Thirteenth International Conference on Domain Decomposition Methods
Editors: N. Debit, M.Garbey, R. Hoppe, J. Périaux, D. Keyes, Y. Kuznetsov

51 Domain Decomposition Method Applied to a Coupling Vibration Problem between Shell and Acoustics

T. Kako[1], H. M. Nasir[2]

Introduction

We consider the numerical method for the structural-acoustic coupling vibration problem between a shell and acoustic fields by the finite element method. The structure is a shell S which encloses a bounded acoustic region Ω_1 and is surrounded by an unbounded acoustic region Ω_2. The structural-acoustic system is described by a coupled problem between the acoustic pressure perturbations of the inner and outer regions and the tangential and normal deformations of the shell. The problem can be regarded as a domain decomposition formulation for the acoustic fields with a generalized Lagrangian multiplier. The normal deformation of the shell acts as the Lagrangian multiplier which is in turn coupled with tangential deformation of the shell. The finite element approximation to the problem results in a block matrix equation. In order to solve this matrix equation by iterative methods, we consider two techniques: one is based on the Schur complement of the block matrix with appropriate preconditioners and the other is a direct iteration with some block preconditioners. We use a descretized version of fictitious domain method to construct the block matrices and use the Krylov subspace iteration method for solving the system of equations [HKNT98]. The fictitious domain is used to obtain preconditioners for the diagonal matrix blocks. The Schur complement technique requires a double iteration whereas the direct iteration techniques requires only a single iteration. We observe that the direct iteration technique with block preconditioners performs well compared to the Schur complement technique.

Let there be two acoustic regions Ω_1 and Ω_2 in $\mathbb{R}^d, d = 2, 3$, separated by a closed shell structure. The domain Ω_1 is bounded and enclosed by the shell and the domain Ω_2 is unbounded (see Fig. 1). Let $\hat{p}_i, \rho_i$ and c_i be the acoustic pressure perturbation, the density of acoustic material and the sound velocity in the domain $\Omega_i, i = 1, 2$ respectively and p^{inc} be the pressure perturbation of an incident wave from the outer region Ω_2. Then the governing equations for the vibration of the system are given by

$$\begin{aligned} \frac{\partial^2 \hat{p}_i}{\partial t^2} - c_i^2 \Delta \hat{p}_i &= 0 && \text{in } \Omega_i, \quad i = 1, 2, \\ \frac{\partial \hat{p}_i}{\partial n} &= -\rho_i \frac{\partial^2 u_n}{\partial t^2} && \text{on } S, \quad i = 1, 2, \\ \rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} + \mathbf{A}\mathbf{u} &= (\hat{p}_1 - \hat{p}_2)|_S \mathbf{n} && \text{on } S, \\ \hat{p}_2 - p^{inc} & \quad \text{is outgoing,} \end{aligned}$$

where $\mathbf{n}$ is the outward unit normal to the shell surface; $\mathbf{u}$ is the vector of the shell deformations; $u_n = \mathbf{u} \cdot \mathbf{n}$ is the shell deformation along the normal $\mathbf{n}$; $\mathbf{A}$ is the shell force operator.

[1]Univ. of Elecro-Comm., Chofu, Tokyo, Japan, kako@im.uec.ac.jp
[2]University of Peradeniya, Sri Lanka, nasirhm@fedu.uec.ac.jp

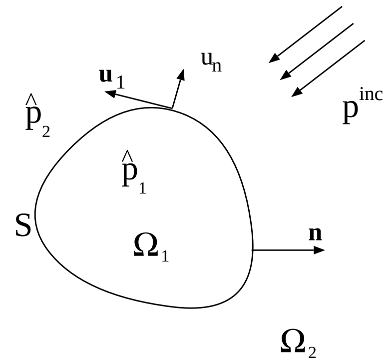

Figure 1: Structural-acoustic coupling

For a general shaped arc shell in two dimensions, the operator $\mathbf{A}$ consists of membrane and flexural components: $\mathbf{A} = D(\mathbf{A}^{memb} + \frac{e^2}{12}\mathbf{A}^{flex})$ with

$$\mathbf{A}^{memb} = \begin{bmatrix} -\partial_s^2 & \partial_s \kappa \\ -\kappa\partial_s & \kappa^2 \end{bmatrix} \text{ and } \mathbf{A}^{flex} = \begin{bmatrix} -A_1^2 & -A_1 A_2 \\ A_2 A_1 & A_2^2 \end{bmatrix}$$

