Thirteenth International Conference on Domain Decomposition Methods
Editors: N. Debit, M.Garbey, R. Hoppe, J. Périaux, D. Keyes, Y. Kuznetsov

5 Aitken-Schwarz algorithm on Cartesian grid

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Introduction

This paper is devoted to the generalization of the Aitken-Schwarz (AS) domain decomposition method (resp. Steffensen-Schwarz (SS)) method introduced in [GTD01]. A solver was first designed to solve linear (resp. nonlinear) elliptic problems in metacomputing framework with a slow communication network. In [GTD01] the domain decomposition was one dimensional domain decomposition of multidimensionnal problems. We extend this domain decomposition to multilevel one dimensional AS (resp. SS) domain decomposition. The AS (resp. SS) method is recursively applied in one different direction at each level. The difficulty is to generate homogeneous Dirichlet boundary conditions at each level of domain decomposition. This problem is solved in AS domain decomposition with the superposition principle when linear problems are solved. A similar shifting technique is also adopted to solve nonlinear problems with SS. Some results on 2D linear and nonlinear problems are given as examples.

The arithmetical complexity of AS is investigated when the inner solver has linear or nonlinear complexity. Notably, a comparison with the best implementation of a fast solvers such as FFT on Poisson problem are given. The stability of the Aitken-Schwarz and Steffensen-Schwarz multilevel domain decomposition methods is investigated with an extensive sensitivity analysis experiment that measures the influence on the convergence history when one systematically perturbed randomly the subproblem solution at the end of each subdomain solve.

The plan of this paper is as follows: section 1 recalls the principles of the Aitken-Schwarz domain decompositions, section 2 describes the extension of the methodology from one dimensional domain decomposition to domain decomposition in several space directions, section 3 comments on the arithmetical complexity of the method, and section 4 comments on the stability of the method. Finally, section 5 gives the conclusions and perspectives.

1 Principles of the Aitken-Schwarz method

We are going to describe briefly the numerical ideas behind the Aitken Schwarz method. We refer to [GTD01] for more details.

For simplicity, we illustrate the concept with the discretized Helmholtz operator $L[u] = \Delta u - \lambda u,\ \lambda > 0$, with a grid that is a tensorial product of one dimensional grids, and a square domain decomposed into strip subdomains.

Let us consider the homogeneous Dirichlet problem $L[U] = f\ in\ \Omega = (0,1),\ U_{|\partial\Omega} = 0$, in one space dimension. We restrict ourselves to a decomposition of Ω into two overlapping

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subdomains $\Omega_1 \bigcup \Omega_2$ and consider the additive Schwarz algorithm [Sch80, Lio88, Lio89].

$$L[u_1^{n+1}] = f \ in \ \Omega_1, \ u_{1|\Gamma_1}^{n+1} = u_{2|\Gamma_1}^{n}, \ L[u_2^{n+1}] = f \ in \ \Omega_2, \ u_{2|\Gamma_2}^{n+1} = u_{1|\Gamma_2}^{n}. \tag{1}$$

with given initial conditions $u_{1|\Gamma_2}^0, u_{2|\Gamma_1}^0$ to start this iterative process.

To simplify the presentation, we assume implicitly in our notations that the homogeneous Dirichlet boundary conditions are satisfied by all intermediate subproblems. This algorithm can be executed in parallel on two computers [Kuz91]. At the end of each subdomain solve, the artificial interfaces $u_{2|\Gamma_1}^n$ and $u_{1|\Gamma_2}^n$ have to be exchanged between the two computers.

In order to avoid as much as possible redundancy in the computation we fix once and for all the overlap between subdomains to be the minimum, i.e of size one mesh. This algorithm can be extended to an arbitrary number of subdomains and is nicely scalable, because the communications linked only subdomains that are neighbors.

However it is one of the worst numerical algorithms to solve the problem, because the convergence is extremely slow. We introduce thereafter a modified version of this Schwarz algorithm so called Aitken-Schwarz that transforms this dead slow iterative solver into a direct fast solver while keeping the scalability of the Schwarz algorithm for a moderate number of subdomains. The idea is as follows.

We observe that the interface operator T,

$$(u_{1|\Gamma_1}^n - U_{\Gamma_1}, u_{2|\Gamma_2}^n - U_{\Gamma_2})^t \rightarrow (u_{1|\Gamma_1}^{n+1} - U_{\Gamma_1}, u_{2|\Gamma_2}^{n+1} - U_{\Gamma_2})^t \tag{2}$$

is *linear*.

