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46 Analysis of a defect correction method for computational aeroacoustics

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Introduction

Many problems of fundamental and practical importance are of multi-scale nature. As a typical example, the velocity field in turbulent transport problems fluctuates randomly and contains many scales depending on the Reynolds number of the flow. In another typical example, which is the main concern of this paper, sound waves are several orders of magnitude smaller than the pressure variations in the flow field that account for flow acceleration. These sound waves are manifested as pressure fluctuations which propagate at the speed of sound in the medium, not as a transported fluid quantity. As a result, numerical solutions of the Navier-Stokes equations which describe fluid motion do not resolve the small scale pressure fluctuations. The direct numerical simulation to include the above multiple scale problems is still an expensive tool for sound analysis [1].

In essence, there are at least three different scales embedded in the flow variables, namely (i) the mean flow, (ii) flow perturbations or aerodynamic sources of sound, and (iii) the acoustic perturbation. While flow perturbation or aerodynamic sources of sound may be easier to recover, it is not true for the acoustic perturbation because of its comparatively small magnitude. ¿From an engineering perspective, much of the larger scales behaviour may be resolved with the state-of-the-art CFD packages which implement various numerical methods of solving Navier-Stokes equations. This paper examines, in more detail, a defect correction method, first proposed in [2], for the recovery of smaller scales that have been left behind. The authors have demonstrated the accurate computation of (i) and (ii) in [3][4][5]. In the present study, a two-scale decomposition of flow variables is considered, i.e. the flow variable U is written as $\bar{u} + u$, where $\bar{u}$ denotes the mean flow and part of aerodynamic sources of sound and u denotes the remaining part of the aerodynamic sources of sound and the acoustic perturbation.

The concept of defect correction [6] has been used in various contexts since the early days. A typical example of defect correction is the computation of a refined approximation to the approximate solution $\bar{x}$ of the nonlinear equation $f(x) = 0$. Since $\bar{x}$ is an approximate solution, the defect may be computed as $-f(\bar{x})$. The idea of a defect correction method is to use a modified/derived version of the original problem such as the one defined by $\bar{f}(x) \equiv f'(\bar{x})(x - \bar{x}) + f(\bar{x}) = 0$. If one replaced $x - \bar{x}$ as v, then v is the correction computed by solving $f'(\bar{x})v = -f(\bar{x})$ and a refined approximation can be evaluated by using $x := \bar{x} + v$. More details in expanding the concept to discretised problems and multigrid methods can be found in [6]. Here, the authors would like to concentrate on using the defect correction concept at the level of the physical problem rather than the discretised problem. For a given mathematical problem and a given approximate solution, the residue or defect may be treated as a quantity to measure how well the problem has been solved. Such information may

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then be used in a modified/derived version of the original mathematical problem to provide an appropriate correction quantity. The correction can then be applied to correct the approximate solution in order to obtain a refined approximate solution to the original mathematical problem.

This paper follows the basic principle of the defect correction as discussed above and applies it to the recovery of the propagating acoustic perturbation. The method relies on the use of a lower order partial differential equation defined on the same computational domain where a residue exists such that the acoustic perturbation may be retrieved through a properly defined coarse mesh.

This paper is organised as follows. First, the derivation of a lower order partial differential equation resulting from the Navier-Stokes equations is given. Truncation errors due to the model reduction are examined. Second, accurate representation of residue on a coarse mesh is discussed. The coarse mesh is designed in such a way as to allow various frequencies of noise to be studied. Suitable interpolation operators are studied for the two different meshes. Third, numerical tests are performed for different mesh parameters to illustrate the concept. Finally, future work is discussed.

The defect correction method

The aim here is to solve the non-linear equation

$$\mathcal{L}\{U\}U := \mathcal{L}\{\bar{u}+u\}(\bar{u}+u) = 0 \tag{1}$$

where $\mathcal{L}\{U\}$ is a non-linear operator depending on U. For simplicity, U is considered to have two different scales of magnitudes as $\bar{u}+u$. Here $\bar{u}$ is the mean flow and u is the acoustic perturbation as described in Section 46. Note that $u \ll \bar{u}$ and that

$$\frac{1}{\delta t}\int_{t_0}^{t_0+\delta t} u dt \to 0$$

with δt much larger than any significant period of the perturbation velocity. The problem here is thus purely related to the scale of magnitude. In the case of sound generated by the motion of fluid, it is natural to imagine $\mathcal{L}$ as the Navier-Stokes operator. For a 2-D problem,

$$\bar{u} = \begin{bmatrix} \bar{\rho} \\ \bar{v}_1 \\ \bar{v}_2 \end{bmatrix} \quad u = \begin{bmatrix} \rho \\ v_1 \\ v_2 \end{bmatrix}$$

where ρ is the density of fluid and v_1 and v_2 are the velocity components along the two spatial axes. Using the summation notation of subscripts, the 2-D Navier-Stokes problem $\mathcal{L}\{u\}u = 0$ is written as

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_j)}{\partial x_j} = 0,$$

$$\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j} + \frac{1}{\rho}\frac{\partial P}{\partial x_i} - \frac{\mu}{\rho}\nabla^2 v_i = 0$$

where P is the pressure and $(\mu/\rho)\nabla^2 v_i$ is the viscous force along i-th axis.

