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21 Analysis of Two-Level Overlapping Additive Schwarz Preconditioners for a Discontinuous Galerkin Method

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Introduction

The Schwarz method refers to a general methodology, based on the idea of divide-and-conquer, for solving the systems of linear algebraic equations resulting from numerical discretizations of partial differential equations. In the past fifteen years extensive research has been done on the method to solve different types of algebraic systems which arise from various discretizations of partial differential equations such as finite difference/element/volume methods, spectral methods and mortar finite element methods (cf. [SBG96, Xu92] and references therein). On the other hand, very few results on the Schwarz method have been known in the literature for discontinuous Galerkin methods (cf. [FK01a, LT00, RVW96]). Discontinuous Galerkin methods use piecewise, totally discontinuous polynomial trial and test function spaces, that is, no continuity constraints are explicitly imposed on the functions across the element interfaces. As a consequence, weak formulations must include jump terms across interfaces and typically penalty terms are (artificially) added to control the jump terms (cf. [Arn82, DD76, Whe78]).

Discontinuous Galerkin methods have several advantages over other types of finite element methods. For example, the trial and test spaces are very easy to construct; they can naturally handle inhomogeneous boundary conditions and curved boundaries; they also allow the use of highly nonuniform and unstructured meshes. In addition, the fact that the mass matrices are block diagonal is an attractive feature in the context of time-dependent problems, especially if explicit time discretizations are used. On the other hand, discontinuous Galerkin methods would seem to be at a disadvantage in view of a relatively larger number of degrees of freedom per element. Therefore, to offset this disadvantage, effective remedies must be found at the level of solution of the systems of algebraic equations.

The objective of this paper is to develop some two-level *overlapping* additive Schwarz preconditioners for a discontinuous Galerkin method for solving second order elliptic problems. In Section 2, the discontinuous Galerkin and some known facts about the method, as well as a trace inequality and a generalized Poincaré inequality for discontinuous, piecewise H^1 functions are recalled. In Section 3, some two-level overlapping additive Schwarz preconditioners are proposed and analyzed for the discontinuous Galerkin method. The main result is to show that the condition numbers of the preconditioned systems are of the order $O(\frac{H}{\delta})$, where H and δ stand for the coarse mesh size and the size of overlaps between subdomains.

This paper is the second in a sequel devoted to developing Schwarz methods for discontinuous Galerkin methods. [FK01a] contains non-overlapping Schwarz methods for discontinuous Galerkin methods. The condition number estimates of the order $O(\frac{H}{h})$ are established and numerical experiments are presented. In [FK01b], Schwarz methods are developed for the discontinuous Galerkin method of Baker [Bak77] for the biharmonic problems.

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Preliminaries

Let $\Omega \subset \mathbf{R}^d,\ d = 2, 3$ be a bounded domain. For the sake of simplicity, we restrict ourselves to the following model problem:

$$
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega, &(1)\\
u &= g \quad \text{on } \partial\Omega. &(2)
\end{aligned}
$$

We remark that although we only consider the above model problem, extension of our construction and analysis of this paper to more general second order elliptic problems can be easily carried out.

The discontinuous Galerkin method to be considered in this paper for discretizing problem (1)–(2) is the one proposed in [Bak77, BJK90]. In this paper, we shall adopt the same notation as that of [BJK90].

Let $\mathcal{T}_h = \{K_i :\ i = 1, 2, \cdots, m_h\}$ be a family of star-like partitions (triangulations) of the domain Ω parametrized by $0 < h \le 1$. Note that $\mathcal{T}_h$ does not have to be geometrically conforming. We define

$\partial K_i =$ the boundary of K_i, $\quad \partial K_{ij} = \partial K_i \cap \partial K_j$, $\quad \partial K_i^e = \partial K_i \cap \partial\Omega$,
$\mathcal{N}_i = \{\ell;\ \text{meas}(\partial K_{i\ell}) > 0\}$, $\quad h_i = \text{diam}(K_i)$, $\quad h_{ij} = \text{diam}(\partial K_{ij})$,
$\tau_{ij} = 1,\ \text{if } i > j; \tau_{ij} = 0,\ \text{if } i \le j.$

We shall refer to $\mathcal{T}_h$ as the "fine" mesh and assume that it satisfies the following assumptions:

(i) The elements of $\mathcal{T}_h$ satisfy the minimal angle condition

(ii) $\mathcal{T}_h$ is locally quasi-uniform, that is if K_j and K_ℓ are adjacent and $\text{meas}(\partial K_{j\ell}) > 0)$, then $h_j \approx h_\ell$.