Therefore, the sequence $(u_{1|\Gamma_1}^n, u_{2|\Gamma_2}^n)$ has pure linear convergence that is, it satisfies the identities:

$$u_{1|\Gamma_2}^{n+1} - U_{|\Gamma_2} = \delta_1(u_{2|\Gamma_1}^n - U_{|\Gamma_1}), \ u_{2|\Gamma_1}^{n+1} - U_{|\Gamma_1} = \delta_2(u_{1|\Gamma_2}^n - U_{|\Gamma_2}), \tag{3}$$

where δ_1 (resp. δ_2) is the damping factor associated to the operator L in subdomain Ω_1 (resp. Ω_2) [GH97]. Consequently

$$u_{1|\Gamma_2}^2 - u_{1|\Gamma_2}^1 = \delta_1(u_{2|\Gamma_1}^1 - u_{2|\Gamma_1}^0), \ u_{2|\Gamma_1}^2 - u_{2|\Gamma_1}^1 = \delta_2(u_{1|\Gamma_2}^1 - u_{1|\Gamma_2}^0), \tag{4}$$

So except if the initial boundary conditions match with the exact solution U at the interfaces, the amplification factors can be computed from the linear system(4). Since $\delta_1\delta_2 \neq 1$ the limit $U_{|\Gamma_i}, i = 1, 2$ is obtained as the solution of the linear system (3). Consequently, this generalized Aitken acceleration procedure gives the *exact* limit of the sequence on the interface Γ_i based on two successive Schwarz iterates $u_{i|\Gamma_i}^n, \ n = 1, 2$, and the initial condition $u_{i|\Gamma_i}^0$. An additional solve of each subproblem (1) with boundary conditions $u_{\Gamma_i}^\infty$ gives the final solution of the ODE problem. We can further improve this first algorithm as follows.

Let (v_1, v_2) be the solution of

$$L[v_1] = 0 \ in \ \Omega_1, \ v_{|\Gamma_1} = 1; \ L[v_2] = 0 \ in \ \Omega_2, \ v_{|\Gamma_2} = 1. \tag{5}$$

We have then $\delta_1 = v_{|\Gamma_2}, \ \delta_2 = v_{|\Gamma_1}$. Consequently δ_1 and δ_2 can be computed before-hand numerically or analytically.

Once (δ_1, δ_2) are known, we need only *one* Schwarz iterate to accelerate the interface and an additional solves for each subproblems. This is a total of two solves per subdomain. The Aitken acceleration thus transforms the additive Schwarz procedure into an *exact* solver regardless of the speed of convergence of the original Schwarz method, and in particular with a minimum overlap.

This Aitken-Schwarz algorithm can be reproduced for multidimensional problems. As a matter of fact, it can be shown [GTD01] that the coefficients of each wave number of the sine expansion of the trace of the solution generated by the Schwarz algorithm has its own rate of *exact* linear convergence.

We can then generalize the one dimensional algorithm to two space dimensions as follows:

- step1 : compute analytically or numerically in parallel each damping factor δ_j^k for each wave number k from the two point one D boundary value problems analogues of (5) with the operator

$$L_k \equiv u_{xx} - (4/h_y^2 \sin^2(k\frac{h_y}{2}) + \lambda)u,$$

  with h_y being the space step in y direction.

- step2: apply one additive Schwarz iterate to the Helmholtz problem with subdomain solver of choice (multigrid, fast Fourier transform, PDC3D, etc...)

- step3:

  - compute the sine expansion $\hat{u}^n_{k|\Gamma_i},\ n = 0,1,\ k = 1..N$ of the traces on the artificial interface $\Gamma_i, i = 1..2$ for the initial boundary condition $u^0_{|\Gamma_i}$ and the solution given by *one* Schwarz iterate $u^1_{|\Gamma_i},\ i = 1,2$.

  - apply generalized Aitken acceleration *separately* to each wave coefficients in order to get $\hat{u}^\infty_{j|\Gamma_i}$.

  - recompose the trace $u^\infty_{j|\Gamma_i}$ in physical space.

- step4: compute in parallel the solution in each subdomains $\Omega_i,\ i = 1,2$ with new inner BCs and subdomain solver of choice.

So far, we have restricted ourselves to domain decomposition with two subdomains. We show in [GTD01], that a generalized Aitken acceleration technique can be applied to an arbitrary number $q > 2$ of subdomains with strip domain decomposition. Our main result is that no matter is the number of subdomains, the total number of subdomain solves required to produce the final solution is still **two**.

However the generalized Aitken acceleration of the vectorial sequences of the sine expansion coefficients of the interface introduces a coupling between all interfaces.

