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22 Optimized Schwarz Methods for Helmholtz Problems

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Introduction

The classical Schwarz algorithm has a long history. It was invented by Schwarz more than a century ago to prove existence and uniqueness of solutions to Laplace's equation on irregular domains. It gained popularity with the advent of parallel computers and was analyzed in depth both at the continuous level and as a preconditioner for discretized problems (see the books by Quarteroni and Valli [QV99] and Smith, Bjørstad and Gropp [SBG96] and references therein). The classical Schwarz algorithm is however not effective for Helmholtz problems, because the convergence mechanism of the Schwarz algorithm works only for the evanescent modes, not for the propagative ones. Nevertheless the Schwarz algorithm has been applied to Helmholtz problems by adding a relatively fine coarse mesh in [CW92] and changing the transmission conditions from Dirichlet in the classical Schwarz case to Robin, as done in [DJR92], [BD97], [Gha97], [dLBFM$^+$98], [MSRKA98] and [CCEW98]. The influence of the transmission conditions on the Schwarz algorithm for the Helmholtz equation has first been studied for a nonoverlapping version of the Schwarz algorithm in [CN98] and for the overlapping case in [GHN00]. We begin this paper by recalling the optimal transmission conditions which lead to the best possible convergence of the Schwarz algorithm and which even work without overlap. These optimal transmission conditions are however non local in nature and thus not ideal for implementations. One therefore approximates the optimal transmission conditions locally. A first result we present is that no matter how one approximates, the new optimized Schwarz method has a better convergence rate than the classical Schwarz method. Then we present a new second order optimized transmission condition for a nonoverlapping variant of the optimized Schwarz method with better asymptotic performance than the one presented in [GMN01]. If h denotes the mesh parameter, then the new method has a convergence rate of $1 - O(h^{1/4})$ whereas the best optimized Schwarz method so far for the Helmholtz equation had a convergence rate of $1 - O(h^{1/2})$, as given in [GMN01].

Classical Schwarz for the Helmholtz Equation

We consider the Helmholtz equation in two dimensions,

$$(\Delta + \omega^2)(u) = f, \quad \text{in } \Omega = \mathbb{R}^2, \tag{1}$$

with Sommerfeld radiation conditions at infinity. We apply the Schwarz algorithm with two overlapping subdomains $\Omega_1 = (-\infty, L] \times \mathbb{R}$, $L > 0$ and $\Omega_2 = [0, \infty) \times \mathbb{R}$ which leads to the Schwarz iteration

$$\begin{array}{rcll} \Delta v^{n+1} + \omega^2 v^{n+1} & = & f, & \text{in } \Omega_1, \\ v^{n+1}(L, y) & = & w^n(L, y), & y \in \mathbb{R} \end{array} \tag{2}$$

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and

$$\begin{aligned} \Delta w^{n+1} + \omega^2 w^{n+1} &= f, && \text{in } \Omega_2, \\ w^{n+1}(0,y) &= v^n(0,y), && y \in \mathbb{R}. \end{aligned} \tag{3}$$

To analyze the convergence rate of this algorithm, we use Fourier analysis. By linearity it suffices to analyze the homogeneous problem, $f(x,y) = 0$, and show convergence to the zero solution. Applying a Fourier transform in the y variable with Fourier parameter k leads to the ordinary differential equations

$$\begin{aligned} \frac{\partial^2 \hat{v}^{n+1}}{\partial x^2} + (\omega^2 - k^2)\hat{v}^{n+1} &= 0, && x < L,\ k \in \mathbb{R}, \\ \hat{v}^{n+1}(L,k) &= \hat{w}^n(L,k), && k \in \mathbb{R}, \\ \frac{\partial^2 \hat{w}^{n+1}}{\partial x^2} + (\omega^2 - k^2)\hat{w}^{n+1} &= 0, && x > 0,\ k \in \mathbb{R}, \\ \hat{w}^{n+1}(0,k) &= \hat{v}^n(0,k), && k \in \mathbb{R}. \end{aligned}$$