Now define the "energy space" E by $E = H^2(K_1) \times H^2(K_2) \times \cdots \times H^2(K_{m_h})$ and the bilinear form $a_h(\cdot,\cdot)$ on $E \times E$ as follows: For $u, v \in E$,

$$
\begin{aligned}
a_h(u,v) = \sum_{K_j \subset \Omega} \Bigg\{ &(\nabla u^{(j)}, \nabla v^{(j)})_{K_j} - \sum_{\ell \in \mathcal{N}_j} \tau_{j\ell} \Bigg[\left\langle \frac{\partial u^{(j)}}{\partial n}, v^{(j)} - v^{(\ell)} \right\rangle_{\partial K_{j\ell}} \\
&+ \left\langle \frac{\partial v^{(j)}}{\partial n}, u^{(j)} - u^{(\ell)} \right\rangle_{\partial K_{j\ell}} - \gamma h_{j\ell}^{-1} \langle u^{(j)} - u^{(\ell)}, v^{(j)} - v^{(\ell)} \rangle_{\partial K_{j\ell}} \Bigg] \\
&- \left\langle \frac{\partial u^{(j)}}{\partial n}, v^{(j)} \right\rangle_{\partial K_j^e} - \left\langle \frac{\partial v^{(j)}}{\partial n}, u^{(j)} \right\rangle_{\partial K_j^e} + \gamma h_j^{-1} \langle u^{(j)}, v^{(j)} \rangle_{\partial K_j^e} \Bigg\} \qquad (3)
\end{aligned}
$$

Here $u^{(j)}$ denotes the restriction of u to the element K_j and $(\cdot,\cdot)_{K_j}$ the L^2 integral over K_j; $\langle u^{(j)}, u^{(\ell)} \rangle_{\partial K_{j\ell}}$ is the L^2 integral over the interface $\partial K_{j\ell}$ of the traces of $u^{(j)}$ and $u^{(\ell)}$. The terms including γ are the so-called penalty terms.

The bilinear form $a_h(\cdot,\cdot)$ induces the following norm on the space E

$$\|v\|_{1,h,\Omega} = \left(\sum_{K_j \subset \Omega} \left\{ ||\nabla v^{(j)}||^2_{0,K_j} + \sum_{\ell \in \mathcal{N}_j} \tau_{j\ell} \left[h_{j\ell} \left| \frac{\partial v^{(j)}}{\partial n} \right|^2_{0,\partial K_{j\ell}} + h_{j\ell}^{-1} |v^{(j)} - v^{(\ell)}|^2_{0,\partial K_{j\ell}} \right] + h_j \left| \frac{\partial v^{(j)}}{\partial n} \right|^2_{0,\partial K_j^e} + h_j^{-1} |v^{(j)}|^2_{0,\partial K_j^e} \right\} \right)^{\frac{1}{2}}. \quad (4)$$

The weak formulation of (1)–(2) is defined as seeking $u \in E \cap H^1(\Omega) \cap H^2_{\text{loc}}(\Omega)$ such that

$$a_h(u, v) = F(v), \qquad \forall v \in E \cap H^1(\Omega) \cap H^2_{\text{loc}}(\Omega), \quad (5)$$

where

$$F(v) = (f, v) - \sum_{\partial K_j^e \subset \partial\Omega} \left\langle g, \frac{\partial v^{(j)}}{\partial n} - \gamma h_j^{-1} v^{(j)} \right\rangle_{\partial K_j^e}.$$

For any integer $r \geq 2$, let $P_{r-1}(D)$ denote the set of all polynomials of degree less than or equal to $r-1$ on D. Define the finite element space V^h as

$$V^h = P_{r-1}(K_1) \times P_{r-1}(K_2) \times \cdots \times P_{r-1}(K_{m_h}).$$

Clearly, $V^h \subset E \subset L^2(\Omega)$. But $V^h \not\subset H^1(\Omega)$. The coercivity and continuity of $a_h(\cdot,\cdot)$ with respect to $\|\cdot\|_{1,h,\Omega}$ norm is summarized in the following lemma.