Suppose (1) may be split and re-written as

$$\mathcal{L}\{\bar{u}+u\}(\bar{u}+u) \equiv \mathcal{L}\{\bar{u}\}\bar{u} + E\{\bar{u}\}u + K[\bar{u},u] \tag{2}$$

where $\mathcal{L}\{\bar{u}\}$ and $E\{\bar{u}\}$ are operators depending on the knowledge of $\bar{u}$ and $K[\bar{u},u]$ is a functional depending on the knowledge of both $\bar{u}$ and u. Following the concept of defect correction, $\bar{u}$ may be considered as an approximate solution to (1). Hence one can evaluate the residue of (1) as

$$R \equiv \mathcal{L}\{\bar{u}+u\}(\bar{u}+u) - \mathcal{L}\{\bar{u}\}\bar{u} = -\mathcal{L}\{\bar{u}\}\bar{u}$$

which may then be substituted into (2) to give

$$E\{\bar{u}\}u + K[\bar{u},u] = R \tag{3}$$

In many cases, $K[\bar{u},u]$ is small and can then be neglected. In those cases, the problem in (3) is a linear problem and may be solved more easily to obtain the acoustics fluctuation u. A non-linear iterative solver is required in order to obtain u for cases when $K[\bar{u},u]$ is not negligible. Finally, to obtain the approximate solution $\bar{u}$, one only needs to solve $\mathcal{L}\{\bar{u}\}\bar{u} = 0$. Expanding $\mathcal{L}\{\bar{u}+u\}(\bar{u}+u) = 0$ for $\mathcal{L}$ being the Navier-Stokes operator and re-arranging we obtain

$$\frac{\partial \rho}{\partial t} + \bar{v}_j \frac{\partial \rho}{\partial x_j} + \bar{\rho}\frac{\partial v_j}{\partial x_j} + [v_j \frac{\partial(\bar{\rho}+\rho)}{\partial x_j} + \rho \frac{\partial(\bar{v}_j + v_j)}{\partial x_j}] = -[\frac{\partial \bar{\rho}}{\partial t} + \bar{v}_j \frac{\partial \bar{\rho}}{\partial x_j} + \bar{\rho}\frac{\partial \bar{v}_j}{\partial x_j}]$$

and

$$\frac{\partial v_i}{\partial t} + \bar{v}_j \frac{\partial v_i}{\partial x_j} + \frac{1}{\bar{\rho}}\frac{\partial P}{\partial x_i} - \frac{\mu}{\bar{\rho}}\nabla^2 v_i \tag{4}$$

$$+[\frac{\rho}{\bar{\rho}}\frac{\partial(\bar{v}_i+v_i)}{\partial t} + (v_j + \frac{\rho}{\bar{\rho}}(\bar{v}_j+v_j))\frac{\partial(\bar{v}_i+v_i)}{\partial x_j}] = -[\frac{\partial \bar{v}_i}{\partial t} + \bar{v}_j\frac{\partial \bar{v}_i}{\partial x_j} + \frac{1}{\bar{\rho}}\frac{\partial \bar{P}}{\partial x_i} - \frac{\mu\nabla^2\bar{v}_i}{\bar{\rho}}]$$

It can be seen that (4) may be written in the form of (3) where

$$E\{\bar{u}\}u = \begin{bmatrix} \frac{\partial \rho}{\partial t} + \bar{v}_j \frac{\partial \rho}{\partial x_j} + \bar{\rho}\frac{\partial v_j}{\partial x_j} \\ \frac{\partial v_i}{\partial t} + \bar{v}_j \frac{\partial v_i}{\partial x_j} + \frac{1}{\bar{\rho}}\frac{\partial P}{\partial x_i} - \frac{\mu}{\bar{\rho}}\nabla^2 v_i \end{bmatrix} \tag{5}$$