where $A_1 = 2\kappa\partial_s + \kappa'$; $A_2 = \partial_s^2 - \kappa^2$; ∂_s and $'$ denote differentiation with respect to the arc length s, κ and e are the curvature and thickness of the shell respectively and $D = E/(1-\nu^2)$ is the flexural rigidity of the shell with Young's modulus E and Poisson ratio $\nu (0 < \nu \leq 1/2)$. Let $\hat{p}_1 = p_1$ and $\hat{p}_2 = p_2 + p^{inc}$. Then, p_1 and p_2 represent the pressures of scattered waves inside and outside respectively. We also assume that the incident wave, the scattering waves and the deformations of the shell are time-harmonic: $f(x,t) = f(x)e^{\mathbf{i}wt}, f = p_1, p_2, p^{inc}$ or $\mathbf{u}$. Then, the problem can be written as follows:

$$-\Delta p_i - k_i^2 p_i = 0, \qquad \text{in } \Omega_i, i = 1, 2, \tag{1a}$$

$$\frac{\partial p_1}{\partial n} = \rho_1 \omega^2 u_n \qquad \text{on } S, \tag{1b}$$

$$\frac{\partial p_2}{\partial n} = \rho_2 \omega^2 u_n - \frac{\partial p^{inc}}{\partial n} \qquad \text{on } S, \tag{1c}$$

$$\mathbf{A}\mathbf{u} - \rho_0 \omega^2 \mathbf{u} = (p_1 - p_2 - p^{inc})|_S \mathbf{n} \qquad \text{on } S, \tag{1d}$$

$$r^{\frac{d-1}{2}} \left(\frac{\partial p_2}{\partial n} - \mathbf{i} k_2 p_2 \right) \longrightarrow 0, \qquad \text{as } r \to \infty, \tag{1e}$$

where $k_i = \omega/c_i, i = 1, 2$ are the wave numbers corresponding to the inner and outer acoustic regions respectively. The last condition is the Sommerfeld radiation condition for the scattering wave p_2 which allows only the out-going waves in the solution for the outer region.

Approximate problem and weak formulation

For the numerical treatment of the problem, introducing an artificial boundary Γ_R, we restrict the unbounded domain Ω_2 into a bounded domain Ω_R and impose an artificial radiation

boundary condition on Γ_R. We choose Γ_R to be a circle or a sphere of radius R for the two or three dimensional problems respectively.
The Sommerfeld radiation condition is then replaced by the radiation boundary condition

$$\frac{\partial p_2}{\partial n} = M p_2 \tag{2}$$

where M is a differential or pseudo-differential operator with respect to the tangent parameter of the boundary Γ_R.
Let us consider the function spaces $V_1 = H^1(\Omega_1), V_2 = H^1(\Omega_2), V_3 = H^1(S)$ and $V_4 = H^2(S)$ as the solution spaces for $p_1, p_2, \mathbf{u}_t$ and u_n respectively. Here, $\mathbf{u} = (\mathbf{u}_t, u_n)$ and $\mathbf{u}_t$ is the vector of tangential deformation.
The weak formulation of the problem (1) can be given as follows:
Find $(p_1, p_2, \mathbf{u}_t, u_n) \in V_1 \times V_2 \times V_3 \times V_4$ such that, for all $(q_1, q_2, \mathbf{v}_t, v_n) \in V_1 \times V_2 \times V_3 \times V_4$

$$\begin{aligned} a_1(p_1, q_1) + \rho_1\omega^2(u_n, q_1)_S &= 0, && \text{(3a)}\\ a_1(p_2, q_2) - m(p_2, q_2)_{\Gamma_R} - \rho_2\omega^2(u_n, q_2)_{\Omega_R} &= (\partial p^{inc}/\partial n, q_2)_S, && \text{(3b)}\\ b(\mathbf{u}, \mathbf{v}) - (p_1, v_n)_S + (p_2, v_n)_S &= -(p^{inc}, v_n)_S && \text{(3c)} \end{aligned}$$

where

$$\begin{aligned} a_1(p_1, q_1) &= (\nabla p_1, \nabla q_1)_{\Omega_1} - k_1(p_1, q_1)_{\Omega_1},\\ a_2(p_2, q_2) &= (\nabla p_2, \nabla q_2)_{\Omega_R} - k_2(p_2, q_2)_{\Omega_R},\\ m(p_2, q_2)_{\Gamma_R} &= \int_{\Gamma_R} (M p_2)\bar{q}_2 ds \quad \text{and} \quad b(\mathbf{u}, \mathbf{v}) = \int_S (\mathbf{A} - \rho_0\omega^2)\mathbf{u}\bar{\mathbf{v}} d\sigma. \end{aligned}$$