Lemma 1 (cf. [BJK90]) *There exists $\gamma_0 > 0$, which only depends on r, such that for $\gamma \geq \gamma_0$*

$$|a_h(u, v)| \leq (1+\gamma)\|u\|_{1,h,\Omega}\|v\|_{1,h,\Omega}, \qquad \forall u, v \in E. \quad (6)$$

$$a_h(v, v) \geq C\|v\|^2_{1,h,\Omega}, \qquad \forall v \in V^h. \quad (7)$$

The discontinuous Galerkin method based on the weak formulation (6) is defined as follows: find $u_h \in V^h$ such that

$$a_h(u_h, v_h) = F(v_h), \qquad \forall v_h \in V^h. \quad (8)$$

We refer to [BJK90] for a detailed exposition on this particular method.

We conclude this section by introducing two inequalities, a trace inequality and a generalized Poincaré inequality, for totally discontinuous piecewise H^1 functions generalizing two well-known inequalities for H^1 functions. These inequalities play a key role for the convergence analysis in the next section.

Let D be a bounded, simply connected star-like domain with diameter H in $\mathbf{R}^d$, $d = 2, 3$ (cf. [BJK90, FK01a]), and $\mathcal{T}_D$ be a family of partitions (triangulations) of D parameterized by $0 < h \leq H$. Let V_D be the space of all piecewise, totally discontinuous H^1 functions over $\mathcal{T}_D$. For a given number $h \leq \delta << H$, let D_δ denote the boundary layer of D with the width δ. That is, $D_\delta = \{x \in D;\ \text{dist}(x, \partial D) \leq \delta\}$. For simplicity, we assume that the boundary ∂D_δ of D_δ is aligned with $\mathcal{T}_D$.

Lemma 2 (cf. [FK01a]) *For any $u \in V_D$, there holds the following trace inequality*

$$|u|_{0,\partial D}^2 \le C[H^{-1}\|u\|_{0,D}^2 + H|u|_{1,h,D}^2], \tag{9}$$

where

$$|u|_{1,h,D}^2 = \sum_{K\in\mathcal{T}_D} \|\nabla u\|_{0,D}^2 + \sum_{\substack{\partial K_{ij}\subset\Omega\\ i>j}} h^{-1}|u^{(i)} - u^{(j)}|_{\partial K_{ij}}^2. \tag{10}$$

Lemma 3 (cf. [FK01a]) *For any $u \in V_D$, the following generalized Poincaré inequality holds.*

$$\|u\|_{0,D_\delta}^2 \le C[\delta H^{-1}\|u\|_{0,D}^2 + \delta(\delta + H)|u|_{1,h,D}^2]. \tag{11}$$

The overlapping Schwarz method

Formulation of the additive Schwarz preconditioners

Let $\mathcal{T}_H$ denote a coarse partition (triangulation) of Ω with the mesh size $H > 0$ and V^H denote the discontinuous Galerkin finite element space of order $r - 1$ associated with the mesh $\mathcal{T}_H$. Suppose that $\mathcal{T}_h$ is obtained as a refinement of $\mathcal{T}_H$ and its members are star-like. Let $\Omega = \cup_{j=1}^J \Omega_j$ be an overlapping decomposition of Ω, where each Ω_j satisfying $\text{diam}(\Omega_j) \approx H$ is a star-like open subdomain of Ω and is aligned with $\mathcal{T}_h$. Moreover, we assume there exist nonnegative C^∞-functions $\{\theta_j\}_{j=1}^J$ such that

$$\sum_{j=1}^J \theta_j = 1 \quad \text{in } \overline{\Omega}, \quad \theta_j = 0 \quad \text{in } \Omega\setminus\Omega_j, \quad \|\nabla\theta_j\|_{L^\infty} \le \frac{1}{\delta}. \tag{12}$$

We also assume that there exist two positive constants C_0 and C_1 such that $C_0 h \le \delta \le C_1 H$. Let $N(x)$ denote the number of subdomains which contain x. We assume that $N_c \equiv \max_{x\in\Omega} N(x)$ is a constant which is independent of h, H, J and δ. Recall that the parameter δ measures the amount of overlaps among the subdomains $\{\Omega_j\}$. For the construction of Ω_j, we refer to [SBG96] and the references therein.