To be more specific, we obtain a given linear system for each wave number k,

$$\tilde{u}^\infty = (Id - P_k)^{-1}(\tilde{u}^{n+1} - P_k\tilde{u}^n). \tag{6}$$

and P_k has the following pentadiagonal structure:

$$\left|\begin{array}{ccccccccc}
0 & \delta_1^r & 0 & 0 & .... & & & & \\
\delta_2^{l,l} & 0 & 0 & \delta_2^{l,r} & ... & & & & \\
\delta_2^{r,l} & 0 & 0 & \delta_2^{r,r} & ... & & & & \\
 & & & & & & & & \\
 & & & ... & \delta_{q-1}^{l,l} & 0 & 0 & \delta_{q-1}^{l,r} & \\
 & & & ... & \delta_{q-1}^{r,l} & 0 & 0 & \delta_{q-1}^{r,r} & \\
 & & & ... & 0 & 0 & \delta_q^r & 0 &
\end{array}\right|$$

But we observe first that this generalized Aitken acceleration processes independently each waves coefficients of the sinus expansion of the interfaces. Second the highest is the frequency k the smallest are the damping factors $\delta_j^{l,l}$, $\delta_j^{l,r}$, $\delta_j^{r,l}$, $\delta_j^{r,r}$. A careful stability analysis of the method shows that

- for low frequencies, we should use the generalized Aitken acceleration coupling all the subdomains.
- for intermediate frequencies, we can neglect this global coupling and implement only the local interaction between subdomains that overlap.
- for high frequencies, we do not use Aitken acceleration because one iteration of the Schwarz algorithm damps the high frequencies error enough.

The algorithm has then the same structure than the two subdomains algorithm presented above. Step 1 and step 4 are fully parallel. Step 2 requires only local communication and scales well with the number of processors. Step 3 requires global communication of interfaces in Fourier space for low wave numbers, and local communications for intermediate frequencies. In addition for moderated number of subdomains, the arithmetic complexity of step 3 that is the kernel of the method is negligible compared to step 2.

Our algorithm can be extended successfully to grids that are tensorial product of one dimensional grids with arbitrary (irregular) space step [BGO00], iterative domain decomposition method such as Dirichlet-Neumann procedure with non-overlapping subdomains or red/black subdomains iterative procedure.

For nonlinear elliptic problems, the Aitken acceleration is no longer exact. the so-called Steffensen-Schwarz variant is then a very efficient numerical method for low order perturbation of constant coefficient linear operators - once again we refer to [GTD01] for more details. We will proceed now with the description of the generalization of the method to domain decomposition in more than one space directions.

2 Multilevel Aitken-Schwarz and Steffensen-Schwarz Domain Decomposition

Let us consider first the linear case and denote L the discrete linear differential operator. For simplicity, we will restrict this presentation to problems in two space dimensions. Once again, we assume homogeneous Dirichlet Boundary conditions on domain Ω. We introduce a first level of domain decomposition into strips in direction x

$$\Omega = \bigcup_{i=1,n_x} \Omega_i,$$

where the $\Omega_i = (x_{i,l}, x_{i,r}) \times (0, \pi)$ are the overlapped rectangles represented in Figure 1.

To proceed with a two dimensional domain decomposition, we introduce a second level of domain decomposition and decompose each subdomain Ω_i into a set of overlapping rectangles in direction y,

$$\Omega_i = \bigcup_{j=1,n_y} \Omega_{i,j},$$

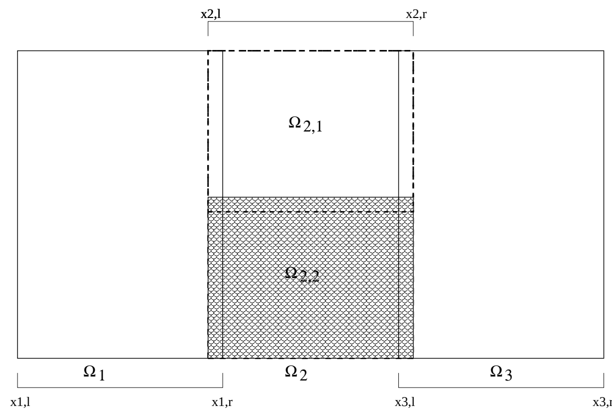

Figure 1: Multilevel Aitken-Schwarz Method principle

The main idea is to apply recursively on each subdomain decomposition level the Aitken-Schwarz algorithm. The difficulty comes from the fact that the Dirichlet boundary conditions of the subdomain at the first level are no more homogeneous Dirichlet boundary conditions. Consequently, the sine expansion operator should not be applied directly to the trace of the interfaces solution generated by this second level of the Schwarz algorithm.