Solving the second equation at step n and inserting the result into the first one we find after evaluating at $x = 0$

$$\hat{v}^{n+1}(0,k) = e^{-2\sqrt{k^2-\omega^2}L}\hat{v}^{n-1}(0,k).$$

Hence the convergence rate of the classical Schwarz method is

$$\rho_{cla} := e^{-2\sqrt{k^2-\omega^2}L}. \tag{4}$$

This shows the main problem of the classical Schwarz method when applied to a Helmholtz problem: while evanescent or high frequency modes, $k^2 > \omega^2$, converge as in the case of Laplace's equation, the propagating or low frequency modes, $k^2 < \omega^2$, do not converge at all, $|\rho_{cla}| = 1$ for those modes. Figure 1 shows the error in a numerical experiment for an example on a domain $\Omega = [0,2] \times [0,1]$ split into two subdomains in the x-direction and $\omega = 10$. The error on the left subdomain is shown as the iteration progresses and one can see that the classical Schwarz algorithm has problems converging because of the low frequency modes, whereas the high frequency modes introduced at the interface by the initial guess are reduced effectively. Figure 2 shows on the left the corresponding convergence rate (4) for this example as a function of the frequency parameter k.

Optimized Schwarz for the Helmholtz Equation

We consider again the Helmholtz equation (1) in two dimensions and we apply a Schwarz algorithm with the same overlapping subdomains $\Omega_1 = (-\infty, L] \times \mathbb{R}$, $L > 0$ and $\Omega_2 = [0,\infty) \times \mathbb{R}$ as before. But this time we do not use Dirichlet transmission conditions, but more general ones,

$$\begin{aligned} \Delta v^{n+1} + \omega^2 v^{n+1} &= f, && \text{in } \Omega_1, \\ (\partial_x + \Lambda_v)(v^{n+1}(L,y)) &= (\partial_x + \Lambda_v)(w^n(L,y)), && y \in \mathbb{R} \end{aligned} \tag{5}$$

and

$$\begin{aligned} \Delta w^{n+1} + \omega^2 w^{n+1} &= f, && \text{in } \Omega_2, \\ (\partial_x + \Lambda_w)(w^{n+1}(0,y)) &= (\partial_x + \Lambda_w)(v^n(0,y)), && y \in \mathbb{R}. \end{aligned} \tag{6}$$

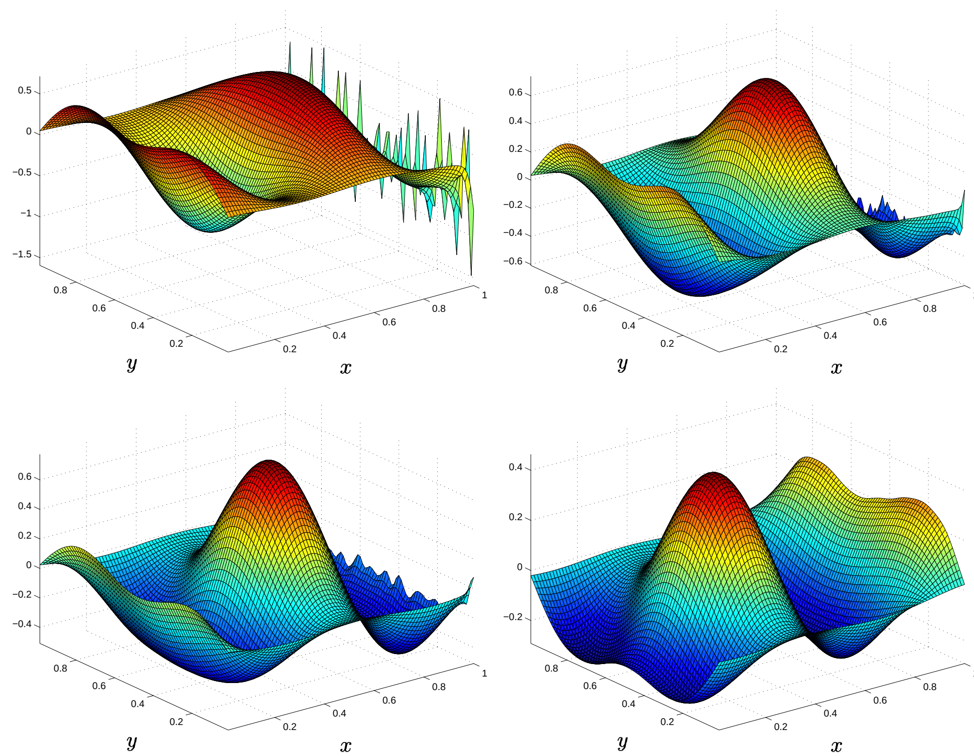

Figure 1: Error in iterations 1, 2, 3 and 8 on the left of the two subdomains of the classical Schwarz algorithm applied to a Helmholtz equation. Clearly the low frequency modes are not effectively reduced by the method.