Introduce the notation

$$\begin{aligned} &\Gamma_j = \partial\Omega_j \cap \partial\Omega, && \Gamma_{jk} = \partial\Omega_j \cap \Omega_k, \\ &\mathcal{N}_j^i = \{\ell \in \mathcal{N}_j;\ \partial K_{j\ell} \subset \Omega_i\}, && \overline{\mathcal{N}}_j^i = \{\ell \in \mathcal{N}_j;\ \partial K_{j\ell} \subset \Gamma_{ik} \text{for some } \Gamma_{ik}\}, \\ &\Omega_j^I = \{x \in \Omega_j;\ x \notin \Omega_k \text{ for all } k \neq j\}, && \Omega_j^\delta = \Omega_j \setminus \Omega_j^I. \end{aligned}$$

It is well-known (cf. [SBG96, Xu92]) that the first step towards constructing the additive Schwarz preconditioners is to have a valid subspace decomposition of the finite element space V^h. For the discontinuous Galerkin method considered in this paper, since $V^h \subset L^2(\Omega)$ and no continuity constrain is imposed for the functions in V^h, it is easy to construct such a space decomposition.

We define the subspace $\{V_j^h\}_{j=1}^J$ associated with the subdomain $\{\Omega_j\}_{j=1}^J$ by

$$V_j^h = \{v_h \in V^h;\ v_h = 0 \text{ in } \Omega\setminus\overline{\Omega}_j\}, \quad j = 1, 2, \cdots, J.$$

In addition to $V_1^h, \cdots, V_J^h$, we now introduce a coarse subspace V_0^h corresponding to $\mathcal{T}_H$. Let the integer $r_{_H}$ be chosen satisfying $2 \le r_{_H} \le r$. Let

$$V_0^h = \prod_{D \in \mathcal{T}_H} P_{r_{_H}-1}(D). \tag{13}$$

It is easy to see that V_0^h is a subspace of V^h. Also, our (theoretical) estimates are valid independent of the choice of $r_{_H}$. Clearly, $V_0^h = V^H$ when $r_{_H} = r$.

It is easy to check that the following space decomposition holds.

$$V^h = V_0^h + V_1^h + V_2^h + \cdots + V_J^h. \tag{14}$$

Having obtained the above space decomposition, the second step requires the construction of a subdomain bilinear form (or a subdomain solver) on each subdomain. To this end, we define $a_i(\cdot,\cdot)$ on $V_i^h \times V_i^h$ to be the restriction of $a_h(\cdot,\cdot)$ on Ω_j for $i = 1, 2, \cdots, J$, and $a_0(\cdot,\cdot) = a_h(\cdot,\cdot)$. Notice that, $a_0(\cdot,\cdot)$ differs from $a_H(\cdot,\cdot)$ only in the choice of the penalty parameter γ on $V_0^h \times V_0^h$.

Now we are ready to define the additive operator

$$T = T_0 + T_1 + \cdots + T_J, \tag{15}$$

where T_j is a projection operator from V^h to V_j^h which is defined by

$$a_j(T_j u, v) = a_h(u, v) \qquad \forall v \in V_j^h,\ j = 0, 1, 2, \cdots, J. \tag{16}$$

The additive Schwarz method is defined by replacing the discrete problem (9) by the equation (cf. [SBG96])

$$Tu = g, \qquad\qquad g = \sum_{j=0}^{J} g_j, \tag{17}$$

where $g_j = T_j u$ is defined as the solution of

$$a_j(g_j, v) = F(v) \qquad \forall v \in V_j^h,\ j = 0, 1, 2, \cdots, J. \tag{18}$$

Condition number estimate for the additive Schwarz method

To estimate the condition number of T, we will use the abstract convergence framework of Schwarz methods given in [SBG96]. To this end, we need some preliminary lemmas, including the decomposition lemma (see Lemma 7).

Let $W_0^H \subset H_0^1(\Omega)$ be the standard P_1 conforming finite element space associated with the coarse mesh $\mathcal{T}_H$. Trivially, $W_0^H \subset V_0^h$. We recall the following approximation property of the finite element space W_0^H.