We introduce therefore a shift denoted v_i in each subdomain Ω_i, in order to retrieve the homogeneous Dirichlet boundary conditions problem on each strip Ω_i.

Let $\otimes$ be the notation for the Kronecker product. In each strip Ω_i, we solve with Aitken-Schwarz the modified problem

$$L[w_i^{n+1}] \;=\; f - L[v_i] \;\; in \;\; \Omega_i \tag{7}$$
$$w_i^{n+1} \;=\; 0 \;\; in \;\; \partial\Omega_i \tag{8}$$

where v_i in matrix notation is defined as

$$v_i = d_i^{-1}.X_l \otimes u_{x=x_{i,l}} + d_i^{-1}.X_r \otimes u_{x=x_{i,r}}, \tag{9}$$

with d_i the size of the strip Ω_i in x direction: $d_i = x_{i,l} - x_{i,r}$, $x_i = (x_{i,l}, .., x_{i,r})$ row vector of the x coordinates of the grid points in $\bar{\Omega}_i$ in increasing order, $X_l = x_i - (x_{i,l}, .., x_{i,l})$ and $X_r = x_i - (x_{i,r}, .., x_{i,r})$, and $u_{x=x_{i,l}}$, $u_{x=x_{i,r}}$ are the column vectors containing the artificial boundary condition.

Table 1 gives the error between the Aitken-Schwarz solution and the discretized exact solution $u(x, y) = (x^2 - 0.25)y(y - 1)$ in maximum norm for a number of subdomains n_x in x-direction varying from 1 to 16 and a number of subdomains n_y in y-direction varying from 2 to 16 and for four global size meshes varying from 34×34 to 258×258 points for the Poisson problem. It exhibits that :

- the methodology gives accurate results close to the machine accuracy (we recall that the test are done with the Matlab software),

- the accuracy reached increases with the number of subdomains especially for large size problem. This is due to the fact that the local system are smaller leading to smaller conditioning number and then round off error in the LU factorization are smaller than in the few-subdomain case.

34×34 points	n_x subdomains				
n_y subdomains	1	2	4	8	16
2	1.9920e-13	2.9296e-14	9.3051e-15	4.0246e-16	7.0777e-16
4	1.2676e-13	2.2225e-14	2.8172e-15	4.0246e-16	7.0777e-16
8	4.2848e-15	1.1623e-15	4.7184e-16	4.0246e-16	7.0777e-16
16	8.5522e-16	2.2421e-16	4.0246e-16	4.0246e-16	7.0777e-16
66×66 points	n_x subdomains				
n_y subdomains	1	2	4	8	16
2	1.0693e-12	3.8589e-13	6.1952e-14	9.1593e-15	1.1380e-15
4	9.3924e-13	4.1277e-13	4.2577e-14	1.1005e-14	1.1380e-15
8	5.3798e-13	1.0418e-13	3.6227e-14	2.2204e-15	1.1102e-15
16	4.8329e-15	2.1164e-15	1.2906e-15	2.1649e-15	1.1380e-15
130×130 points	n_x subdomains				
n_y subdomains	1	2	4	8	16
2	7.7432e-12	2.3652e-12	1.2857e-12	8.1442e-14	1.6535e-14
4	6.2317e-12	1.6914e-12	6.3427e-13	1.3916e-13	1.9729e-14
8	2.6878e-12	1.2031e-12	3.5410e-13	1.7667e-13	7.2026e-15
16	1.0170e-12	4.1206e-13	5.3402e-14	2.3787e-14	5.9119e-15
258×258 points	n_x subdomains				
n_y subdomains	1	2	4	8	16
2	4.3664e-11	1.3513e-11	2.1562e-12	1.6181e-12	1.6694e-13
4	2.9052e-11	6.6053e-12	2.8467e-12	1.1419e-12	3.3179e-13
8	2.0213e-11	6.7309e-12	4.5298e-12	1.3563e-12	2.1001e-13
16	1.9160e-11	4.4557e-12	1.8232e-12	3.2840e-13	8.4238e-14

Table 1: Error with respect of the number of subdomains

Secondly, let us consider the nonlinear case. The problem to be solved can be written formally as

$$A(u)u = F \tag{10}$$

We do not have anymore the superposition principle as in the linear case, but we can still use the same shift to recover at the first level of domain decomposition homogeneous Dirichlet boundary conditions. We set:

$$A[w_i^{n+1} + v_i] = F \; in \; \Omega_i \tag{11}$$

$$w_i^{n+1} = 0 \; in \; \partial\Omega_i \tag{12}$$

where v_i is defined as in (9).