$$K[\bar{u},u] = \begin{bmatrix} v_j \frac{\partial(\bar{\rho}+\rho)}{\partial x_j} + \rho\frac{\partial(\bar{v}_j+v_j)}{\partial x_j} \\ \frac{\rho}{\bar{\rho}}\frac{\partial(\bar{v}_i+v_i)}{\partial t} + (v_j + \frac{\rho}{\bar{\rho}}(\bar{v}_j+v_j))\frac{\partial(\bar{v}_i+v_i)}{\partial x_j} \end{bmatrix} \tag{6}$$

$$R = \begin{bmatrix} -[\frac{\partial \bar{\rho}}{\partial t} + \bar{v}_j \frac{\partial \bar{\rho}}{\partial x_j} + \bar{\rho}\frac{\partial \bar{v}_j}{\partial x_j}] \\ -[\frac{\partial \bar{v}_i}{\partial t} + \bar{v}_j \frac{\partial \bar{v}_i}{\partial x_j} + \frac{1}{\bar{\rho}}\frac{\partial \bar{P}}{\partial x_i} - \frac{\mu}{\bar{\rho}}\nabla^2\bar{v}_i] \end{bmatrix} \equiv -\mathcal{L}\{\bar{u}\}\bar{u} \tag{7}$$

¿From the knowledge of physics of fluids, the acoustic perturbations ρ and v_j are of very small magnitude (this is not true for their derivatives), therefore, K may be considered negligible due to the reason that any feedback from the propagating waves to the flow may be completely ignored, except in some cases of acoustic resonance, which we are not concerned with here.

Hence the equation $E\{\bar{u}\}u = R$, with E given by (5), which is known as the linearised Euler equation, can be solved in an easier way. The numerics and the techniques involved here are often referred to as Computational AeroAcoustics (CAA) methods.

The remaining question is to obtain the approximate solution $\bar{u}$ to the original problem (2). It is well known that CFD analysis packages provide excellent methods for the solution of $\mathcal{L}\{\bar{u}\}\bar{u} = 0$. Therefore one requires to use a Reynolds averaged Navier-Stokes package supplemented with turbulence models such as [7, 8] to provide a solution of $\bar{u}$. One requires $\bar{u}$ to be as accurate as possible to capture all the physics of interest, such as flow turbulence and the presence of vortices.

The use of a CFD analysis package effectively solves $\mathcal{L}\{\bar{u}\}\bar{u} = 0$ instead of $\mathcal{L}\{\bar{u}+u\}(\bar{u}+u) = 0$. Following the concept of truncation error in a finite difference method, the truncation error due to the removal of the perturbation part of the flow variable may be defined by

$$\tau = \mathcal{L}\{\bar{u}+u\}(\bar{u}+u) - \mathcal{L}\{\bar{u}\}(\bar{u}+u) \tag{8}$$

Using the relation $\mathcal{L}\{\bar{u}\}(\bar{u}+u) = \mathcal{L}\{\bar{u}\}\bar{u} + E\{\bar{u}\}u$, the truncation error in the present context is thus given by

$$\tau = K[\bar{u}, u] \tag{9}$$

Note that this truncation error is not related to the discretisation of continuous model.

A two-level multigrid method

In order to simulate accurately the approximate solution, $\bar{u}$, to the original problem, $\mathcal{L}U = 0$, the QUICK differencing scheme [9] is used which produces sufficiently accurate results of $\bar{u}$ for the purpose of evaluating the residue as defined in (7). A sufficiently fine mesh has to be used in order to preserve vorticity motion. However, much coarser mesh may be used for the numerical solutions of linearised Euler equations [3, 4, 5]. It certainly has to obey the Courant limit and also to account for the fact that the acoustic wavelength may be larger than a typical flow feature which needs to be resolved, e.g. a travelling vortex [10]. The present defect correction method requires to calculate the residue on the CFD mesh and to transfer these residuals onto the acoustic mesh. Physically, the residue is effectively the sound source that would have disappeared without the proper retrieval technique as discussed in this paper. Let h denote the mesh to be used in the Reynolds averaged Navier-Stokes solver. Instead of evaluating $\bar{u}$, one would solve the discretised approximation $\mathcal{L}_h\bar{u}_h = 0$ to obtain $\bar{u}_h$. The residue on the fine mesh h can be computed as $\mathcal{L}\bar{u}_h$ by means of a higher order approximation [5]. Let H denote the mesh for the linearised Euler equations solver. Again instead of evaluating u, one would solve the discretised approximation $E_H\{\bar{u}_H\}u_H = R_H$ to obtain u_H. Here R_H is the projection of R onto the mesh H. Let $I_{\{h,H\}}$ be a restriction operator to restrict the residue computed on the fine mesh h to the coarser mesh H. The restricted residue can then be used in the numerical solutions of linearised Euler equations. Therefore the two-level numerical scheme is (for non-resonance problems) :