The operators Λ_v in (5) and Λ_w in (6) are linear operators in the y direction along the interface which we will try to determine to obtain optimal performance of the Schwarz algorithm. Using Fourier analysis like in the case of the classical Schwarz algorithm and setting $f(x, y) = 0$, we obtain the iteration in the Fourier transformed domain

$$\begin{array}{rcll} \dfrac{\partial^2 \hat{v}^{n+1}}{\partial x^2} + (\omega^2 - k^2)\hat{v}^{n+1} & = & 0, & x < L,\ k \in \mathbb{R}, \\ (\partial_x + \lambda_v(k))(\hat{v}^{n+1}(L, k)) & = & (\partial_x + \lambda_v(k))(\hat{w}^n(L, k)), & k \in \mathbb{R}, \\ \dfrac{\partial^2 \hat{w}^{n+1}}{\partial x^2} + (\omega^2 - k^2)\hat{w}^{n+1} & = & 0, & x > 0,\ k \in R, \\ (\partial_x + \lambda_w(k))(\hat{w}^{n+1}(0, k)) & = & (\partial_x + \lambda_w(k))(\hat{v}^n(0, k)), & k \in \mathbb{R}. \end{array}$$

Solving the second equation at step n and inserting the result into the first equation we find after evaluating at $x = 0$

$$\hat{v}^{n+1}(0, k) = \frac{\lambda_v(k) - \sqrt{k^2 - w^2}}{\lambda_v(k) + \sqrt{k^2 - w^2}} \cdot \frac{\lambda_w(k) + \sqrt{k^2 - w^2}}{\lambda_w(k) - \sqrt{k^2 - w^2}} e^{-2\sqrt{k^2 - \omega^2} L} \hat{v}^{n-1}(0, k)$$

and hence the convergence rate of the new Schwarz method is

$$\rho_{opt} := \frac{\lambda_v(k) - \sqrt{k^2 - w^2}}{\lambda_v(k) + \sqrt{k^2 - w^2}} \cdot \frac{\lambda_w(k) + \sqrt{k^2 - w^2}}{\lambda_w(k) - \sqrt{k^2 - w^2}} e^{-2\sqrt{k^2-\omega^2}L}, \tag{7}$$

where we can choose the symbols $\lambda_v(k)$ and $\lambda_w(k)$ of the linear operators Λ_v and Λ_w along the interface to influence the performance of the new Schwarz method.

An Optimal Schwarz Method

There is a best choice for the free parameters in the convergence rate (7) of the new Schwarz method: choosing $\lambda_v(k) = \sqrt{k^2 - w^2}$ and $\lambda_w(k) = -\sqrt{k^2 - w^2}$, the convergence rate becomes zero for all values of the frequency parameter k and hence the method converges in 2 iterations. In addition for this choice the convergence rate is independent of the overlap, the exponential factor in (7) is irrelevant and hence the Schwarz method can be used without overlap as well. One can show that this result generalizes to convergence in N iterations if N subdomains in strips are employed [NR95]. But for real computations, we do not want to depend on Fourier transforms, we want to do the computations as usual on a given finite element or finite difference mesh. Hence we need the inverse transform of the optimal transmission conditions, $\lambda_{vw}(k) = \pm\sqrt{k^2 - w^2}$. Unfortunately, this inverse transform leads to nonlocal operators Λ_{vw} in the y variable, because of the square root in their symbol. Even though such non-local operators can be implemented by using a convolution on the boundary, it is much more cumbersome than to implement local transmission conditions. If the symbol of the optimal transmission conditions was a polynomial in k however, then the operator in real space would be local, because a polynomial in k transforms into derivatives in real space, and derivatives are local operators. Therefore, instead of using the best possible transmission conditions, we introduce local approximations to those conditions which are easy to implement. One can either choose a Taylor expansion about a low frequency to improve the low frequency behavior of the algorithm or, even better, optimize the approximation for the performance of the algorithm by making ρ_{opt}, the natural measure of performance, as small as possible. This leads to the new class of optimized Schwarz methods.