The solution for one Schwarz iterate on the subdomain Ω_i is obtained as

$$u_i^{n+1} = w_i^{n+1} + v_i \quad (13)$$

To illustrate the two level domain decomposition algorithm, we consider the Bratu problem which represents a simplified model of combustion written as follows:

$$\Delta u(x,y) + exp(\lambda u(x,y)) = 0, \ \ (x,y) \in \Omega = [0,1]^2, \lambda \geq 0, \quad (14)$$

$$u(x,y) = 0, \ \ (x,y) \in \partial\Omega. \quad (15)$$

The discrete operator on a regular stencil of space step h_x in x direction and h_y in y direction is:

$$-\frac{u_{i+1,j} + u_{i-1,j} - 2u_{i,j}}{h_x^2} - \frac{u_{i,j+1} + u_{i,j-1} - 2u_{i,j}}{h_y^2} + exp(\lambda u_{i,j}).$$

This operator is a nonlinear and nonseparable discrete operator. We use a Newton scheme to solve each nonlinear subdomain problem. The solution of the linear systems inside the Newton loop are obtained either by sparse LU or Preconditioned Conjugate gradient method with uncomplete LU. The acceleration procedure is described in [GTD01]. To be more precise, since the nonlinearity of the discrete operator is a second order perturbation of the Laplacian, we use the same acceleration procedure as in the Poisson problem case; that is, we compute the diagonal approximation of the matrix of acceleration P [GTD01] based on three successive Schwarz iterates.

Figures 2 and 3 give the convergence history of the Steffenson-Schwarz multilevel domain decomposition on the 2D Bratu problem with $n_x \times n_y = 3 \times 4$ subdomains and $\lambda = 6$. The convergence history is given for two problem sizes, namely 30×29 and 90×89. For the smaller problem the size of overlap is one mesh cell, but for the 3 times larger problem we have used 3 mesh cells overlap. The convergence to the exact discrete solution of the problem, at the outer loop level i.e the Steffensen-Schwarz iteration between Ω_i strips, -see 2- and inside each strips -see 3- with the second level of Steffensen-Schwarz iteration seems to be almost independent of the number of grid points provided that the size of the overlap between subdomains in each space direction stays the same.

The stop criterion for the Newton loop (resp. the Steffensen-Schwarz iterative procedure inside strips) was that the difference between two successive iterates was less than 10^{-9} (resp. 10^{-7}).

3 Arithmetical complexity

For the Helmholtz or Poisson operator case, the arithmetic complexity of the Aitken Schwarz method can be easily given analytically, provided the arithmetic complexity of the linear solver used in each subdomain is given.

Let us assume for simplicity that the arithmetic complexity of a fast sinus transform or its inverse of a vector of size N is $5Nlog_2(N)$. For strip domain decomposition with n_x subdomains, and a problem of global size $N_x \times N_y$, the Aitken acceleration requires the sinus transform and its inverse of the artificial interfaces at two iteration levels. It results into $20\ (n_x - 1)\ N_y log_2(N_y)$ operations. The solution of the pentadiagonal linear system corresponding to the acceleration procedure itself cost $36\ N_y\ (n_x - 1)$ operations. We recall that we need to solve each subdomain problem twice.

If one uses a sparse Gaussian elimination for each subdomain linear solve, the overall arithmetic complexity is therefore approximately

$$6\ n_x\ N_x\ N_y\ (\frac{N_x}{n_x} + 3)^2\ +\ 20\ (n_x - 1)\ N_y log_2(N_y)\ +\ 36\ N_y\ (n_x - 1)$$

If one uses a fast Poisson or Helmholtz solver, the arithmetic complexity becomes approximately

$$20\ n_x\ N_y\ (\frac{N_x}{n_x}+2) \left(\log_2(\frac{N_x}{n_x} + 2) + \log_2(N_y)\right) + 20\ (n_x-1)\ N_y log_2(N_y) + 36\ N_y\ (n_x-1).$$

This complexity analysis can be extended to the two level domain decomposition method described in this paper. We have summarized in Figure 4 and Figure 5 the result of this analysis. The efficiency of our solver increases when the number of subdomains n_y in the second space direction increases from 1 to 16. Our two level domain decomposition method speedup significantly the sparse Gaussian solver, but stays at best 50% slower than a fast Poisson solver.

Our methodology is not therefore the best Poisson solver in terms of arithmetic complexity, but as shown in [BGH$^+$] its parallel efficiency in distributed computing with slow network is very good, as opposed to the parallel efficiency of fast Poisson solver based on Fast Fourier Transform algorithm that requires global transpose of matrices.