Solve $\mathcal{L}_h\bar{u}_h = 0$
$R_H := -I_{\{h,H\}}\mathcal{L}\bar{u}_h$
$\bar{u}_H := I_{\{h,H\}}\bar{u}_h$
Solve $E_H\{\bar{u}_H\}u_H = R_H$
$U_H := \bar{u}_H + u_H$

Here U_H denotes the discretised approximation of the resultant solution on mesh H. Note that R_H cannot be computed as $\mathcal{L}I_{\{h,H\}}\bar{u}_h$ because $\mathcal{L}$ is a non-linear operator.
In the actual implementation, a pressure-density relation which also defines the speed of sound c in air is used:

$$\frac{\partial P}{\partial \rho} = c^2 \approx 1.4\frac{\bar{P}}{\bar{\rho}} \tag{10}$$

and the first component of the linearised Euler equations in (5) becomes

$$\frac{\partial P}{\partial t} + \bar{v}_j\frac{\partial P}{\partial x_j} + \bar{\rho}c^2\frac{\partial v_j}{\partial x_j} = -c^2[\frac{\partial \bar{\rho}}{\partial t} + \bar{v}_j\frac{\partial \bar{\rho}}{\partial x_j} + \bar{\rho}\frac{\partial \bar{v}_j}{\partial x_j}] \tag{11}$$

The purpose of this substitution is to make sure that the new fluctuations P and v_i do not contain a hydrodynamic component, and hence can be resolved on regular Cartesian meshes [4] which is essential for the accurate representation of the acoustic waves or the fluctuation quantity u. On the other hand, an unstructured mesh may be used to obtain $\bar{u}_h$. The two different meshes overlap one another on the computational domain. The computational domain for the linearised Euler equations is not necessarily the same as the one for the CFD solutions. It must be large enough to contain at least the longest wavelength of a particular problem under consideration or a number of wavelengths where propagation is of interest. The numerical example as shown in Section 46 does not contain any complicating solid objects, the restriction operator $I_{\{h,H\}}$ may then be chosen as an arithmetic averaging process [10].

Numerical experiments with various grid parameters

The propagation of the following one-dimensional pulse is considered: an initial pressure distribution with a peak in the origin generates two opposite acoustic waves in both directions. The exact solution of this problem (12) can be verified by substitution in the linearised Euler equations.

$$\begin{aligned} P &= f(x-ct) + f(x+ct) \\ \bar{\rho}cv_1 &= f(x-ct) - f(x+ct) \\ f(x) &= \begin{cases} \frac{A}{2}(1+\cos 2\pi\frac{x}{\lambda}), |x| < \frac{\lambda}{2} \\ 0, |x| \geq \frac{\lambda}{2} \end{cases} \end{aligned} \tag{12}$$

Here A is the amplitude and λ is the wavelength of the two sound waves that start from the origin ($x = 0$) at $t = 0$. The example was reported in [2]. This paper provides a detailed numerical study on various aspects of the grid parameters being used in the two-level method. The CFD domain is of 12 wavelengths and the CAA domain is of 14 wavelengths.
The effects of the following parameters on the solution accuracy are studied. These parameters are (a) the ratio H:h, (b) number of points per wavelength, and (c) the restriction operator for residual transfer from fine grid to coarse grid. In all cases, the norm $||P_H - P||_\infty$ is compared. Here P_H is the approximation obtained on the coarse mesh (CAA) after correction and P is the exact solution of the pressure variable.
Let δt_h and δt_H be the step lengths in the temporal axis for the CFD mesh and the CAA mesh respectively. Figure 1 shows the effect on the accuracy for Case (a). Here δt_h and δt_H are

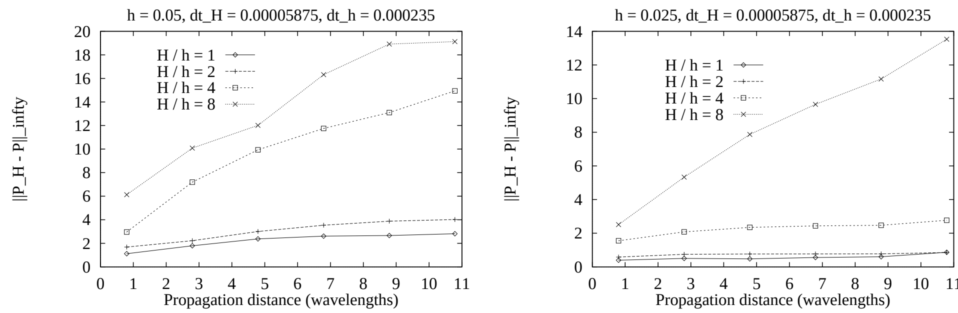

Figure 1: The effect of mesh ratio $H{:}h$ on the accuracy.