Optimized Schwarz Methods

We introduce local approximations of the best transmission operator,

$$\Lambda_v = +(\alpha_1 + \beta_1 k^2), \text{ and } \Lambda_w = -(\alpha_2 + \beta_2 k^2),$$

where $\alpha_j, \beta_j \in \mathbb{C}$, $j = 1, 2$. Note that we do not include a first order term because the Helmholtz operator is symmetric. For non-symmetric problems one would include the first order term as well. The case $\beta_j = 0$ leads to Robin transmission conditions and gives us four coefficients to optimize the performance (two complex numbers α_1 and α_2). If $\beta_j \neq 0$ we obtain transmission conditions including second order tangential derivatives which gives us eight coefficients to optimize the performance of the algorithm. In the sequel we restrict our analysis for simplicity to the special case where $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$, for which the convergence rate of the optimized Schwarz method can be simplified to

$$\rho_{opt} = \left(\frac{\alpha + \beta k^2 - \sqrt{k^2 - w^2}}{\alpha + \beta k^2 + \sqrt{k^2 - w^2}} \right)^2 e^{-2\sqrt{k^2-\omega^2}L}. \tag{8}$$

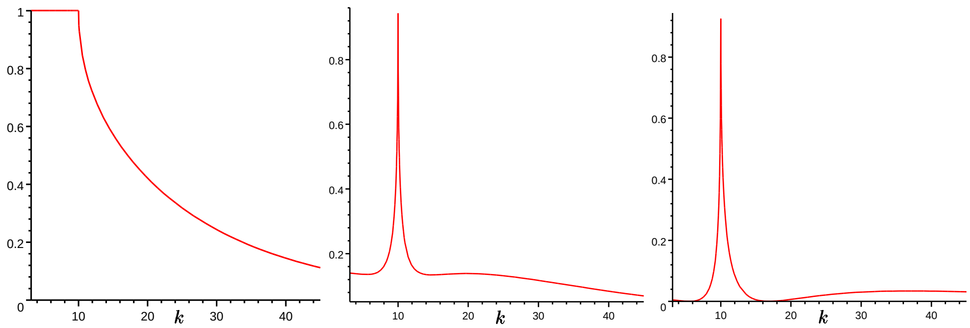

Figure 2: Comparison of the convergence rate of classical Schwarz on the left with optimized Schwarz using Robin transmission conditions in the middle and second order optimized transmission conditions on the right.

This cuts the number of optimization parameters in half and simplifies the optimization, at the cost of not finding the best possible second order transmission conditions. For symmetric positive definite problems the difference is investigated in [Gan00] and is found to be significant.

Theorem 1 *If* $\Re(\alpha), \Im(\alpha), \Re(\beta), \Im(\beta) \geq 0$ *then the optimized Schwarz method always converges faster than the classical Schwarz method.*

Proof We have to show under the conditions of the theorem that $\rho_{opt}(k)$ given in (8) is smaller or equal to $\rho_{cla}(k)$ given in (4) for all k. The only difference between the two convergence rates is the additional factor in front of the exponential in (8). But the modulus of this factor is

$$\frac{(\Re(\alpha) + \Re(\beta)k^2 - \Re(\sqrt{k^2 - w^2}))^2 + (\Im(\alpha) + \Im(\beta)k^2 - \Im(\sqrt{k^2 - w^2}))^2}{(\Re(\alpha) + \Re(\beta)k^2 + \Re(\sqrt{k^2 - w^2}))^2 + (\Im(\alpha) + \Im(\beta)k^2 + \Im(\sqrt{k^2 - w^2}))^2} \leq 1$$

if the real and imaginary parts of α and β are non-negative, which completes the proof. ■

This indicates that one should not use the classical Schwarz method any longer, whatever one does to the coefficients in the transmission conditions, the optimized Schwarz method will work better than the classical Schwarz method. Figure 2 shows a comparison of the convergence rates of the classical Schwarz method and optimized Schwarz methods with Robin and second order transmission conditions as a function of the frequency parameter k. Note that for optimized Schwarz methods the low frequency modes converge as well, not just the high frequency ones. Only at the resonance frequency $k^2 = \omega^2$ the convergence rate equals one for optimized Schwarz methods. This is however not a problem when optimized Schwarz is used as a preconditioner for a Krylov subspace method, since such a method easily corrects one bad mode in the spectrum.