We introduce finite dimensional subspaces V_{ih} of $V_i, i = 1, 2, 3, 4$ respectively and consider the approximate weak formulation, i.e., the finite element method:
Find $(p_{1h}, p_{2h}, \mathbf{u}_{th}, u_{nh}) \in V_{1h} \times V_{2h} \times V_{3h} \times V_{4h}$ such that for all
$(q_1, q_2, \mathbf{v}_t, v_n) \in V_{1h} \times V_{2h} \times V_{3h} \times V_{4h}$,

$$\begin{aligned} a_1(p_{1h}, q_1) + \rho_1\omega^2(u_{nh}, q_1)_S &= 0, && \text{(4a)}\\ a_1(p_{2h}, q_2) - m(p_{2h}, q_2)_{\Gamma_R} - \rho_2\omega^2(u_{nh}, q_2)_{\Omega_R} &= (\partial p^{inc}/\partial n, q_2)_S, && \text{(4b)}\\ b(\mathbf{u}_h, \mathbf{v}) - (p_{1h}, v_n)_S + (p_{2h}, v_n)_S &= -(p^{inc}, v_n)_S. && \text{(4c)} \end{aligned}$$

By choosing bases for the function spaces and writing $p_{1h}, p_{2h}, \mathbf{u}_{th}$ and u_{nh} with respect to these bases, we obtain the block matrix equation as follows:

$$\begin{bmatrix} M_1 & 0 & 0 & -\rho_1\omega^2 L_1^T \\ 0 & M_2 & 0 & \rho_2\omega^2 L_2^T \\ 0 & 0 & A & B^T \\ -L_1 & L_2 & B & C \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ U_t \\ U_n \end{bmatrix} = \begin{bmatrix} 0 \\ F \\ 0 \\ G \end{bmatrix} \tag{5}$$

where each block corresponds to the sesquilinear form and its entries are given with respect to the chosen base functions.
The matrices M_1 and M_2 are constructed by the finite element discretization of fictitious domains. For the inner bounded domain Ω_1, we consider a rectangular region such that Ω_1 is

included in it. We discretize the rectangular domain by a uniform orthogonal rectangular grid. Then, the nodes close to the boundary of Ω_1 are moved onto the boundary so that the new locally modified partition is topologically equivalent to the orthogonal grid partition. Then, the modified rectangles are triangulated such that the resulting triangles satisfy a regularity condition. The computational domain for the inner region is then obtained by discarding the extended portion in the rectangular fictitious domain.
Similarly, for the outer domain Ω_R, we enlarge the domain towards inside the inner boundary of Ω_R so that it makes an annulus including the inner boundary of Ω_R. We discretize the annulus by a uniform orthogonal polar grid. The nodes near the inner boundary of Ω_R are modified as in the case of inner domain (see Fig. 2).

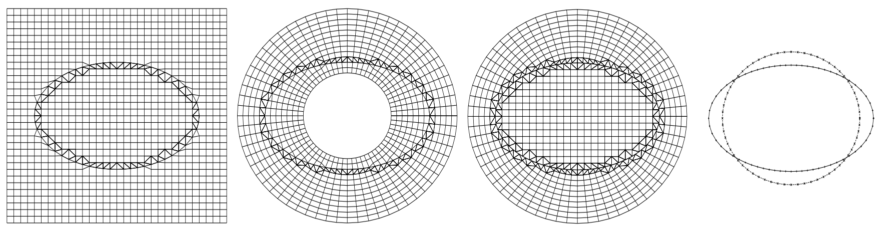

Figure 2: Fictitious domains and partitioning

Preconditioners for the matrices M_1 and M_2 that are constructed by the fictitious domain method are obtained by using the enlarged fictitious domain itself. We explain the construction of the preconditioner for the inner region. The one for the outer region follows analogously.
The unmodified orthogonal mesh is used to obtain a matrix by using the same weak formulation on the fictitious domain. This will give a matrix $\mathbf{N}$ which we write in a block form as follows:

$$\mathbf{N}_1 = \begin{bmatrix} N_{11} & N_{12} \\ N_{12}^T & N_{22} \end{bmatrix}$$

where the matrix N_{11} corresponds to the nodes on the inner region, but not moved; the matrix N_{22} corresponds to the nodes outside the inner region. The matrix $\mathbf{N}_1$ is obviously larger in size than the original matrix M_1 which corresponds to the moved nodes on the inner boundary. When we want to solve a matrix equation of the form

$$M_1 P_1 = F_1,$$

we enlarge the system as follows:

$$\mathbf{M}_i = \begin{bmatrix} M_i & N_{12} \\ 0 & N_{22} \end{bmatrix} \begin{bmatrix} P_1 \\ Q_1 \end{bmatrix} = \begin{bmatrix} F_1 \\ 0 \end{bmatrix}.$$

The two system of equations are equivalent in the sense that the solution P_1 is the same for both systems. Hence, we solve the enlarged system using the Krylov subspace iteration method with the matrix $\mathbf{N}_1$ as a preconditioner. For more details of the fictitious domain method see [HKNT98].