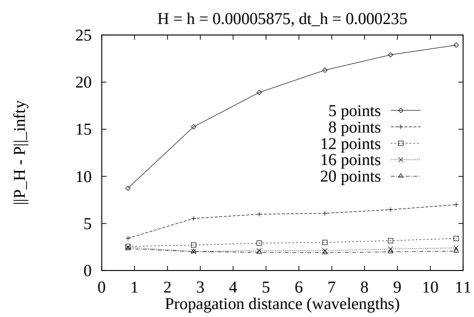

Figure 2: The effect of number of grid points per wavelength on the accuracy.

chosen to be 0.000235 and 0.00005875 respectively. Two different mesh sizes for the CFD are chosen and they are 0.05 and 0.025. It can be seen that when h is not fine enough, say h = 0.05, to resolve some of the physics, it is still possible to use the mesh $H = 2h$ or $H = h$ to recover the small scale signal. If a finer mesh was used, say $h = 0.025$, it is possible to use $H \leq 4h$. This property essentially links with the Courant number of the coarse mesh for CAA [5], i.e. H, and is also confirmed in the test performed for Case (b).

Figure 2 shows the effect on the accuracy for Case (b). The most accurate solution may be achieved with more than 12 grid points per wavelength, e.g. 16 or more grid points. This confirms the theoretical study based on Courant limits as discussed in [5]. For number of grid points per wavelength less than 12, the accuracy deteriorates very fast.

Figure 3 shows the effect on the accuracy for Case (c). The restriction operators being used in this test to transfer the function g_h onto the coarse mesh H includes

$$\text{3 point formula: } I_{\{h,2h\}} g_h = \frac{1}{4}(g_{i-1} + 2g_i + g_{i+1})$$

$$\text{5 point formula: } I_{\{h,4h\}} g_h = \frac{1}{12}(g_{i-2} + 2g_{i-2} + 6g_i + 2g_{i+1} + g_{i+2})$$

$$\text{7 point formula: } I_{\{h,6h\}} g_h = \frac{1}{16}(g_{i-3} + 2g_{i-2} + 3g_{i-1} + 4g_i + 3g_{i+1} + 2g_{i+2} + g_{i+3})$$

$$\text{9 point formula: } I_{\{h,8h\}} g_h = \frac{1}{48}(g_{i-4} + 2g_{i-3} + 6g_{i-2} + 8g_{i-1} + 14g_i + 8g_{i+1} + 6g_{i+2} + 2g_{i+3} + g_{i+4})$$

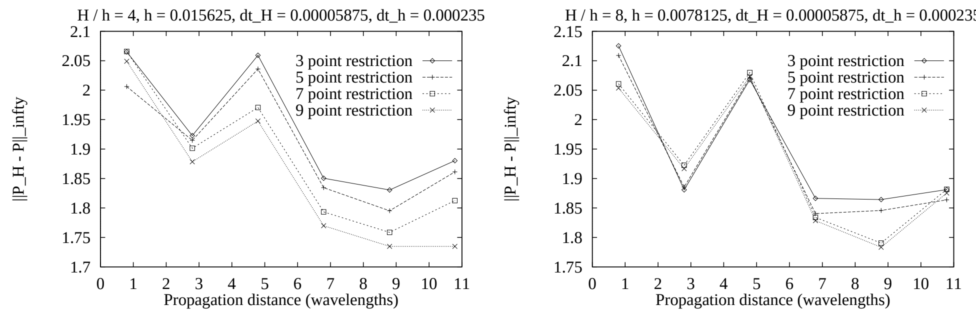

Figure 3: The effect of restriction operators on the accuracy.

For very fine CFD mesh, one can retrieve the small scale signal even on a relatively coarse mesh. In the present study, with $h = 0.0078125$ one can use $H \leq 8h$ while still maintaining the accuracy. The accuracy exhibited by using the coarse mesh $H = 8h = 0.0625$ is compatible with the result for Case (a) as depicted in Figure 1.

Conclusions

This paper provides a numerical method for the retrieval of sound signals using the defect correction method. A detailed numerical experiments to examine various grid parameters are provided. Truncation error of solving $\mathcal{L}\{\bar{u}\}\bar{u} = 0$ instead of $\mathcal{L}\{\bar{u} + u\}\bar{u} + u = 0$ is derived. The authors are currently applying the present method to sound propagation in vortex-vortex interactions.

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Figure 4: Hello

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