partial n}(v - z(0))\ d\sigma\ +\ \int_{\Sigma_\epsilon} \frac{\partial v}{\partial n}(u - w(0))\ d\sigma\ +\ \delta \int_{\Sigma_\epsilon} (u - w(0))(v - z(0))\ d\sigma.$
And the adapted hybrid formulation states then as

$$\text{Find } (u,w) \in \dot{W} \text{ such that} \tag{8}$$

$$B_\delta((u,w),(v,z)) = \int_{\Omega^1} f\ v\ dx\ +\ \epsilon\ \int_{\Lambda^2} g\ z\ d\sigma \quad \forall (v,z) \in \dot{W} \tag{9}$$

Since $w \in H^2(\Lambda^2)$, if $u \in H^{\sigma+1}(\Omega^1),\ 0 < \sigma \leq 1$, the analysis carried in this context with minor adaptation for a standard P_1- finite element discretization as performed in the previous section and $\delta = \dfrac{1}{h_1}$, gives the following error estimate,

$$\|(u - u_h, w - w_h)\|_\delta \leq\ C\ ({h_1}^\sigma\ |u|_{1+\sigma,\Omega^1}\ +\ h_2\ |w|_{2,\Lambda^2})$$

with the constant C independent of ϵ.
Since w is the opproximation of u^ϵ on Ω^2_ϵ, it is clear that if the error
$\|w(0) - u^\epsilon(0,.))\|_{1/2,\Sigma^\epsilon}$ is small, then the error $|u - u^\epsilon|_{1,\Omega^1}$ is also small. This is due to the fact that $e = u - u^\epsilon$ is the weak solution of the following elliptic equation

$$\begin{cases} -\Delta e &= 0 & \text{in } \Omega^1 \\ e &= 0 & \text{on } \Gamma_0 \\ \dfrac{\partial e}{\partial n} &= 0 & \text{on } \Gamma_1 \setminus \Sigma_\epsilon \\ e &= w(0) - u^\epsilon(0,.) & \text{on } \Sigma_\epsilon \end{cases}$$

More precisely, we have

$$|u - u^\epsilon|_{1,\Omega^1} \leq C\|w(0) - u^\epsilon(0,.))\|_{1/2,\Sigma_\epsilon}$$

where C is a constant independent of ϵ.
The following plots of the solution on the one dimensional subdomain Λ^2 illustrate this remark.

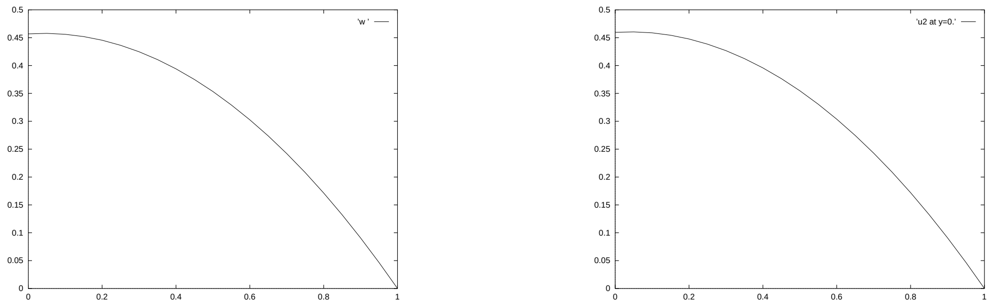


Figure 1: Plots of the solution w and $u_\epsilon(.,0)$for different values of ϵ, on subdomain Λ^2.

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