Schur Complement Method

The Schur complement of the block matrix with respect to its last block is obtained by solving the block matrix equation (5) for the vector component U_2:

$$\left[C - BA^{-1}B^T - \rho_1\omega^2 L_1 M_1^{-1} L_1^T - \rho_2\omega^2 L_2 M_2^{-1} L_2^T\right] U_2 = G - M_2^{-1} F \tag{6}$$

This matrix can then be solved numerically by using the Krylov subspace method. The terms involving matrix inverses in this Schur complement are computed based on the fictitious domain method with preconditioners obtained from the fictitious domains.

Direct Iteration Method

In this method, we directly use the Krylov subspace iteration procedure to solve the block matrix equation (5). For this purpose, the block matrix equation is enlarged to the the one with the size corresponding to that of their fictitious domain preconditioners as follows:

$$\begin{bmatrix} \mathbf{M}_1 & 0 & 0 & -k^2\mathbf{L}_1^T \\ 0 & \mathbf{M}_2 & 0 & k^2\mathbf{L}_2^T \\ 0 & 0 & A & B^T \\ -\mathbf{L}_1 & \mathbf{L}_2 & B & C \end{bmatrix} \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{U}_t \\ \mathbf{U}_n \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{F} \\ \mathbf{0} \\ \mathbf{G} \end{bmatrix}$$

where the matrices in bold symbols are the enlarged matrices of their counterparts in the block matrix equation (5).

The preconditioner used for this method is based on the preconditioning technique by Bramble and Pasciak [BP88] which is given as follows:

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ -\mathbf{L}_1 & \mathbf{L}_2 & B & -I \end{bmatrix} \begin{bmatrix} \mathbf{N}_1^{-1} & 0 & 0 & 0 \\ 0 & \mathbf{N}_2^{-1} & 0 & 0 \\ 0 & 0 & A_0^{-1} & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

where the matrix A_0 is the preconditioner for the matrix A based on the fictitious shell domain (see Fig. 2). The first matrix is an elementary pre-multiplication matrix which makes the preconditioned matrix symmetric.

Numerical Results

We present in this section the results of the implementation of the method. All computations were carried out on VT-Alpha5, 533Mhz, 512MB RAM with Linux operating system environment with double precision arithmetic using object oriented C++ codes.

We test the two iterative methods in the last two sections for a two dimensional shell-acoustic coupling problem. The shell is a circular arc of radius $r_0 = 1$. The densities of the acoustic material in both inner and outer acoustic regions are the same $\rho_1 = \rho_2 = 1$. The artificial boundary for the outer acoustic region is a circle of radius $R = 2$ is chosen. The incident wave is a plane wave with wave number $k = \pi$.

In the Schur complement methods, each iteration step requires the matrix inverses of M_1, M_2 and A_0. These are performed by an inner iteration using the fictitious domain method. Hence, each iteration step of the Schur complement matrix equation involves other iterations. Table 1 shows the number of iterations and times for both Schur complement and direct iteration methods.

Method	Outer Iter.	Inv. Mat. Multiplications			time (sec.)
		M_1	M_2	A	
Schur Complement	23	1276	267	23	80.45
Direct iteration	35	35	35	35	3.20

Table 1: Performances of Schur complement and direct iteration methods

The Schur complement method has 23 outer iteration steps each of them has inner iterations. The total numbers of inner iterations are 1276 for M_1, 267 for M_2, and 23 for A and the total time for the iterations is 80.45sec. For the case of A, the preconditioner is A_0 the same as A, because the shell is circular. Hence, it has only one iteration per step.

For the direct preconditioning method, the total iterations required to achieve the same result is 35. Each iterative step requires one matrix multiplication with the preconditioned matrices for M_1, M_2 and A_0. Hence the total numbers of iterations is 35 for each matrices. The time required for the iterations is 3.20sec.

Figure 3 shows the real part of the scattering waves for circular and elliptic shell cases. The radius of the artificial boundary is $R = 2$. The incident wave is a plane wave coming from left along the x-axis direction with wave numbers $k = 2\pi$ and 3π. The radius of the shell is $r_0 = 1$ and the major and minor axes of the elliptic shell are $2a = 3.2$ and $2b = 2$.