Lemma 4 (cf. [Cia78]) *For any $\psi \in H^2(\Omega) \cap H^1_0(\Omega)$, the following estimate holds*

$$\inf_{v \in W_0^H} a_h(\psi - v, \psi - v)^{\frac{1}{2}} = \inf_{v \in W_0^H} \|\nabla(\psi - v)\|_{0,\Omega} \le CH\|\psi\|_{2,\Omega}. \tag{19}$$

Next, for any function $u \in V^h$, we define $P_H u$ to be the projection of u into W_0^H with respect to $a_h(\cdot,\cdot)$, that is,

$$a_h(P_H u, v) = a_h(u, v) \qquad \forall v \in W_0^H. \tag{20}$$

The operator P_H satisfies the following stability and approximation properties.

Lemma 5 *There exists a positive constant C, which is independent of h, J, δ and H, such that*

$$a_h(P_H u, P_H u) \le a_h(u,u). \tag{21}$$

$$\|u - P_H u\|_{0,\Omega} \le CHa_h(u,u)^{\frac{1}{2}}. \tag{22}$$

To save space, we omit the proof and refer to [FK01a] for a proof of similar type.

For each $K \in \mathcal{T}_h$, let Π_K denote the usual interpolation operator to the polynomial space $P_{r-1}(K)$ as defined in the conforming finite element methods. Define the interpolation operator $\Pi_h : \prod_{j=1}^{m_h} C^0(\overline{K_j}) \longrightarrow V^h$ by

$$\Pi_h \phi = \Pi_K \phi \quad \text{in } K, \quad \forall K \in \mathcal{T}_h,\ \forall \phi \in \prod_{j=1}^{m_h} C^0(\overline{K_j}) \tag{23}$$

For any $u \in V^h$, we introduce the following decomposition of u

$$u = u_0 + u_1 + \cdots + u_J, \quad u_j \in V_j^h, \tag{24}$$

where

$$u_0 = P_H u, \quad u_j = \Pi_h[\theta_j(u - P_H u)],\ j = 1, 2, \cdots, J. \tag{25}$$

We emphasize that the operator P_H is only needed in the analysis and does not contribute to the construction of the computational method.

Lemma 6 *For any $u \in V^h$, let $u_j \in V_j^h$ be defined as above. Then there is a positive constant C which is independent of h, J, δ and H such that*

$$a_i(u_i, u_i) \le C\left(\|u - P_H u\|^2_{1,h,\Omega_i} + \frac{1}{\delta^2}\|u - P_H u\|^2_{0,\Omega_i^\delta}\right),\ i = 1, 2, \cdots, J, \tag{26}$$

Proof: Let $w = u - u_0$. Since $\theta_i = 0$ in $\Omega \setminus \Omega_i$, $u_i = 0$ on every $\partial K_{j\ell} \subset \partial\Omega_i \setminus \partial\Omega$. By the definition of $a_i(\cdot,\cdot)$ and the Schwarz inequality we have

$$a_i(u_i, u_i) \le C \sum_{K_j \subset \Omega_i} \left\{ \|\nabla u_i^{(j)}\|^2_{0,K_j} + \sum_{\ell \in \mathcal{N}_j^i} \tau_{j\ell} \left[h_{j\ell} \left| \frac{\partial u_i^{(j)}}{\partial n} \right|^2_{0,\partial K_{j\ell}} \right.\right.$$
$$\left.\left. + h_{j\ell}^{-1} |u_i^{(j)} - u_i^{(\ell)}|^2_{0,\partial K_{j\ell}} \right] + h_j \left| \frac{\partial u_i^{(j)}}{\partial n} \right|^2_{0,\partial K_j^e} + h_j^{-1} |u_i^{(j)}|^2_{0,\partial K_j^e} \right\}. \tag{27}$$

Let $\overline{\theta}_{ij}$ be the average of θ_i over an element $K_j \subset \Omega_i$. It is known that (cf. [SBG96])

$$\|\theta_i - \overline{\theta}_{ij}\|_{L^\infty(K_j)} \leq \begin{cases} Ch_j\delta^{-1} & \forall K_j \in \Omega_i^\delta, \\ 0 & \forall K_j \in \Omega_i^I. \end{cases} \tag{28}$$

For each term on the right hand side of (27) we have the following estimate.