An Optimized Schwarz Method without Overlap

We optimize now the coefficients α and β in (8) for the case of no overlap, $L = 0$. For the continuous problem we would need to optimize for all frequency parameters $k \in \mathbb{R}$ which would lead to convergence problems as $k \longrightarrow \pm\infty$. But in a numerical computation, the frequency range is bounded, from below by the smallest frequency $k_{\min}$ relevant to the subdomain and from above by the largest frequency $k_{\max}$ supported by the numerical grid. The largest frequency $k_{\max}$ is of order π/h. We therefore have to solve the optimization problem

$$\min_{\alpha,\beta\in\mathbb{C}}\left(\max_{k\in(k_{\min},\omega_-)\cup(\omega_+,k_{\max})}\left|\frac{\alpha+\beta k^2-\sqrt{k^2-w^2}}{\alpha+\beta k^2+\sqrt{k^2-w^2}}\right|^2\right) \tag{9}$$

where ω_- and ω_+ are parameters to be chosen to exclude the single mode with convergence rate one at the resonance frequency $k^2=\omega^2$. We have the following asymptotic convergence result

Theorem 2 *There exist parameters $\alpha,\beta\in\mathbb{C}$ such that the asymptotic convergence rate of the optimized Schwarz method is*

$$\rho_{opt}=1-4\left(\frac{\pi}{\sqrt{d\omega(2\omega-d\omega)}}\right)^{1/4}h^{1/4}+O(h^{1/2})$$

where $d\omega:=\omega_+-\omega=\omega-\omega_-$.

The proof of this result is beyond the scope of this short paper, since it involves the asymptotic solution of the min-max problem (9). But it is important to notice that the classical Schwarz method does not converge without overlap, not even in the symmetric positive definite case. If the overlap is of order h, then the convergence rate of classical Schwarz is $1-O(h)$ in the symmetric positive definite case. The optimized Schwarz method without overlap converges even for the indefinite case at the much better rate of $1-O(h^{1/4})$ except for the resonance mode. The numerical results in the next section show that the optimized Schwarz method used as a preconditioner for a Krylov methods exhibits a convergence rate of nearly $1-O(h^{1/8})$, gaining almost the expected square-root from Krylov acceleration.

Numerical Results

We chose the model problem of a tube,

$$\begin{aligned}
\Delta u+\omega^2 u &= f & & 0<x,y<1,\\
u &= 0 & & 0<x<1,\ y=0,1,\\
\frac{\partial u}{\partial x}-i\omega u &= 0 & & x=0,\ 0<y<1,\\
-\frac{\partial u}{\partial x}-i\omega u &= 0 & & x=1,\ 0<y<1.
\end{aligned}$$

and two nonoverlapping subdomains $\Omega_1=[0,1/2]\times[0,1]$, $\Omega_2=[1/2,1]\times[0,1]$. For experiments with overlap, see [GHN00]. Table 1 shows the number of iterations required to converge to a desired tolerance $10e-6$ using optimized Schwarz as a preconditioner for GMRES and compares this to a non-optimized local approximation of the optimal transmission conditions using a Taylor expansion for low frequencies.

Figure 3 shows the asymptotic convergence rate in h achieved by the optimized Schwarz method. Note how Krylov acceleration gives almost the additional square-root, $\rho_{opt}=1-O(h^{1/8})$ as one can expect in ideal situations. It would have been interesting to do the experiment for $h=1/1600$, but the case $h=1/800$ shown constitutes already a complex linear system with 640'000 unknowns and is at the limit of current workstation capacities.

h	1/50	1/100	1/200	1/400	1/800
Taylor Order 2	25	32	38	46	57
Optimized Order 2	10	10	10	11	13

Table 1: Optimized Schwarz second order transmission conditions compared to a simpler second order Taylor approximation of the optimal transmission conditions for low frequencies.

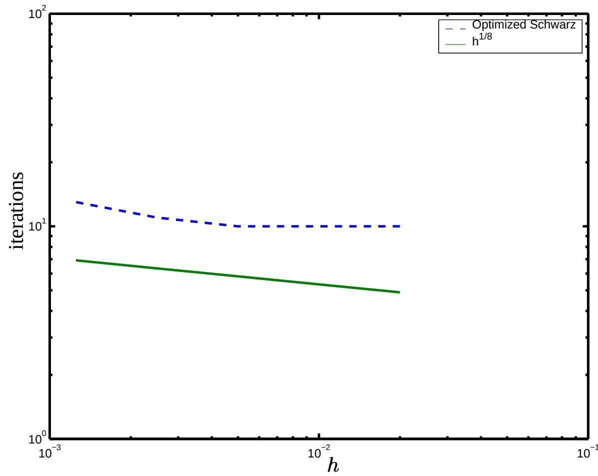

Figure 3: Asymptotic convergence rate of the second order optimized Schwarz method without overlap.

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