We proceed now with some experimental measurement of the arithmetic complexity of our two levels domain decomposition with the Bratu problem. We have compared different iterative procedures for the same final global accuracy of the solution: the error in maximum norm between the exact solution of the discrete problem and the final iterate is about 10^{-4}.

The linear subdomain solver inside the Newton loop is either sparse Gaussian elimination or conjugate gradient with incomplete LU preconditioning. We select the most efficient solver in our experiments, and typically the direct linear solver is preferred when the subdomains are narrow strips.

Figure 6 reports on the domain decomposition performance with $n_x = 8$ or $n_x = 16$ strip subdomains compared to the iterative solver with no domain decomposition i.e $n_x = 1$. The problem's size is $N_x \times N_y$ with $N_x = 81$, and $N_y = 11,\ 21,\ 41,\ 81$. We get good performances only if the strips are narrow enough and N_y is large. Once again the Steffensen-Schwarz algorithm for such problem becomes a very efficient algorithm for large problems. The two level domain decomposition efficiency follows the same principle. In addition, the parallel efficiency of this algorithm in metacomputing situation has been demonstrated in [BGH$^+$].

Now we proceed with some remarks on the stability of this method.

4 Sensitivity analysis

For the linear case, and when the acceleration matrix P_k are known analytically, the additional source of unstabilities in the Aitken Schwarz algorithm may come from the linear solve of (6). Let us restrict ourselves to uniform strip domain decomposition with minimum overlap and denote

$$\begin{pmatrix} \delta_1 & 0\ 0\ \delta_2 \\ \delta_2 & 0\ 0\ \delta_1 \end{pmatrix} \tag{16}$$

the generic subblock of P_k for a given wave number k. The conditioning number of $Id - P_k$ for the Helmholtz operator, is bounded by [GTD]:

$$cond(Id - P_k) \leq 2\left(\frac{1}{1-\delta_1} + \frac{\delta_2\ (1-\delta_1)^{-2}}{1-\delta_2\ (1-\delta_1)^{-1}}\right)$$

with

$$\delta_1 = \sinh(\sqrt{\lambda} h_x)/\sinh(\sqrt{\lambda} d_x),\ \delta_2 = \sinh(\sqrt{\lambda}(d_x - h_x))/\sinh(\sqrt{\lambda} d_x),$$

where d_x is the size of the Ω_i strip in x direction. The conditioning number is then of order h_x^{-1} for $\lambda = 0(1)$. A direct numerical simulation to test the sensitivity of our algorithm to perturbation on the RHS of the linear differential problem confirms the good stability properties of the one-level and two-level Aitken-Schwarz method. The linear stability of the solvers deteriorates very slowly as the number of subdomains increases, as expected.

The sensitivity analysis of the Steffensen-Schwarz method for nonlinear elliptic problems is more challenging, because P_k^i is approximately reconstructed from the sequence of 3 Schwarz iterates:

$$\begin{pmatrix} \hat{u}_{i-1}^{r,n+3} - \hat{u}_{i-1}^{r,n+2} & \hat{u}_{i-1}^{r,n+2} - \hat{u}_{i-1}^{r,n+1} \\ \hat{u}_{i+1}^{l,n+3} - \hat{u}_{i+1}^{l,n+2} & \hat{u}_{i+1}^{l,n+2} - \hat{u}_{i+1}^{l,n+1} \end{pmatrix} = P_k^i \begin{pmatrix} \hat{u}_{i}^{l,n+2} - \hat{u}_{i}^{l,n+1} & \hat{u}_{i}^{l,n} - \hat{u}_{i}^{l,n} \\ \hat{u}_{i}^{r,n+2} - \hat{u}_{i}^{r,n+1} & \hat{u}_{i}^{r,n} - \hat{u}_{i}^{r,n} \end{pmatrix} \tag{17}$$

where the $\hat{u}_i^{l,n}$ and $\hat{u}_i^{r,n}$ stand for the sine expansion coefficients of the left and right interfaces solution in Ω_i.

In particular there is no guarantee that (17) system is well posed. In our implementation, the Steffensen acceleration algorithm is applied only to waves for which this system is not badly conditioned or possibly singular. We have undertaken an extensive sensitivity analysis experiment that measures the influence on the convergence history of our algorithm when one systematically perturbed randomly the subproblem solution at the end of each subdomain solve. The test for a given domain decomposition and a given number of grid points was realized 50 times, and we checked by doubling the number of runs the sensitivity of the result. Figure 7 shows a representative average measure of the error that was obtained as a function of the norm of the perturbation. We looked at square domains with $n_x = 2$ ('o' curves), $n_x = 4$ ('+' curves), $n_x = 8$ ('v' curves) and $n_x = 16$ ('*' curves). We checked the influence of the number of points in y direction, for these four different cases. It should be noticed that the standard deviation from the mean in these experiments are of the same order than the mean of errors. These results seems to provide some confidence in the robustness of our method.