$$\begin{aligned} \|\nabla u_i^{(j)}\|_{0,K_j}^2 &\leq \|\nabla w^{(j)}]\|_{0,K_j}^2 + Ch_j^{-2}\|\Pi_{K_j}[(\theta_i - \overline{\theta}_{ij})w^{(j)}]\|_{0,K_j}^2 \\ &\leq \|\nabla w^{(j)}]\|_{0,K_j}^2 + \begin{cases} C\delta^{-2}\|w^{(j)}\|_{0,K_j}^2 & \text{if } K_j \in \Omega_i^\delta, \\ 0 & \text{if } K_j \in \Omega_i^I. \end{cases} \end{aligned} \tag{29}$$

$$|u_i^{(j)} - u_i^{(\ell)}|_{0,\partial K_{j\ell}}^2 = |\Pi_{K_{j\ell}}[\theta_i(w^{(j)} - w^{(\ell)})]|_{0,\partial K_{j\ell}}^2 \leq C|w^{(j)} - w^{(\ell)}|_{0,\partial K_{j\ell}}^2. \tag{30}$$

$$\begin{aligned} \left|\frac{\partial u_i^{(j)}}{\partial n}\right|_{0,\partial K_{j\ell}}^2 &\leq \left|\frac{\partial w_i^{(j)}}{\partial n}\right|_{0,\partial K_{j\ell}}^2 + Ch_j^{-1}\|\nabla\Pi_{K_j}[(\theta_i - \overline{\theta}_{ij})w^{(j)}]\|_{0,K_j}^2 \\ &\leq \left|\frac{\partial w_i^{(j)}}{\partial n}\right|_{0,\partial K_{j\ell}}^2 + \begin{cases} Ch_j^{-1}\delta^{-2}\|w^{(j)}\|_{0,K_j}^2 & \text{if } K_j \in \Omega_i^\delta, \\ 0 & \text{if } K_j \in \Omega_i^I. \end{cases} \end{aligned} \tag{31}$$

$$\left|\frac{\partial u_i^{(j)}}{\partial n}\right|_{0,\partial K_j^e}^2 \leq C\left(\left|\frac{\partial w_i^{(j)}}{\partial n}\right|_{0,\partial K_j^e}^2 + h_j^{-1}\delta^{-2}\|w^{(j)}\|_{0,K_j}^2\right). \tag{32}$$

$$\left|u_i^{(j)}\right|_{0,\partial K_j^e}^2 = \left|\Pi_{K_j^e}[\theta_i w^{(j)}]\right|_{0,\partial K_j^e}^2 \leq C\left|w^{(j)}\right|_{0,\partial K_j^e}^2 \tag{33}$$

Finally, the estimate (26) follows from (27), (29)–(33) and the definition of $\|w\|_{1,h,\Omega_i}$.
The following lemma follows directly from applying Lemma 6 and Lemma 3 on each Ω_j.

Lemma 7 *For any $u \in V^h$, let $u_j \in V_j^h$ be as in (25). There is a positive constant C which is independent of h, J, δ and H such that*

$$\sum_{j=0}^{J} a_i(u_j, u_j) \leq C\frac{H}{\delta}a_h(u, u). \tag{34}$$

It is trivial to show the next lemma (cf. [FK01a]).

Lemma 8 *There holds the following identity.*

$$a_h(v_j, v_j) = a_j(v_j, v_j), \quad \forall v_j \in V_j^h, \quad j = 0, 1, \cdots, J. \tag{35}$$

Using a coloring argument (cf. [SBG96]), it is easy to show the following lemma.

Lemma 9 *Let u and u_i be as in (25). Let $0 \leq \mathcal{E}_{ij} \leq 1$ to be the minimal values such that*

$$|a_h(u_i, u_j)| \leq \mathcal{E}_{ij} a_h(u_i, u_i)^{\frac{1}{2}} a_h(u_j, u_j)^{\frac{1}{2}}, \quad i, j = 1, 2, \cdots, J. \tag{36}$$

Then there holds the following estimate

$$\rho(\mathcal{E}) \leq N_c + 1. \tag{37}$$

We are now ready to establish the main theorem of this paper.

Theorem 1 *There exists a positive constant C which is independent of h, J, δ and H such that there holds the estimate*

$$cond(T) \leq C(2 + N_c) H \delta^{-1}. \tag{38}$$

Proof: The estimate (38) follows immediately from Lemma 7–9 and Lemma 3 of Chapter 5 of [SBG96] with $C_0^2 = O(H\delta^{-1})$, $\omega = 1$ and $\rho(\mathcal{E}) = 1 + N_c$.

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