Conclusion

The structural-acoustic coupling problem between a shell and inner and outer acoustic fields is considered by the finite element method. Fictitious domain method is used to discretize the acoustic domains and the resulting block matrix equation is solved by a Krylov subspace iteration methods.

Two schemes are used: A Schur complement method and a direct block iteration method. The Schur complement method requires a double iteration while the direct block iteration method needs a single iteration.

The block iteration method performs very well in terms of the numbers of matrix multiplications and computing time.

References

[Ber96]Michel Bernadou. *Finite Element Methods for Thin Shell Problems*. John Wiley & Sons, 1996. Translated by Claude Andrew James.

[BP88]James H. Bramble and Joseph E. Pasciak. A preconditioning technique for indefinite systems resulting from mixed approximations of elliptic problems. *Mathematics of Computation*, 50(181):1–17, 1988.

[BW90]Christoph Börgers and Olof B. Widlund. On finite element domain imbedding methods. *SIAM J. Numer. Anal.*, 27(4), August 1990.

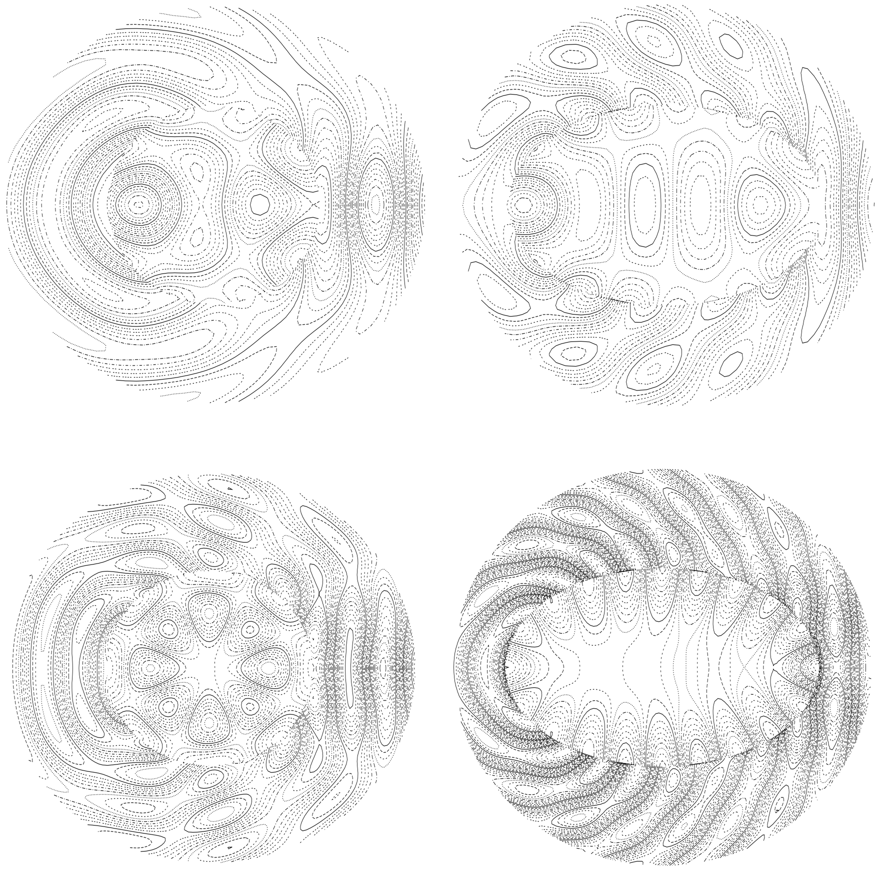

Figure 3: Scattering waves: circular and elliptic cases

[Cia78]Philippe G. Ciarlet. *The Finite Element Method for Elliptic Problems*. North-Holland, Amsterdam, 1978.

[Ern96]Oliver G. Ernst. A finite-element capacitance matrix method for exterior helmholtz problems. *Numer. Math.*, 75(2):175–204, 1996.

[Fre93]R. W. Freund. A transpose-free quasi-minimal residual algorithm for non-hermitian linear systems. *SIAM J. Sci. Comput.*, 14(2):470–482, 1993.

[HKNT98]E. Heikkola, Y. A. Kuznetsov, P. Neittaanmaki, and J. Toivanen. Fictitious domain methods for the numerical solution of two-dimesional scattering problems. *J. Comput. Phys.*, 145:89–109, 1998.

[Li98]Deng Li. *Study on Finite Element Approximation and Perturbation Analysis of Eigenvalue Problems for Structural-Acoustic Coupled System*. PhD thesis, The University of Electro-Communications, Japan, March 1998.