5 Conclusion

We have extended our result on Aitken like acceleration of the Schwarz algorithm presented in [GTD01], to two level domain decomposition and further investigated the arithmetic complexity and stability of our algorithm.

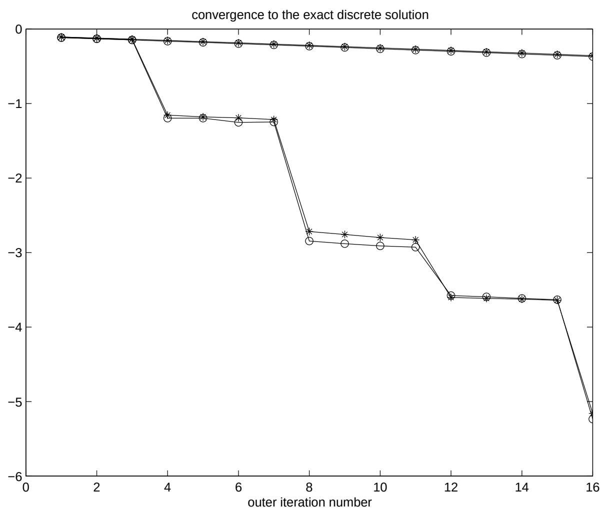

Figure 2: Convergence of the first level of Steffensen-Schwarz iterative solver, 'o' for 30×29 problem size, '*' for 90×89 problem size.

Further extension of this method to irregular meshes or non-matching grids are presently under investigation -see [BGO00] for example. We have shown in this paper that our technique is robust and numerically efficient, for the Helmholtz operator or weakly nonlinear high order perturbation of this operator such as the operator in Bratu problem. The main interest of our methodology lies however in its application to large scale metacomputing. The LIONS project [BGH$^+$] demonstrates the rather unique ability of our algorithm to provide numerical and parallel efficiency for a PDE solver with several hundred of processors distributed on several heterogeneous large-scale parallel computers in Europe linked with a slow network.

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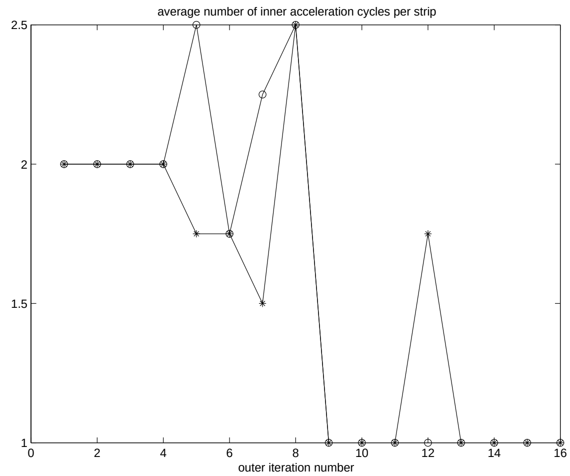

Figure 3: Averaged number of Steffensen acceleration cycles inside each strip subdomain at each iteration number of the outer loop, 'o' for 30×29 problem size, '*' for 90×89 problem size.

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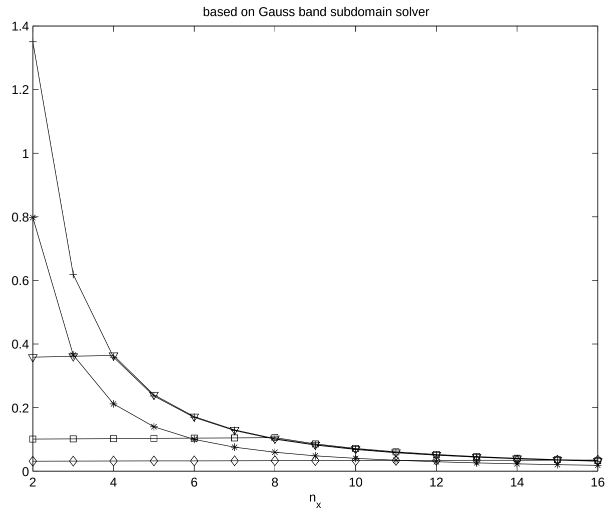

Figure 4: Arithmetic complexity for the linear case assuming sparse Gauss subdomain solver, '*' for $n_y = 1$, '+' for $n_y = 2$, triangles for $n_y = 4$, square for $n_y = 8$, diamond for $n_y = 16$.

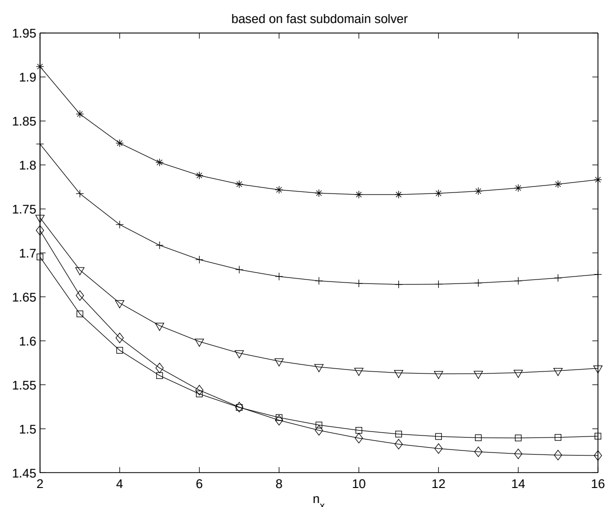

Figure 5: Arithmetic complexity for the linear case assuming fast Poisson subdomain solver '*' for $n_y = 1$, '+' for $n_y = 2$, triangles for $n_y = 4$, square for $n_y = 8$, diamond for $n_y = 16$.

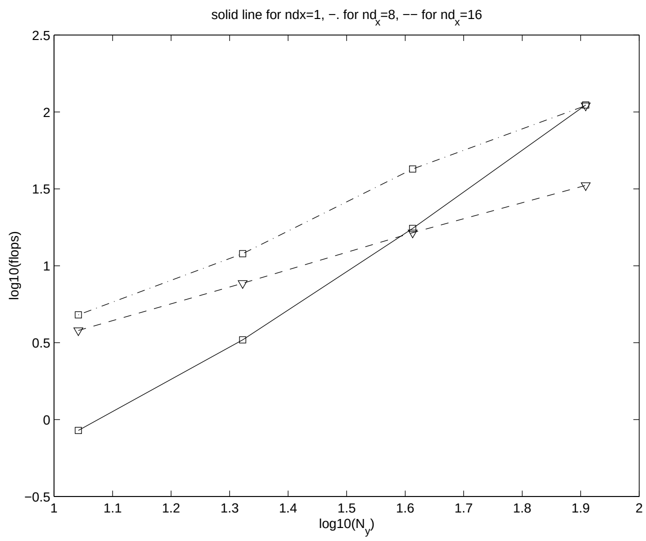

Figure 6: Arithmetic complexity for the nonlinear case, solid line for $n_x = 1$, '.-' for $n_x = 8$, dashed line for $n_x = 16$.

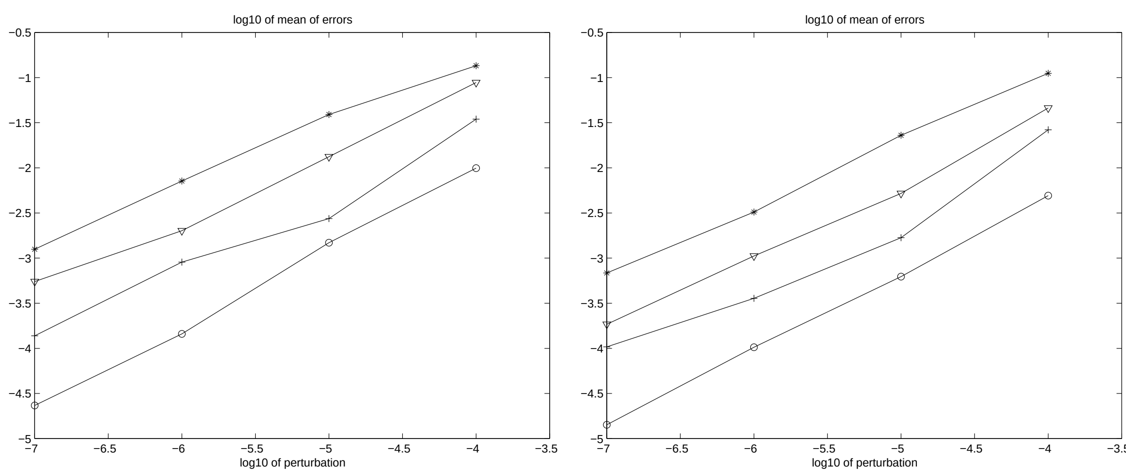

Figure 7: $N_x = 80,\ N_y = 80$, overlap is one mesh cell.